MATH 308. Differential Equations

Lecture 19: Nonlinear autonomous systems. Linearization at equilibriums.

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1. The system of the form $x' = f(x, y), y' = g(x, y)$ is called autonomous system of ODEs: there is no dependence on t in the right-hand side. Reminder: vector field and phase portrait.

   Equilibrium (singular point, critical point, zero) of the autonomous system is a point (x_0, y_0) such that $f(x_0, y_0) = 0, g(x_0, y_0) = 0$. Then $x(t) = x_0, y(t) = y_0$ is a constant solution of the system.

2. Example: mathematical pendulum with friction, $x'' = -\sin x - 0.1x'$: turning this into a system of two first-order equations $x' = y, y' = -\sin x - 0.1y$, finding equilibriums (answer: $(x, y) = (0, 0)$ and $(\pi, 0)$, pendulum hanging down or pointing up.).

3. Linearization of a system near the lower equilibrium $x = 0, y = 0$: since $\sin x \approx x$ near zero, the system is close to a linear system $x' = y, y' = -x - 0.1y$. This is a spiral sink. The pendulum will oscillate near the lower equilibrium.

   Linearization near the upper equilibrium $x = \pi, y = 0$: since $\sin x \approx -(x - \pi)$ near $x = \pi$, the system is close to a linear system $(x - \pi)' = y, y' = (x - \pi) - 0.1y$ on two variables $\tilde{x} = x - \pi$ and y. This linear system is a saddle. The pendulum can not stay near this equilibrium for long since it is unstable.

Plotting the phase portraits of these linear systems and the phase portrait of the nonlinear system, discussing the motion of the pendulum.