MATH 618. Complex Analysis

Final Exam

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Deadline: Tuesday May 7, 10 pm

Each item of each problem is 4 pts. Four lowest scores are dropped.

1. (a) Prove that there exists a sequence of polynomials p_n that converges uniformly to $\sin z$ on $|z| \le 1$ and to $\cos z$ on $|z-3| \le 1$.

   (b) Show that this sequence is not uniformly bounded on $|z-1| \le 2$.

2. Let γ_1, γ_2 be circles that start at 0.5 and go around 0 and 1 in a counterclockwise direction. Consider the multivalued function $\sqrt{\log z}$. For the branch of this function that has a real value at 0.5, find its analytic continuations along the loops $\gamma_1\gamma_2$ and $\gamma_2\gamma_1$.

3. (a) Estimate from above the number of zeros of the function $e^z - z$ in the disc $|z| < r$.

   (b) Use the proof of the Hadamard's factorization theorem to find $\sum a_n^{-2}$ where a_n are all zeros of this function[1]

4. Let $a_k \to \infty$, $c_k \in \mathbb{C}$. Show that there exists an analytic function on $\mathbb{C} \setminus \{a_1, a_2, \dots\}$ with essential singularities at a_k that has residues c_k at a_k.

   Hint: mimick the proof of the Mittag-Leffler's theorem.

[1] This is a_n and not $|a_n|$.

5. Analytic functions f_n converge uniformly on compact subsets of $0 < |z| < 1$ to a non-constant function f. Which of the following is true? Prove or provide a counterexample.

   (a) If all f_n have essential singularities at zero, then f has an essential singularity at zero.

   (b) If all f_n have removable singularities at zero, then f has a removable singularity at zero.

   (c) If each $\log f_n$ has a well-defined analytic branch in $0 < |z| < 1$, then the same holds for $\log f$.

6. Analytic functions $f_n \colon \mathbb{D} \to \mathbb{C}$ form a normal family in $O(\mathbb{D})$ and satisfy $f_n(0.5) = 1$.

   (a) For any $r < 1$, show that the number of zeros of f_n in $|z| < r$ is bounded.

   (b) Show that the sequence $f_n(z)/f_n(0)$ can be non-normal in $C(\mathbb{D}, \overline{\mathbb{C}})$ (even if $f_n(0) \neq 0$).

   (c) If f_n are nowhere zero, show that the sequence $f_n(z)/f_n(0)$ is normal in $O(\mathbb{D})$.

7. (a) Show that we can always find a holomorphic function f in the unit disc such that $\operatorname{Re} f$ continuously extends to the boundary of the unit disc and equals a given continuous function on this boundary.

   (b) Can we always solve the same problem as in (a) for the annulus?

   (c) Find an analytic function in the unit disc such that for $\phi \in (-\pi, \pi)$, $\phi \neq 0$, we have $\lim_{r \to 1} \operatorname{Re} f(re^{i\phi}) = sign(\phi)$.

8. (a) Show that a mapping f that takes a strip $-a < \operatorname{Im} z < a$ into itself (not necessarily one-to-one) cannot take 0 to 0 and $b_1 \in \mathbb{R}$ to $b_2 \in \mathbb{R}$, $|b_2| > |b_1|$.

   (b) Show that a mapping f that takes an annulus $1 < |z| < 2$ into itself (not necessarily one-to-one) cannot take $\sqrt{2}$ to $\sqrt{2}$ and $\sqrt{2}i$ to $-\sqrt{2}$.

Hint: look at $g^{-1}fg$ where g is a covering map that takes a strip into this annulus (make sure to prove that this composition does not branch).