MATH 308. Differential Equations

Lecture 9: Second-order equations (Sec. 3.4, 3.5, 3.7)

Nataliya Goncharuk

Texas A & M

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Criterion for y_1, y_2 to form a fundamental set

Theorem

Given two solutions y_1, y_2 of $y'' + p(t)y' + q(t)y = 0$, the following are equivalent:

- *y_1 and y_2 form a fundamental system of solutions;*
- *y_1 and y_2 are not proportional;*
- *their Wronskian is not zero at some point;*
- *their Wronskian is not zero at all points.*

Proof of the Criterion

∘ If solutions are proportional, $y_2 = cy_1$, their Wronskian is $cy_1y_1' - cy_1y_1' = 0$ everywhere and they do not form a fundamental set.
∘ If solutions form a fundamental set, they must be not proportional. Now check $W(t_0) \neq 0$ at some point t_0. $W(t) \neq 0$ will follow due to the Abel's formula.

Proof of the Criterion (continued)

Proving that if solutions form a fundamental set, then $W(t_0) \neq 0$ at some point t_0.

- $y(t) = C_1 y_1 + C_2 y_2$ gives all solutions, thus
- $y(t) = C_1 y_1 + C_2 y_2$ can satisfy any initial condition A, B:
- $y(t_0) = C_1 y_1(t_0) + C_2 y_2(t_0) = A$,
  $y'(t_0) = C_1 y_1'(t_0) + C_2 y_2'(t_0) = B$
- We can always solve for C_1, C_2, thus LHS are not proportional.
- $y_1(t_0) : y_1(0) \neq y_2(t_0) : y_2'(t_0)$.
- $y_1(t_0) y_2'(t_0) - y_1'(t_0) y_2(t_0) \neq 0$.
- $W(t_0) \neq 0$, q.e.d

Solving second-order equations with constant coefficients

$$y'' + ay' + by = 0 \to \lambda^2 + a\lambda + b = 0 \text{ (characteristic equation)}$$

Theorem

- *If the characteristic equation has 2 real roots* $\lambda_1 \neq \lambda_2$*, then the general solution is*

$$y(t) = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t}$$

- *If the characteristic equation has 2 complex roots* $\lambda_{1,2} = m \pm in$*, then the general solution is*

$$y(t) = C_1 e^{mt} \cos nt + C_2 e^{mt} \sin nt$$

- *NEW: If the characteristic equation has only 1 real root* $\lambda = r$*, then the general solution is*

$$y(t) = C_1 e^{rt} + C_2 t e^{rt}$$

Example

If the characteristic equation has only 1 real root r, then

$$y(t) = C_1 e^{rt} + C_2 t e^{rt}$$

is a general solution of $y'' + ay' + by = 0$.

Example

Solve the equation $y'' + 2y' + y = 0$ with initial conditions $y(0) = 0, y'(0) = 1$.

(A) $te^t + 1$;

(B) te^t;

(C) te^{-t};

(D) $-te^{-t}$.

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Proof: reduction of order for $y'' - 2ry' + r^2 y = 0$

- Let the characteristic equation be $\lambda^2 - 2r\lambda + r^2 = 0$, with the only root r. Then $y(t) = e^{rt}$ is one solution.
- Find a solution in the form $y(t) = v(t)e^{rt}$:
- $(v''e^{rt} + 2rv'e^{rt} + vr^2e^{rt}) - 2r(v'e^{rt} + vre^{rt}) + r^2 \cdot (ve^{rt}) = 0$ thus
- $v'' = 0$
- Thus $v(t) = C_1 + C_2 t$.
- General solution: $y(t) = e^{rt}(C_1 + C_2 t)$.

Reduction of order: *if we know one solution f of the equation, we can always find the second one in the form $v \cdot f$. See HW4 Task 3 or the end of Sec. 3.4.*

Damped oscillators (Sec. 3.7)

Damped oscillator:

$$my'' = -ky - cy'$$

m is mass, k is spring tension, c is damping coefficient.

Characteristic equation: $\lambda^2 + \frac{c}{m}\lambda + \frac{k}{m} = 0$

[Using https://www.desmos.com/calculator/q8vvkeo0cf to show graphs of solutions in the case when we have 2 real or 2 complex roots]

Underdamped oscillator – complex roots;

Critically damped oscillator – 1 real root;

Overdamped oscillator – 2 real roots.

Example

Which of the following oscillators are underdamped?

(A) $y'' + 4y' + 9y = 0$, (B) $y'' + 5y' + 9y = 0$,

(C) $y'' + 6y' + 9y = 0$, (D) $y'' + 7y' + 9y = 0$.

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Forced oscillators

Forced (undamped) oscillator:

$$my'' = -ky + F_{out}(t)$$

m is mass, k is spring tension, $F_{out}(t)$ is an outer force.
This is a nonhomogeneous equation.

Definition

The equation

$$y'' + p(t)y' + q(t)y = r(t)$$

with nonzero $r(t)$ is called a nonhomogenous linear second-order equation.

Superposition principle does not work for nonhomogeneous equations.

Structure of solutions

Theorem

If y_p is a particular solution of $y'' + p(t)y' + q(t)y = r(t)$ and y_c is the general solution of $y'' + p(t)y' + q(t)y = 0$ **(complementary function)**, *then*

$$y = y_c + y_p$$

is the general solution of $y'' + p(t)y' + q(t)y = r(t)$.

Example

$$y'' + y = t + 1$$

- $y_p = t + 1$ is a particular solution.
- $y_c = c_1 \sin t + c_2 \cos t$ is the general soltuion of $y'' + y = 0$.
- $$y(t) = y_p + y_c = 1 + t + c_1 \sin t + c_2 \cos t.$$

Example

Example ($y'' + y = t + 1$:)

$y_c = c_1 \sin t + c_2 \cos t$ — general solution of $y'' + y = 0$,
$y_p = 1 + t$ — particular solution of $y'' + y = t + 1$,
$y = y_c + y_p = 1 + t + c_1 \sin t + c_2 \cos t$.

Example ($y'' = y + e^{2t}$:)

- Guess a solution in the form $y_p(t) = ke^{2t}$ (find appropriate k).
- Solve the corresponding homogeneous equation.
- Write out the general solution.
  (A) $y = ke^{2t} + c_1 e^t + c_2 e^{-t}$;
  (B) $y = \frac{1}{3} e^{2t} + c_1 e^t + c_2 e^{-t}$;
  (C) $y = e^{2t} + c_1 e^t + c_2 e^{-t}$;
  (D) There is no solution of the form ke^{2t}.

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Method of undetermined coefficients: Guessing one solution y_p

$$y'' + ay' + by = f(t)$$

Guessing table

suppose $f(t) is$	then try for y_p
$ce^{\lambda t}$	$ke^{\lambda t}$
$c \sin at$ or $c \cos at$	$k \sin at + l \cos at$
Polynomial $p(t)$	Polynomial $q(t)$, same degree
$p(t)e^{\lambda t} \sin at$	$q_1(t)e^{\lambda t} \sin at + q_2(t)e^{\lambda t} \cos at$

If some terms of your guess appear in the complementary function, it will not work. **(Resonant case).**
Rule: multiply your guess by t. If situation persists, multiply by t^2

Example: forced oscillator (Sec. 3.8), $f(t) = \sin \omega t$

$y'' = -y + \sin \omega t$

- $y_c = c_1 \cos t + c_2 \sin t$
- Our guess: $y_p(t) = a \sin \omega t + b \cos \omega t$
- Substitute:
  $-a\omega^2 \sin \omega t - b\omega^2 \cos \omega t = -a \sin \omega t - b \cos \omega t + \sin \omega t$
- Find a, b: $-a\omega^2 = -a + 1, -b\omega^2 = -b$,
- thus $b = 0, a = \frac{1}{1-\omega^2}$,
- $$y(x) = \frac{1}{1 - \omega^2} \sin \omega t + c_1 \cos t + c_2 \sin t$$

Resonant case: $\omega = \pm 1$, terms of y_p appear in y_c.
Then find y in the form $at \sin \omega t + bt \cos \omega t$.