MATH 308. Differential Equations

Lecture 8. Autonomous equations; Existence and Uniqueness Theorem

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September 13, 2022

Existence & Uniqueness Theorem

For the initial value problem

$$\frac{dy}{dx} = f(x, y), \qquad y(x_0) = y_0$$

with continuous f and $\frac{\partial f}{\partial y}$, the solution *exists* and is *unique* on *some* interval that contains x_0.

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[Plotting solution curves]
We cannot predict the domain before solving the equation.

Existence & Uniqueness Theorem

First-order Linear Equations

For a *first-order linear* *differential equation*

$$y' + p(x)y = q(x)$$

with given initial conditions, the solution *exists* and is *unique* on the *same* interval where the coefficients $p(x), q(x)$ are well-defined and are continuous.

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Example

$$xy' + y = e^x, \qquad y(1) = 5.$$

What is the domain of its solution?

(A) $\mathbb{R}$; (B) $x > 0$; (C) $x > 1$; (D) IDK.

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Non-existence: $y(0) = 1$ would be a bad initial condition, with no solution, since $p(0), q(0)$ are undefined.

Existence & Uniqueness Theorem

Non-uniqueness example: Liquid in a barrel

Liquid flows from a barrel at the rate proportional to the square root of the height of the remaining liquid,

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Are assumptions of the Existence and Uniqueness Theorem satisfied for all t_0, h_0?

(A) yes; (B) no.

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Interpretation of the Theorem in terms of solution curves

For any initial condition, solution exists $=$ Solution curves fill all the domain where f, df/dy are continuous.

For any initial condition, solution is unique $=$ Solution curves do not intersect or merge in the domain where f, df/dy are continuous.

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We can solve them using separation of variables.

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Equilibrium state = equilibrium = critical point = singular point = equilibrium point.

Direction field and phase portrait

What can we say about solutions without solving the equation? [Discussion on a whiteboard: solution curves and the phase line for the logistic equation $y' = 2y(2 - 3y)$, cf. Fig.2.5.6 in the textbook.]

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Conclusion : all positive solutions tend to 1.5 (carrying capacity) as $x \to +\infty$.

Example

Let $y(x)$ be the solution of $y' = y^2(y-2)(y-3)$ with initial condition $y(0) = 2.5$. Plot the phase line and find $\lim_{x\to\infty} y(x)$.

(A) 0

(B) 2

(C) 3

(D) ∞

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Stability of an equilibrium state

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Example

$y' = y^2(y-2)(y-3)$:

- $0, 2, 3$ are equilibriums
- Conclusion: 0 is semistable, 2 is stable, 3 is unstable.