# MATH 308. Differential Equations

## Lecture 24: Power series.

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1. $\mathcal{L}\{f * g\} = \mathcal{L}\{f\}\mathcal{L}\{g\}$. Proof:

$$\begin{aligned}\mathcal{L}\{f\} \cdot \mathcal{L}\{g\} &= \int_0^\infty e^{-sx} f(x)dx \cdot \int_0^\infty e^{-sy} g(y)dy \\ &= \int_0^\infty \int_0^\infty e^{-s(x+y)} f(x)g(y)dxdy \\ &= \int_{t=0}^\infty \int_{x=0}^{x=t} e^{-st} f(x)g(t-x)dxdt \\ &= \int_{t=0}^\infty e^{-st} \left( \int_{x=0}^{x=t} f(x)g(t-x)dx \right) dt = \mathcal{L}\{f * g\}.\end{aligned}$$

   Here we denote $t = x + y$ and replace the integration $\int_0^\infty \int_0^\infty \dots dxdy$ by the integration $\int_{t=0}^\infty \int_{x=0}^{x=t} \dots dxdt$.

2. Power series is the expression $\sum_{k=0}^{+\infty} a_k (x - x_0)^k$ where $a_k$ are numbers. The function $f(x) = \sum_{k=0}^{+\infty} a_k(x - x_0)^k$ is called the sum of power series.

   The series converges on some interval $|x - x_0| < R$ and diverges for $|x - x_0| > R$: $R$ is called the radius of convergence. $R$ could be 0 or $\infty$.

   Examples: polynomials are finite power series; geometric progression $1 + ax + a^2x^2 + \cdots = \frac{1}{1-ax}$ for $|ax| < 1$; series for the exponential

$1 + x + \frac{x^2}{2!} + \cdots = e^x$ for all $x$. Using power series to approximate $e^1 = e$: $1 + 1 + 1/2 + 1/6 + 1/24 \approx 2.71$

3. Taylor series: for many functions,

   $$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \frac{f'''(x_0)}{3!}(x - x_0)^3 + \ldots$$

   for $x \approx x_0$. This is the Taylor series for the function.

   The power series coincides with the Taylor series of its sum.

   Examples: Find Taylor series at $x_0 = 0$ for $e^x$. Answer: since all derivatives of $e^x$ at zero are 1, we get $e^x = 1 + x + \frac{x^2}{2!} + \ldots$.

   Find Taylor series at $x_0 = 0$ for $\sin x$. Answer: since derivatives of $\sin x$ at zero are $0, 1, 0, -1, 0, 1, -, -1, \ldots$, we get $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots$.

   In both cases, Taylor series converge for all $x$ to the function. This is not always the case: the Taylor series for the function $e^{-1/x^2}$ is zero.

4. Operations with power series: addition, multiplication, substitution, differentiation and integration. All operations work in the same way as for polynomials, but are only valid inside the interval of convergence $|x - x_0| < R$.

   Examples:

   Addition: $\frac{1}{1-x} + e^x = 1 + x + x^2 + \cdots + 1 + x + \frac{x^2}{2!} + \cdots = 2 + 2x + (1 + \frac{1}{2})x^2 + \ldots$

   Multiplication: $\frac{1}{(1-x)^2} = \frac{1}{1-x} \cdot \frac{1}{1-x} = (1 + x + x^2 + \ldots)(1 + x + x^2 + \ldots) = 1 + 2x + 3x^2 + 4x^3 + \ldots$;

   Substitution: substituting $2x^2$ instead of $x$ into the series for $\frac{1}{1-x}$, we get

   $\frac{1}{1-2x^2} = 1 + (2x^2) + (2x^2)^2 + (2x^2)^3 + \cdots = 1 + 2x^2 + 4x^4 + 8x^6 + \ldots$. The new series converges for $|2x^2| < 1$, i.e. $|x| < \frac{1}{\sqrt{2}}$ (radius of convergence is $R = \frac{1}{\sqrt{2}}$)

   Differentiation: $\frac{1}{(1-x)^2} = (\frac{1}{1-x})' = (1 + x + x^2 + \ldots)' = 1 + 2x + 3x^2 + 4x^3 + \ldots$, cf. above; this series converges for $|x| < 1$;

Integration: $-\ln(1-x) = \int \frac{1}{1-x}dx = \int(1+x+x^2+\dots)dx = C + x + \frac{x^2}{2} + \frac{x^3}{3} \dots$. This series converges for $|x| < 1$.

5. Solving ODEs with power series.

   Idea: we substitute $y(x) = \sum a_k x^k$ into the equation $y' = f(x,y)$, equate terms with same powers of $x$, and find $a_0, a_1, \dots$

   Note that $a_0 = y(0)$ and $a_1 = y'(0)$: this is the way to use initial conditions.

   Example: $y' = y$, $y(0) = 1$.

   Setting $y(x) = a_0 + a_1 x + a_2 x^2 + \dots$ and equating to $y'(x) = a_1 + 2a_2 x + 3a_3 x^2 + \dots$, we get $a_0 = a_1, a_1 = 2a_2, a_2 = 3a_3, \dots$.

   Since $y(0) = a_0 = 1$, we get $a_1 = 1, a_2 = \frac{1}{2}, a_3 = \frac{1}{2 \cdot 3}, a_4 = \frac{1}{2 \cdot 3 \cdot 4}$ etc., thus $y(x) = 1 + x + \frac{x^2}{2!} + \dots = e^x$.

   Example: $y'' = xy, y(0) = 1, y'(0) = 0$ (Airy's equation).

   - Substitute into the equation:

     $$2a_2 + 3 \cdot 2a_3 x + 4 \cdot 3a_4 x^2 + 5 \cdot 4a_5 x^3 + \dots = a_0 x + a_1 x^2 + a_2 x^3 + \dots$$

   - Equate coefficients:

     $$2a_2 = 0, \quad 2 \cdot 3a_3 = a_0, \quad 3 \cdot 4a_4 = a_1, \quad 4 \cdot 5a_5 = a_2 \dots$$

   - Use initial conditions: $a_0 = y(0) = 1$, $a_1 = y'(0) = 0$.
   - Find $a_2, a_3, \dots$:

     $$a_2 = 0, \quad a_3 = \frac{a_0}{2 \cdot 3} = \frac{1}{2 \cdot 3}, \quad a_4 = \frac{a_1}{3 \cdot 4} = 0,$$

     $$a_5 = \frac{a_2}{4 \cdot 5} = 0, a_6 = \frac{a_3}{5 \cdot 6} = \frac{1}{2 \cdot 3 \cdot 5 \cdot 6}, \dots$$

   - $\boxed{y(x) = 1 + \frac{1}{2 \cdot 3}x^3 + \frac{1}{2 \cdot 3 \cdot 5 \cdot 6}x^6 + \dots}$.

6. Radius of convergence.

   Ratio test: If $\lim_{n\to+\infty} |a_{n+1}/a_n| \to r$, then $R = 1/r$.

   Informal explanation: in this case, $a_n$ grows as the geometric progression $a_n \approx cr^n$, and the series $1 + rx + r^2x^2 + \cdots = \frac{1}{1-rx}$ converges for $|rx| < 1$ i.e. $|x| < R$.

   To use the ratio test for the Taylor series of $y(x)$ from the previous example, we first set $z = x^3$. The series becomes $1+\frac{1}{2\cdot3}z+\frac{1}{2\cdot3\cdot5\cdot6}z^2+\ldots$. Thus $a_n/a_{n-1} = \frac{1}{3n(3n-1)} \to 0$, and $R = \infty$: this power series always converges.