MATH 308. Differential Equations

Lecture 13: Linear systems – eigenvalue method; complex eigenvalues.

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1. Discussing restrictions of the eigenvalue method: complex eigenvalues and repeated roots.

2. Linear algebra: complex eigenvalues and eigenvectors. Example: $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, finding eigenvalues and eigenvectors (answer: eigenvalues $\pm i$, eigenvectors $(1, \pm i)$).

3. Real-valued solutions of $x' = Ax$ with complex eigenvalues $\lambda_{1,2} = a \pm ib$ and eigenvectors $\xi_{1,2} = u \pm iv$: transforming $x(t) = c_1 e^{\lambda_1 t}\xi_1 + c_2 e^{\lambda_2 t}\xi_2$ into the real form

$$x(t) = d_1 e^{at}(u \cos bt - v \sin bt) + d_2 e^{at}(u \sin bt + v \cos bt).$$

Equivalently,

$$x(t) = d_1 \mathrm{Re}\,(e^{\lambda_1 t} v_1) + d_2 \mathrm{Im}\,(e^{\lambda_1 t} v_1).$$

Example: $x_1' = x_2, x_2' = -x_1$: we get $x(t) = d_1(\cos t \begin{pmatrix} 1 \\ 0 \end{pmatrix} - \sin t \begin{pmatrix} 0 \\ 1 \end{pmatrix}) + d_2(\sin t \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \cos t \begin{pmatrix} 0 \\ 1 \end{pmatrix})$, which is equivalent to what we obtained using eliminetion method, $x(t) = c_1 \begin{pmatrix} \cos t \\ -\sin t \end{pmatrix} + c_2 \begin{pmatrix} \sin t \\ \cos t \end{pmatrix}$.

4. Example: $A = \begin{pmatrix} 3 & -2 \\ 1 & 1 \end{pmatrix}$, solving, making conclusions about behavior.

   Answer: eigenvalues $2 \pm i$, eigenvectors $(2, 1-i)$ and $(2, 1+i)$, thus $x(t) = d_1 e^{2t}(\cos t \begin{pmatrix} 2 \\ 1 \end{pmatrix} - \sin t \begin{pmatrix} 0 \\ -1 \end{pmatrix}) + d_2 e^{2t}(\sin t \begin{pmatrix} 2 \\ 1 \end{pmatrix} + \cos t \begin{pmatrix} 0 \\ -1 \end{pmatrix})$.

   All solutions $x(t)$ tend to infinity and oscillate.

5. Complex eigenvalues for $n \times n$ matrices: $x(t) = c_1 e^{\lambda_1 t}\xi_1 + \dots c_n e^{\lambda_n t}\xi_n$; if some of the eigenvalues are pairs of complex conjugate, e.g. $\lambda_{1,2} = a \pm ib$, then we replace the first two summands with $d_1 \text{Re}\,(e^{\lambda_1 t}\xi_1) + d_2 \text{Im}\,(e^{\lambda_1 t}\xi_1)$ as before.

   Computer-assisted finding of eigenvalues and eigenvectors using sympy eigenvals() and eigenvects().

6. Types of phase portraits for $x' = Ax$ where A is 2×2:

   for real eigenvalues, $x(t) = c_1 e^{\lambda_1 t}\xi_1 + c_2 e^{\lambda_2}\xi_2$

   - $\lambda_1 < 0, \lambda_2 < 0$ — node sink, stable;
   - $\lambda_1 > 0, \lambda_2 > 0$ — node source, unstable;
   - $\lambda_1 < 0, \lambda_2 > 0$ — saddle, unstable;

   (plotting using Wolframalpha).

   In all cases, the picture contains phase curves that are rays along v_1, v_2, given by $x(t) = \pm e^{\lambda_1 t} v_1$ and $x(t) = \pm e^{\lambda_2} v_2$.

   for complex eigenvalues $\lambda_{1,2} = a \pm ib$:

   - $a = 0$ center,
   - $a < 0$ spiral sink,
   - $a > 0$ spiral source.

   (to be discussed next time).