Homework 1 Solutions

MATH 663: Complex Dynamics and the Mandelbrot Set

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1. For the map $z \to 1/z^2$ on $\overline{\mathbb{C}}$, find all attracting periodic orbits, their multipliers, and attracting basins. Find the Julia set.

   Solution: The second iterate of the map is $f^2(z) = 1/(1/z^2)^2 = z^4$. The map f^2 has superattracting fixed points at $0, \infty$, with attracting basins $|z| < 1$ and $|z| > 1$ respectively. Periodic orbits of f^2 on the unit circle are all repelling.

   We conclude that f has only one attracting periodic orbit, namely $0 \to \infty \to 0$. Its multiplier equals $(f^2)'(0) = (z^4)'_0 = 0$. Its attracting basin is $\overline{\mathbb{C}} \setminus \{0\}$. The Julia set is the boundary of the attracting basin, namely $\{|z| = 1\}$.

2. For the map $z \to z^2$, show that the point $z \neq 0, \infty$ is eventually periodic if and inly if $z = e^{2\pi ip/q}$. Show that the orbit of $z \neq 0, \infty$ is periodic if and only if $z = e^{2\pi ip/q}$, q is odd.

   Solution:

   If the orbit is eventially periodic, then $z^{2^n} = z^{2^m}$ and thus z is a root of unity of order $2^n - 2^m$, $z = e^{2\pi ip/q}$ for $q = 2^n - 2^m$. If $z = e^{2\pi ip/q}$, then z has finitely many different images (all of them of the form $z = e^{2\pi ik/q}$) and thus some of them coincide, $f^n(z) = f^m(z)$.

   The orbit is periodic if and only if $z^{2^n} = z$ for some n, i.e. z is a root of unity of order $2^n - 1$. Thus q divides $2^n - 1$, which is only possible if q

is odd. On the other hand, any odd number q is a divisor of $2^n - 1$ for some n. Indeed, the sequence of remainders of 2^n, $n = 0, 1, \ldots$ with respect to q must repeat itself at some point; it cannot have a pre-period since the map $n \to 2n$ is one-to-one on the set of remainders. Thus this sequence must be periodic, and the remainder $2^0 = 1$ must appear again, which means that $2^n - 1$ for some n will be divisible by q.

3. Chebyshev polynomials T_n, $\deg T_n = n$, are given by $T_n(\cos\phi) = \cos(n\phi)$.

   (a) Prove that the map $h(z) = 0.5(z + \frac{1}{z})$ is a semi-conjugacy between the Chebyshev polynomial and the map $z \to z^n$: we have $T_n(h(z)) = h(z^n)$. Use this to prove that the Julia set for T_n, $n \geq 2$, is $[-1, 1]$.

   (b) Will the fixed points of T_n coincide with the images of the fixed points of z^n under h? Will critical points of T_n be images of critical points of z^n under h?

   (c) Using linear conjugacy with T_2, find the Julia set of $z^2 - 2$, which corresponds to the leftmost point $c = -2$ of the Mandelbrot set.

   Solution. (a) Let $z = e^{iw}$. Then $h(z) = 0.5(z + 1/z) = 0.5(e^{iw} + e^{-iw}) = \cos w$. Thus $T_n(h(z)) = \cos nw$ and $h(z^n) = 0.5(e^{inw} + e^{-inw}) = \cos nw$. Since any nonzero complex number has the form e^{iw}, we have $P_n(h(z)) = h(z^n)$ i.e. h semi-conjugates T_n to z^n.

   Note that the map $0.5(z + 1/z)$ takes the unit circle to $[-1, 1]$, and it takes the set $|z| > 1$ to $\mathbb{C} \setminus [-1, 1]$. For any point z in $\{|z| > 1\}$, we have $\lim z^n = \infty$. Thus $(T_n)^k(h(z)) = h(z^{n^k})$ tends to infinity, i.e. for any $z \in \mathbb{C} \setminus [-1, 1]$, its orbit under T_n tends to infinity. We conclude that $\mathbb{C} \setminus [-1, 1]$ is a part of the attracting basin of ∞, and does not belong to the Julia set.

   On the other hand, for any point z in $\{|z| = 1\}$, the orbit under $z \to z^n$ stays bounded. Thus $(T_n)^k(h(z))$ is bounded, i.e. for any $z_0 \in [-1, 1]$, its orbit under T_n is bounded. Since there are points in any neighborhood of z_0 with unbounded orbits, the family $\{(T_n)^k\}$ is not normal and z belongs to the Julia set. Alternatively, we can say that $[-1,1]$ is on the boudnary of the attracting basin and thus belongs to the Julia set.

Hence the Julia set of T_n is $[-1,1]$.

(b) Note that images of fixed points of z^n under h are fixed points of T_n, since $z^n = z$ implies $T_n(h(z)) = h(z^n) = h(z)$. However, not all fixed points of T_n are obtained this way. Since h is surjective, for any fixed point w_0 of T_n, we can write $w_0 = h(z_0)$; we have $T_n(h(z_0)) = h(z_0) = h(z_0^n)$, but this does not imply $z^n = z$ since h is not one-to-one.

This leads to a counterexample. Take $|w_0| = 1$ and $w_0^n = \bar{w}_0 = 1/w_0$ (say, $w_0 = e^{2\pi i/(n+1)}$). Then w_0 is not a fixed point of $z \to z^n$, but $h(w_0) = \cos 2\pi/(n+1)$ is a fixed point of T_n: $T_n(\cos 2\pi/(n+1)) = \cos 2\pi n/(n+1) = \cos(-2\pi/(n+1)) = \cos 2\pi/(n+1)$.

Critical points of T_n are not images of critical points of z^n, since z^n has only two critical points $0, \infty$, and the polynomial T_n of degree n has $n-1$ finite critical points. This happens because if we differentiate $T_n(h(z)) = h(z^n)$, we get $T_n'|_{h(z)} \cdot h'(z) = h'|_{z^n}(z^n)'$, and the left-hand is zero if $(z^n)'$ is zero OR if h' is zero. Since h has two critical points $-1, 1$, the polynomial T_n will have critical points whenever $h'(z^n) = 0$ but $h'(z) \neq 0$, i.e. at the points of the form $h(e^{2\pi ik/n})$ and $h(e^{2\pi ik/n + \pi i/n})$, $k = 1, \ldots, n-1$ (these are the $n-1$ real critical points of T_n).

(c) We have $T_2(z) = 2z^2 - 1$, thus $2 \cdot P(0.5z) = z^2 - 2$. So the map $z \to 2z$ conjugates P_2 to $z^2 - 2$, and the Julia set for $z^2 - 2$ is the interval $[-2, 2]$.