MATH 618. Complex Analysis

Midterm test

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Deadline: Wednesday Feb 28, 10 pm

All problems are out of 10 pts, one lowest score is dropped from the computation.

Here and below, $\mathbb{D}$ is the open unit disc: $\mathbb{D} = \{|z| < 1\}$.

1. Write out a formula for an entire function with a zero of order n at each point n for $n \in \mathbb{N}$.

2. Write out a formula for a meromorphic function that has a pole at each point n for $n \in \mathbb{N}$, with a singular part equal to $\frac{n}{z-n}$.

3. Prove that if $f_n \to f$ in $M(G)$ and f has a pole at z_0 of order k, then for any small ε, for sufficiently large n, the function f_n has k poles in $B_\varepsilon(z_0)$ counting with multiplicities.

4. An analytic map $f\colon \mathbb{D} \to \mathbb{C}$ does not assume real positive values and takes 0 to -1. Find the maximal possible value of $|f'(0)|$.

5. Let f be analytic in the annulus $3 < |z-2| < 4$ and have no analytic extension to any larger connected open set. Let K_n be a sequence of compacts $K_n = \{3 + 1/n < |z-2| < 4 - 1/n\}$.

   Show that if Runge theorem is used to construct a sequence of rational functions R_n with poles at 0 and ∞ such that $|f - R_n| < 1/n$ on K_n, then the coefficientls of their (finite) Laurent series expansions

   $$R_n(z) = \cdots + a_{n,-1}z^{-1} + a_{n,0} + a_{n,1}z + \ldots$$

converge, $a_{n,k} \to c_k$ for any k. However, show that the sum of the "limit" Laurent series $\sum_{k=-\infty}^{\infty} c_k z^k$ is not equal to f on any open set.

6. Use the uniqueness part of the Riemann Mapping Theorem to show that if a simply connected domain G is symmetric with respect to zero, then any biholomorphic mapping f that takes G to $\mathbb{D}$ and 0 to 0 is odd, $f(-z) = -f(z)$.

7. Construct an entire function with prescribed values at points $a_n \to \infty$. Namely, for any sequence of complex numbers $a_n \to \infty$ and any complex numbers b_n, find an entire function with $f(a_n) = b_n$.

   Hint: let g be an entire function with simple zeros at a_n. Show that

$$f(z) = \sum_{n=1}^{\infty} g(z) \frac{e^{\gamma_n (z-a_n)}}{z-a_n} \cdot \frac{b_n}{g'(a_n)}$$

   converges for some choice of $\gamma_n \in \mathbb{C}$ to an analytic function and satisfies $f(a_n) = b_n$.