

MATH 617. Complex Analysis

Exam 1

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Deadline: Monday Oct 2, 2023, 10 pm

RULES: Solve and submit any 5 out of the following 7 problems. Students can use their class notes, any textbooks on Complex Analysis, and other online resources as they work on the term test. Students may not collaborate and may not ask online any questions related to the test problems.

Task 1. (10 pt) Compute $\sum_{n=0}^{\infty} \frac{\cos n}{2^n}$.

Task 2. (10 pt) The function $f: M \rightarrow \mathbb{C}$ is called locally Lipschitz on a set M if for any point $z \in \overline{M}$, there exists a disc $B_\varepsilon(z)$, $\varepsilon > 0$, such that f is Lipschitz on $B_\varepsilon(z) \cap M$.

Prove that if f is locally Lipschitz on a compact set M , then f is Lipschitz on M .

Task 3. (10 pt) Using the definition of continuity, prove that $g(z) = \frac{1}{z}$ is continuous at 0 as a mapping of $\overline{\mathbb{C}}$ (we assume $g(0) = \infty$).

Task 4. (10 pt) Suppose that a function f is \mathbb{C} -differentiable on an open connected set U . Show that if f only takes real values, then f is constant.

Task 5. (10 pt) Find all $z \in \mathbb{C}$ such that e^z is real and $\log z$ is purely imaginary.

Task 6 (5+5 pts). A sequence $\{a_n\}$ satisfies the recurrent relation

$$a_n = \frac{a_{n-3}}{n(n-1)};$$

also, $a_0 = 1, a_1 = 0, a_2 = 0$.

(a) Show that the series converges for all z .

Hint: write $f(z) = g(z^3)$ for a certain power series g and use the ratio test for g .

(b) Show that the sum of the series $f(z) = \sum_{n=0}^{\infty} a_n z^n$ satisfies the differential equation $f'' = zf$ (the Airy equation).

Task 7 (8+2 pts). (a) Find a conformal bijection f that takes a set

$$\Omega = \{z \mid |z| < 1 \text{ and } |z - 0.5| > 0.5\}$$

to the unit disc $\{|z| < 1\}$.

(b) Show that your map f does not extend continuously to 1.