MATH 308. Differential Equations

Quiz 4 solutions

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Task 1. Find the first 4 coefficients a_0, a_1, a_2, a_3 of the power series solution $y(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots$ of the following differential equation:

$$y'' + y' + y = 0, \qquad y(0) = 1, y'(0) = 1.$$

We get
$y(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots$
$y'(x) = a_1 + 2a_2 x + 3a_3 x^2 + \dots$
$y''(x) = 2a_2 + 6a_3 x + \dots$
thus

$$y'' + y' + y = (a_0 + a_1 + 2a_2) + (a_1 + 2a_2 + 6a_3)x + \dots.$$

Equating coefficients to zero, we get

$$a_0 + a_1 + 2a_2 = 0, \quad a_1 + 2a_2 + 6a_3 = 0, \dots$$

Initial conditions imply $a_0 = y(0) = 1$ and $a_1 = y'(0) = 1$. Thus

$$a_2 = -\frac{a_0 + a_1}{2} = -1, \quad a_3 = -\frac{a_1 + 2a_2}{6} = \frac{1}{6}.$$

We conclude that $y(x) = 1 + x - x^2 + \frac{1}{6}x^3 + \dots$

One can use the sum notation to get the general form of the recurrence relation:

$$0 = y'' + y' + y = \sum_{n=2}^{\infty} a_n n(n-1)x^{n-2} + \sum_{n=1}^{\infty} a_n n x^{n-1} + \sum_{n=0}^{\infty} a_n x^n =$$
$$\sum_{n=0}^{\infty} a_{n+2}(n+2)(n+1)x^n + \sum_{n=0}^{\infty} a_{n+1}(n+1)x^n + \sum_{n=0}^{\infty} a_n x^n,$$

thus

$$a_{n+2}(n+2)(n+1) + a_{n+1}(n+1) + a_n = 0$$

or

$$a_{n+2} = -\frac{a_{n+1}(n+1) + a_n}{(n+1)(n+2)}$$

but this was not required in the problem.