MATH 308. Differential Equations

Lecture 7: Second-order equations (Sec. 3.1-3.3)

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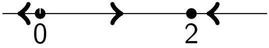
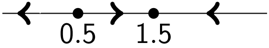
Overfishing

Logistic model for population growth: $y' = ky(c - y)$.
If a certain number of fish is caught: $y' = ky(c - y) - h$.

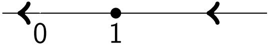
- $y' = 2y - y^2$:



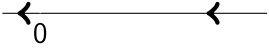
- $y' = 2y - y^2 - 0.75$:



- $y' = 2y - y^2 - 1$:



- $y' = 2y - y^2 - 1.1$:



Second-order linear ODEs

Definition

An equation of the form

$$y'' + p(t)y' + q(t) = r(t)$$

is called a *second-order linear differential equation*.

Comment

If we have $M(t)y'' + N(t)y' + S(t)y = U(t)$, we can divide by $M(t)$.

Initial conditions:

$$y(t_0) = a, \qquad y'(t_0) = b.$$

Spring-mass oscillator

A ball attached to a spring oscillates according to

$$my'' = F_{tension} = -ky \text{ (Hooke's law).}$$

Damped oscillator: $my'' = -ky - cy'$.
$y(t_0) = y_0$ and $y'(t_0) = v_0$ are initial position and initial velocity of the system.

Example

Consider the equation $y'' = -y$. Which of the following functions are solutions of this equation? [multiple choice]

(A) e^t

(B) e^{-t}

(C) $\cos t$

(D) $\cos t + 2\sin t$

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General solution: $y(t) = C_1 \sin t + C_2 \cos t$.
If initial conditions are provided, find C_1, C_2 to satisfy them.

Existence and Uniqueness. Superposition Principle

Theorem (Existence and Uniqueness Theorem)

Let p, q, r be continuous functions. Then the initial value problem

$$y'' + p(t)y' + q(t)y = r(t), \qquad y(t_0) = y_0, \qquad y'(t_0) = v_0$$

has a unique solution for any t_0, y_0, v_0.

Now set $r(t) = 0$ (homogeneous equations)

Superposition principle

If y_1 and y_2 are solutions of $y'' + p(t)y' + q(t)y = 0$, then $C_1y_1 + C_2y_2$ is also a solution for any constants C_1, C_2.

Equations with constant coefficients. Characteristic equation

Take $\quad y'' + ay' + by = 0 \quad$ with constant a, b.

- When $y(t) = e^{\lambda t}$ is a solution of $y'' + ay' + by = 0$?
- Answer: when λ is a root of the *characteristic equation* $\lambda^2 + a\lambda + b = 0$.

Theorem

If the characteristic equation has 2 real roots $\lambda_1 \neq \lambda_2$, then

$$y(t) = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t}$$

is a general solution of $y'' + ay' + by = 0$.

Proof:

- (1) Such $y(t)$ is a solution (superposition principle);
- (2) Such $y(t)$ can satisfy all initial conditions;
- (3) Thus it gives all solutions (Existence&Uniqueness Thm).

Examples

Example

Solve $y'' - 4y' + 3y = 0$ with initial conditions $y(0) = 2$, $y'(0) = 4$.

Characteristic equation $\lambda^2 - 4\lambda + 3 = 0$

Roots $\lambda_1 = 1$, $\lambda_2 = 3$

General solution $C_1 e^t + C_2 e^{3t}$

Finding parameters $C_1 + C_2 = 2$, $C_1 + 3C_2 = 4$, hence $C_1 = C_2 = 1$.

Example

Solve $y'' - 4y = 0$ with initial conditions $y(0) = 0$, $y'(0) = 4$.

(A) $e^{2t} - e^{-2t}$;

(B) $\frac{1}{2}e^{4t} - \frac{1}{2}e^{-4t}$;

(C) $e^{4t} - 1$;

(D) e^{2t}.

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What can go wrong?

- complex roots: $y'' + 4y = 0$, $y'' + y' + y = 0$;
- repeated root: $y'' = 0$, $y'' + 2y' + y = 0$.