

MATH 618. Complex Analysis

Homework 4

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Deadline: Monday Feb 19, 10 pm

In this home assignment, the lattice Λ is given by $\Lambda = \{n + m\omega\}_{n,m \in \mathbb{N}}$ for some fixed complex number $\omega \in \mathbb{H}$.

1. Define a Weierstrass elliptic function by

$$\wp(z) = \frac{1}{z^2} + \sum_{\lambda \in \Lambda \setminus \{0\}} \left(\frac{1}{(z - \lambda)^2} - \frac{1}{\lambda^2} \right)$$

Prove that this series converges uniformly on compact subsets of $\mathbb{C} \setminus \Lambda$.

2. Prove that \wp is even and doubly periodic: $\wp(z) = \wp(-z)$, $\wp(z + 1) = \wp(z)$, and $\wp(z + \omega) = \wp(z)$.
3. Prove that the Laurent series for \wp on a neighborhood of zero has the form

$$\wp(z) = \frac{1}{z^2} + 3G_4 z^2 + 5G_6 z^4 + 7G_8 z^6 + \dots$$

with

$$G_k = \sum_{\lambda \in \Lambda \setminus \{0\}} \frac{1}{\lambda^k}$$

4. Show that the doubly periodic function $(\wp')^2 - 4\wp^3 - g_2\wp$ with $g_2 = 60G_4$ has no poles. Derive that this function is constant:

$$(\wp')^2 = 4\wp^3 + g_2\wp + g_3.$$

for some constant g_3 (in fact, $g_3 = 140G_6$).

The last problem means that the map $z \mapsto (\wp(z), \wp'(z))$ parametrizes the torus-shaped two-dimensional surface $w^2 = 4z^3 + g_2z + g_3$ in the four-dimensional space (z, w) , $z, w \in \mathbb{C}$.

Practice problems: Task 1, 2 (all items) on p.206