Final Exam

MATH 308 Sec 508
Texas A&M University, College Station

Nataliya Goncharuk

December 9, 2022, 3:00 - 5:00 pm (120 min).

Honor the Aggie Code: "An Aggie does not lie, cheat, or steal or tolerate those who do."

Your Name:

Your UIN:

Task 1. (10 pts) Find the general solution and the solution with initial condition $y(0) = 1$ for the following equation:

$$(x+1)y' + (y+2) = 0.$$

Task 2. (10+3+2 pts) The motion of the oscillator is described by the following equation:

$$y'' + 4y' + 5y = \sin t$$

(a) Find the general solution of this equation.

(b) Show that for any two solutions of this equation y_1 and y_2, their difference $y_1(t) - y_2(t)$ tends to zero as $t \to +\infty$.

(c) Find the (only) periodic solution of this equation.

Scratch paper for Task 2

Task 3. (15 pts) Find the solution of the initial value problem

$$y'' + 4y' + 5y = 2\delta(t-5), \qquad y(0) = 0, y'(0) = 1.$$

Task 4. (15 pts) Find the power series at zero $y(x) = \sum_{n=0}^{\infty} a_n x^n$ for the solution $y(x)$ of the equation

$$y' - xy = 1$$

with initial condition $y(0) = 1$.

Task 5. (5+10+10+10 pts) (a) Find the matrix exponential e^{At} where the matrix A is $A = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}$.

(b) Solve the system

$$x_1' = x_1 + 2x_2, x_2' = 4x_1 + 3x_2,$$

with initial condition $x_1(0) = 3, x_2(0) = 3$.

(c) Classify the linear system $x' = Ax$ as saddle, node (sink or source), spiral sink or source, or center. Make a conclusion: will all solutions of this system tend to zero as $t \to +\infty$?

(d) Find the general solution of the system

$$x_1' = x_1 + 2x_2, x_2' = 4x_1 + 3x_2 + 3e^{5t}.$$

Task 6. (5 +5 pts) (a) Find all a, b, c such that the function $x(t) = a + bt + ce^{-t}$ is the solution of the equation

$$x'(t) = t - x.$$

(b) A lazy cat is chasing a mouse. The mouse is running away at a constant speed 1 meter per second, and its position at time t is t meters from the start. The position of the cat is $x(t)$; the cat speeds up if the distance to the mouse increases. The velocity of the cat satisfies

$$x'(t) = t - x(t).$$

At time $t = 0$, the cat is 2 meters behind the mouse. Find $x(t)$ (hint: you may use (a)). Will the cat ever catch the mouse?

End of the exam. The Table of Laplace transforms appears at the end of the exam script.

Scratch paper for Task 6

Scratch paper

Scratch paper

$f(t)$	$F(s) = \mathcal{L}\{f\}(s)$	
1	$\frac{1}{s}$, $s > 0$	
e^{at}	$\frac{1}{s-a}$, $s > a$	
$t^n,\ n = 1, 2, \ldots$	$\frac{n!}{s^{n+1}}$, $s > 0$	
$\sin{(at)}$	$\frac{a}{s^2+a^2}$, $s > 0$	
$\cos{(at)}$	$\frac{s}{s^2+a^2}$, $s > 0$	
$\sinh{(at)}$	$\frac{a}{s^2-a^2}$, $s > \lvert a \rvert$	
$\cosh{(at)}$	$\frac{s}{s^2-a^2}$, $s > \lvert a \rvert$	
$e^{at}\sin bt$	$\frac{b}{(s-a)^2+b^2}$, $s > a$	
$e^{at}\cos bt$	$\frac{s-a}{(s-a)^2+b^2}$, $s > a$	
$t^n e^{at},\ n = 1, 2, \ldots$	$\frac{n!}{(s-a)^{n+1}}$, $s > a$	
$u_c(t) = \begin{cases} 0 & t < c \\ 1 & t \geq c \end{cases}$	$\dfrac{e^{-cs}}{s}$, $s > 0$	for $c \geq 0$
$u_c(t)f(t-c)$	$e^{-cs}F(s)$	for $c \geq 0$
$e^{ct}f(t)$	$F(s-c)$	
$f(ct)$	$\frac{1}{c}F(\frac{s}{c})$	for $c > 0$

$f(t)$	$F(s) = \mathcal{L}\{f\}(s)$	
$(f * g)(t) = \int_0^t f(t-\tau)g(\tau)d\tau$	$F(s)G(s)$	
$\delta(t-c)$	e^{-cs}	for $c \geq 0$
$f^{(n)}(t)$	$s^n F(s) - s^{n-1} f(0) - \cdots - f^{n-1}(0)$	
$(-t)^n f(t)$	$F^{(n)}(s)$	

Do not write your solutions here — this page will not be graded.