

MATH 618. Complex Analysis

Homework 6

Nataliya Goncharuk, natasha_goncharuk@tamu.edu

Deadline: Wednesday Mar 20, 10 pm

1. (4 pt) Let the topological space $(\mathbb{R}, \mathcal{T})$ be defined as follows: open sets are \emptyset, \mathbb{R} , and any subset of \mathbb{R} with a finite complement. Answer the following questions:
 - a) Is this topological space Hausdorff?
 - b) Is it true that any converging sequence has a unique limit?
 - c) Is this space compact?
 - d) Show that if a function $f: (\mathbb{R}, \mathcal{T}) \rightarrow \mathbb{R}$ is continuous (in the topology in the image is the standard topology of \mathbb{R}), then it is a constant function.
2. (3 pt) Let $p(z, w)$ be a polynomial such that $\text{grad } p \neq 0$ whenever $p(z, w) = 0$. Using projections to z and w as local charts, show that $\{p(z, w) = 0\}$ has a structure of an analytic manifold.
Hint: use the (real) Implicit Function theorem.
3. (3 pt) Let $\omega \in \mathbb{H}$. Use the Argument principle to show that any meromorphic function with $f(z + 1) = f(z)$ and $f(z + \omega) = f(z)$ has equal amount of zeros and poles (counting with multiplicities) on a torus $\mathbb{C}/\mathbb{Z} + \omega\mathbb{Z}$.

Derive that the Weierstrass function $\wp: \mathbb{C}/\mathbb{Z} + \omega\mathbb{Z} \rightarrow \mathbb{C}$ takes every value exactly twice, except the four values ∞ , $\wp(0.5)$, $\wp(0.5\omega)$, and $\wp(0.5 + 0.5\omega)$.

Hint: to check that there are no other exceptional values, apply the Argument principle to \wp' to see that it only has three zeros.

4. (3 extra pts) (a) Show that $\phi(z) = e^{2\pi iz}$ is a biholomorphism between \mathbb{H}/\mathbb{Z} and $\mathbb{D} \setminus \{0\}$.

(b) For any analytic function $f: \mathbb{H} \rightarrow \mathbb{H}$ such that $f(z+1) = f(z) + 1$, show that the function $\phi f \phi^{-1}: \mathbb{D} \setminus \{0\} \rightarrow \mathbb{D} \setminus \{0\}$ has a removable singularity at zero, with nonzero derivative;

(c) Derive that in assumptions of (b), the limit

$$\lim_{y \rightarrow +\infty} f(x + iy) - (x + iy)$$

exists and does not depend on x .

Practice problems: p. 244 ex. 2, 4, 5.