# MATH 308. Differential Equations

## Lecture 8: Second-order equations (Sec. 3.2-3.3)

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# Reminder: Characteristic equation – real roots

$$y'' + ay' + by = 0 \to \lambda^2 + a\lambda + b = 0 \text{ (characteristic equation)}$$

## Theorem

*If the characteristic equation has 2 real roots $\lambda_1 \neq \lambda_2$, then*

$$y(t) = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t}$$

*is a general solution of $y'' + ay' + by = 0$.*

## Example

Solve $y'' - 4y' + 3y = 0$ with initial conditions $y(0) = 2$, $y'(0) = 4$.

Characteristic equation $\lambda^2 - 4\lambda + 3 = 0$

Roots $\lambda_1 = 1$, $\lambda_2 = 3$

General solution $y(t) = C_1 e^t + C_2 e^{3t}$

Finding parameters $C_1 + C_2 = 2$, $C_1 + 3C_2 = 4$, hence $C_1 = C_2 = 1$.

# Complex roots

## Example

Solve $y'' + 2y' + 5y = 0$.

Characteristic equation $\lambda^2 + 2\lambda + 5 = 0$

Roots $\lambda_{1,2} = \frac{-2\pm\sqrt{-16}}{2} = -1 \pm 2i$.

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- Definition:

$$e^{m+in} = e^m(\cos n + i \sin n)$$

- If the characteristic equation has complex roots $\lambda = m \pm in$, then we have a complex solution

$$y(t) = e^{\lambda t} = e^{(m+in)t} = e^{mt}(\cos(nt) + i\sin(nt))$$

- Its real and imaginary part are real solutions:

$$y(t) = e^{mt}\cos(nt) \text{ and } y(t) = e^{mt}\sin(nt)$$

- The general solution:

$$y(t) = C_1 e^{mt}\cos(nt) + C_2 e^{mt}\sin(nt)$$

# Examples: complex roots

**Example**

Solve $y'' + 2y' + 5y = 0$.

Characteristic equation $\lambda^2 + 2\lambda + 5 = 0$

Roots $\lambda_{1,2} = \frac{-2\pm\sqrt{-16}}{2} = -1 \pm 2i$.

General solution $C_1 e^{-t}\cos 2t + C_2 e^{-t}\sin 2t$.

**Example**

Solve $y'' + ky = 0$ for $k > 0$.

(A) $C_1 e^{kt} + C_2$;

(B) $C_1 \sin\sqrt{k}t + C_2\cos\sqrt{k}t$;

(C) $C_1 \sin kt + C_2 \cos kt$;

(D) $C_1 e^{\sqrt{k}t} + C_2 e^{-\sqrt{k}t}$.

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# Sympy method dsolve()

`https://live.sympy.org/`

The command dsolve(diff(f(t), t, 2) - diff( f(t), t) - f(t)) means "solve $\frac{d^2 f}{dt^2} - \frac{df}{dt} - f(t) = 0$".
Examples:

$$y' = -y^2, y'' + y = 0, t^2 y'' + y = 0, y'' - ty = 0.$$

# Fundamental set of solutions

$$y'' + p(t)y' + q(t)y = 0$$

### Definition

The pair $(y_1, y_2)$ is called a fundamental set of solutions if

$$y(t) = C_1 y_1(t) + C_2 y_2(t)$$

is a general solution.

- $(e^t, e^{3t})$ is a fundamental set of solutions of the equation $y'' - 4y' + 3y = 0$.
- $(e^t + e^{3t}, -e^t)$ is another fundamental set of solutions.
- $(e^t + e^{3t}, 2e^t + 2e^{3t})$ is not a fundamental set of solutions.

### Theorem (Criterion for $y_1, y_2$ to form a fundamental set)

*Two solutions $y_1, y_2$ form a fundamental set if and only if they are not proportional.*

# Fundamental set of solutions: Examples

$$y'' + p(t)y' + q(t)y = 0$$

**Definition:** The pair $(y_1, y_2)$ is called a fundamental set of solutions if

$$y(t) = C_1 y_1(t) + C_2 y_2(t)$$

is a general solution.

### Theorem (Criterion for $y_1, y_2$ to form a fundamental set)

*Two solutions $y_1, y_2$ form a fundamental set if and only if they are not proportional.*

### Example

Which of the following is a fundamental set of solutions for the equation $y'' = 0$?

(A) $(1, t)$;

(B) $(t + 1, 2t - 1)$;

(C) $(t + 1, 2t + 2)$;

(D) $(t, 0)$.

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# Wronskians for $y'' + p(t)y' + q(t)y = 0$

**Definition:** The *Wronskian* of solutions $y_1$, $y_2$ is $W(t) = y_1 y_2' - y_1' y_2$.

Theorem (Abel's formula)

*For any two solutions $y_1$, $y_2$ we have*

$$W'(t) = -p(t)W(t), \qquad \textit{thus } W(t) = k \exp\left(\int (-p(t))dt\right)$$

Remark

If $k = 0$, then $W(t) = 0$ everywhere. If $k \neq 0$, then $W(t)$ is not zero at any point where $p(t)$ is defined.

Example

Find the Wronskian of any two solutions of the equation $ty'' - y' - ty = 0$.

(A) $kt$; (B) $k \ln t$; (C) $ke^{-t}$; (D) $k$.

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# Proof of Abel's formula

$$y'' + p(t)y' + q(t)y = 0$$

Theorem (Abel's formula)

$$W'(x) = -p(t)W(t)$$

*for any two solutions $y_1, y_2$.*

Proof.

$$\begin{aligned}(y_1y_2' - y_1'y_2)' &= y_1'y_2' + y_1y_2'' - y_1''y_2 - y_1'y_2' \\ &= y_1y_2'' - y_1''y_2 \\ &= y_1(-p(t)y_2' - q(t)y_2) - (-p(t)y_1' - q(t)y_1)y_2 \\ &= -p(t)(y_1y_2' - y_1'y_2) \\ &= -p(t)W(t).\end{aligned}$$

□

# Criterion for $y_1, y_2$ to form a fundamental set

Given two solutions $y_1$, $y_2$ of $y'' + p(t)y' + q(t)y = 0$, the following are equivalent:

- $y_1$ and $y_2$ form a fundamental system of solutions;
- $y_1$ and $y_2$ are not proportional;
- their Wronskian is not zero at some point;
- their Wronskian is not zero at all points.

# Proof of the Criterion

∘ If solutions are proportional, $y_2 = cy_1$, their Wronskian is $cy_1 y_1' - cy_1 y_1' = 0$ everywhere and they do not form a fundamental set.
∘ If solutions form a fundamental set, they must be not proportional. Now check $W(t_0) \neq 0$ at some point $t_0$.

- $y(t) = C_1 y_1 + C_2 y_2$ can satisfy any initial condition $A, B$:
- $y(t_0) = C_1 y_1(t_0) + C_2 y_2(t_0) = A$,
  $y'(t_0) = C_1 y_1'(t_0) + C_2 y_2'(t_0) = B$
- We can always solve for $C_1$, $C_2$, thus LHS are not proportional.
- $y_1(t_0) : y_1(0) \neq y_2(t_0) : y_2'(t_0)$.
- $y_1(t_0) y_2'(t_0) - y_1'(t_0) y_2(t_0) \neq 0$.
- $W(t_0) \neq 0$, q.e.d