MATH 308. Differential Equations

Lecture 7: Higher-order equations

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Item 0: discussing task 2(b) of HW2 (the bifurcation in the family $y' = y^2 - 4y - c$, where equilibriums collide and disappear) and the bifurcation in the family $y' = ay - y^3$ where an equilibrium splits into three.

1. $$y^{(n)} = f(x, y, y', \dots, y^{(n-1)})$$

   is called the nth order differential equation. Second Newton's law is a second-order ODE (examples: ball attached to a spring: $mx'' = -kx$, adding liquid friction: $mx'' = -kx - cx'$, swings: $x'' = -\frac{g}{l}\sin x$ where x is the angle with the vertical line).

   Initial conditions for nth order differential equations have the form $y(x_0) = y_0, y'(x_0) = y_1, \dots, y^{(n-1)}(x_0) = y_{n-1}$. For mechanical problems, we get $x(t_0) = x_0, x'(t_0) = v_0$ — initial position and velocity.

2. **Existence and Uniqueness Theorem** for nth order equations.

   If the function $f(z_1, \dots z_n)$ is continuous and its derivatives $\frac{\partial f}{\partial z_2}, \frac{\partial f}{\partial z_3} \dots$ are continuous around the initial point $(x_0, y_0, y_1, \dots, y_{n-1})$, then the solution of the equation $y^{(n)} = f(x, y, \dots)$ with initial conditions $y(x_0) = y_0, y'(x_0) = y_1, \dots, y^{(n-1)}(x_0) = y_{n-1}$ exists, is defined on some interval $(x_0 - \delta, x_0 + \delta)$, and is unique.

   Example: $f(z_1, z_2, z_3) = z_2 + z_1 z_3$ is continuously differentiable everywhere, thus the theorem applies to $y'' = f(x, y, y') = y + xy'$.

3. Linear homogeneous n-th order equation:

$$y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_0(x)y = 0$$

(linear with respect to y and its derivatives).

Superposition principle: if two functions y_1, y_2 are solutions of the linear homogeneous equation, then $(c_1y_1 + c_2y_2)$ is also a solution, where c_1, c_2 are constants. Proof: substitute y_1, y_2 into the equation (we get zero since they are solutions), and add these expressions with coefficients c_1, c_2.

Example: $y'' = -y$. Solutions: $\sin x, \cos x$ (guessing), thus $y(x) = c_1 \cos x + c_2 \sin x$ is also a solution for any c_1, c_2; for any initial conditions $y(0) = a, y'(0) = b$, we can find appropriate c_1, c_2, namely $c_1 = a, c_2 = b$; thus we have found all solutions due to Existence and Uniqueness theorem.

Example: $y'' = y$. Solutions: e^x, e^{-x} (guessing), thus $y(x) = c_1e^x + c_2e^{-x}$ is also a solution for any c_1, c_2; again, we can satisfy any initial conditions (no proof today), thus we have found all solutions.

4. Linear homogeneous equations with constant coefficients:

$$y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_0y = 0.$$

A function $y = e^{\lambda x}$ is a solution if and only if λ is a root of the **characteristic equation** $\lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_0 = 0$. (Poll: which functions $y = e^{\lambda x}$ are solutions of the equation $y'' + 4y' + 3y = 0$? Answer: substituting, get $\lambda^2 e^{\lambda x} + 4\lambda e^{\lambda x} + 3e^{\lambda x} = 0$, thus $\lambda^2 + 4\lambda + 3 = 0$, i.e. $\lambda = -1, -3$).

Using Superposition principle to find all solutions: If the characteristic polynomial has n different roots $\lambda_1, \ldots \lambda_n$, then the general solution of the equation is

$$\boxed{y(x) = c_1e^{\lambda_1 x} + \ldots c_ne^{\lambda_n x}}$$

(Example: for $y'' + 4y' + 3y = 0$, the general solution is $y(x) = c_1e^{-x} + c_2e^{-3x}$).

Proofs are postponed: we need to prove that any initial condition can be satisfied.

5. **Complex roots of characteristic polynomials.** Define $e^{ib} = \cos b + i \sin b$ (complex exponential). Complex exponentials are differentiated normally, $(e^{\lambda x})' = \lambda e^{\lambda x}$, thus the previous formula works well.

   But we will simplify it to get rid or complex terms: if $\lambda = a \pm ib$ are complex roots of the characteristic polynomial, we have

   $$y(x) = c_1 e^{(a+ib)x} + c_2 e^{(a-ib)x} + \cdots = c_1 e^{ax}(\cos bx + i \sin bx) + c_2 e^{ax}(\cos bx - i \sin bx) + \cdots$$
   $$= d_1 e^{ax} \cos bx + d_2 e^{ax} \sin bx + \ldots$$

   where $d_1 = c_1 + c_2, d_2 = ic_1 - ic_2$. Since d_1, d_2 are arbitrary constants, we can just write these real terms $d_1 e^{ax} \cos bx + d_2 e^{ax} \sin bx + \ldots$ in the general formula instead of complex exponentials $c_1 e^{(a+ib)x} + c_2 e^{(a-ib)x}$.

   Example: $y'' = -y$, solutions are $y(x) = c_1 e^{ix} + c_2 e^{-ix} = d_1 \cos x + d_2 \sin x$.

6. Repeated roots of the characteristic equation also produce a problem; we postpone this to the next time.