MATH 308. Differential Equations

Lecture 18: Repeated eigenvalues for homogeneous systems $x' = Ax$.

Nataliya Goncharuk, natasha_goncharuk@tamu.edu

1. Different bases (coordinate systems) in linear algebra. To switch between bases, they use the transition matrix P: columns are components of vectors from the new basis f_1, f_2 (expressed in the old basis e_1, e_2). Then (old coordinates)= P(new coordinates). In the new basis, the matrix A turns into $\tilde{A} = P^{-1}AP$.

   Example: transition matrix from the standard basis $e_1 = (1,0), e_2 = (0,1)$ to a new basis $f_1 = (1,1), f_2 = (1,-1)$ is $P = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$. The vector with coordinates (x,y) turns into $P^{-1}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{x+y}{2} \\ \frac{x-y}{2} \end{pmatrix}$ in the new basis. The matrix $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ becomes diagonal, $\tilde{A} = P^{-1}AP = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, in the new basis.

   This makes sense, since $x \to Ax$ is the transformation $(x,y) \to (y,x)$, i.e. the reflection across the line $x = y$; in the new basis, it becomes the reflection across f_1 given by $(\tilde{x}, \tilde{y}) \mapsto (\tilde{x}, -\tilde{y})$.

2. Theorem from Linear Algebra: Let A be a matrix with repeated eigenvalues. In a certain basis (called Jordan basis), the matrix A becomes a Jordan form (block-diagonal with Jordan cells on the diagonal): $\tilde{A} = PAP^{-1}$ is a Jordan form.

Jordan cell has same value λ on the diagonal, and 1-s right above the diagonal. Here are 2x2 and 3x3 Jordan cells: $\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$, $\begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix}$

Sympy can compute both P and $\tilde{A}$ (denoted J in sympy), see the previous class.

3. Computing the matrix exponential for Jordan cells: if $\tilde{A} = \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix}$, then
$e^{\tilde{A}t} = e^{\lambda I} e^{\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} t}$ and we have $e^{\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} t} = I + \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} t + \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}^2 \frac{t^2}{2!} + \dots = I + \begin{pmatrix} 0 & t & 0 \\ 0 & 0 & t \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & t^2/2! \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & t & t^2/2! \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix}$
thus $e^{\tilde{A}t} = \begin{pmatrix} e^{\lambda t} & te^{\lambda t} & \frac{t^2}{2} e^{\lambda t} \\ 0 & e^{\lambda t} & te^{\lambda t} \\ 0 & 0 & e^{\lambda t} \end{pmatrix}$

   In the initial basis, $e^{At} = P^{-1} e^{\tilde{A}t} P$ where $\tilde{A}$ is formed by Jordan cells and P is the transition matrix to a Jordan basis.

4. **The general solution of $x' = Ax$ in the repeated eigenvalues case.**

   Columns of the matrix exponential are solutions, thus if $\xi_1, \xi_2, \dots, \xi_n$ are Jordan basis vectors that correspond to a $n \times n$ Jordan cell with λ on the diagonal, we get

$$x(t) = c_1 e^{\lambda t} \xi_1 + c_2 e^{\lambda t} (t\xi_1 + \xi_2) + c_3 e^{\lambda t} (\frac{t^2}{2!} \xi_1 + t\xi_2 + \xi_3) + \dots$$

   and we should add such summands for all existing Jordan cells.

5. Examples. $A = \begin{pmatrix} 3 & 1 \\ -1 & 1 \end{pmatrix}$: Jordan basis (using sympy) $\xi_1 = (1, -1)$ and $\xi_2 = (1, 0)$, thus

$$x(t) = c_1 e^{\lambda t} \xi_1 + c_2 e^{\lambda t}(t\xi_1 + \xi_2)$$

$A = \begin{pmatrix} 6 & 5 & -2 & -3 \\ -3 & -1 & 3 & 3 \\ 2 & 1 & -2 & -3 \\ -1 & 1 & 5 & 5 \end{pmatrix}$: Jordan basis (using sympy): $\xi_1 = (4, -3, 2, -1)$ and $\xi_2 = (1, 0, 0, 0)$ corresponding to one 2x2 cell $\begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$, and $v_1 = (5, -3, 1, 1)$ and $v_2 = (0, 1, 0, 0)$ corresponding to the other 2x2 cell $\begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$, thus

$$x(t) = c_1 e^{2t} \xi_1 + c_2 e^{2t}(t\xi_1 + \xi_2) + c_3 e^{2t} v_1 + c_4 e^{2t}(t v_1 + v_2)$$

6. Definition: ξ is called the generalized eigenvector of A with eigenvalue λ if $(A - \lambda I)^k \xi = 0$ for some k. We will see below that the Jordan basis is formed by eigenvectors and generalized eigenvectors.

   Finding a Jordan basis.

   Suppose that the Jordan form of the 2x2 matrix A is $\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$. Then the Jordan basis ξ_1, ξ_2 satisfies $(A - \lambda I)\xi_2 = \xi_1, (A - \lambda I)\xi_1 = 0$. We can use these identities to find ξ_1, ξ_2:

   - either we find ξ_1 (this is an eigenvector due to the second relation) and solve the linear system $(A - \lambda I)\xi_2 = \xi_1$;

   - or we choose any ξ_2 and find ξ_1 from $(A - \lambda I)\xi_2 = \xi_1$. Since $(A - \lambda I)^2 = 0$ (in the Jordan basis, thus in any basis as well), the identity $(A - \lambda I)\xi_1 = 0$ will hold automatically.

   This is the general way to find a Jordan basis for 2x2 matrices.

   Note that $(A - \lambda I)^2 \xi_2 = 0$, so ξ_2 is the generalized eigenvector.

Example: for $A = \begin{pmatrix} 3 & 1 \\ -1 & 1 \end{pmatrix}$, setting $\xi_2 = (1, 0)$, we get $\xi_1 = (A - 2I)\xi_2 = (1, -1)$. This coincides with the basis computed by sympy.

Or we can choose $\xi_2 = (1, 2)$, and get $\xi_1 = (A - 2I)\xi_2 = (3, -3)$.

Remark: ξ_1 is always the same modulo proportionality, and there is a lot of freedom when you choose ξ_2.

7. Another example: If the Jordan form of the 3x3 matrix A is $\begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix}$, then the Jordan basis ξ_1, ξ_2, ξ_3 satisfies $(A - \lambda I)\xi_3 = \xi_2, (A - \lambda I)\xi_2 = \xi_1, (A - \lambda I)\xi_1 = 0$, and we can use this to find ξ_k.

   Example: $A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 1 \\ 1 & -1 & 1 \end{pmatrix}$ with repeated eigenvalue 2. Choosing $\xi_3 = (1, 0, 0)$, we find $\xi_2 = (A - 2I)\xi_3 = (0, 0, 1)$ and $\xi_1 = (A - 2I)\xi_2 = (0, 1, -1)$. This is a Jordan basis.