

MATH 618. Complex Analysis

Homework 7

Nataliya Goncharuk, natasha_goncharuk@tamu.edu

Deadline: Wednesday Apr 3, 10 pm

1. Show that the function $e^z - z$ attains any value infinitely many times. What does this imply about the limit of the integral $\int_{|z|=R} \frac{e^z - 1}{e^z - z} dz$ as $R \rightarrow \infty$?
2. Find all $n \in \mathbb{N}$ such that for non-constant entire functions f, g we can have $f^n(z) + g^n(z) = 1$ for all $z \in \mathbb{C}$.
3. Let $G = \{0 < |z| < \varepsilon\}$ be a punctured disc, and suppose that an analytic function $f: G \rightarrow \mathbb{C}$ with an essential singularity at 0 has Laurent series $f(z) = \sum c_k z^k$ at zero.

Write out the Laurent series for the functions $f(\frac{z}{n})$. Explain why the sequence $f(\frac{z}{n})$ has no convergent subsequence in $H(G)$.

(This argument can replace the Maximum Modulus principle in the proof of the Great Picard Theorem.)

Hint: what happens to the Laurent series if functions converge uniformly on compact subsets of G to a finite analytic function?