Homework 2

MATH 663: Complex Dynamics and the Mandelbrot Set

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Deadline: Wednesday Feb 5, 10 pm

The homework is out of 10 pts, points above 10 are extra credit.

1. (3 pt) Find Julia sets for all Moebius maps $f(z) = \frac{az+b}{cz+d}$.

   Hint: prove that a Moebius map has either 1 or 2 fixed points. Using conjugacy with a Moebius map, place these points at ∞ (resp. 0 and ∞).

   Solution. Assume that the Moebius map is not identical (otherwise its Julia set is empty).

   If ∞ is a fixed point of f, then $f(z) = az + b$.

   If $a \neq 1$, conjugacy with $h(z) = z - b/a$ will turn it into $f(z) = az$. If $|a| > 1$, zero is a repellor and ∞ is an attractor. Since $|z| \neq 0$ is a basin of infinity, the Julia set coincides with the repellor. If $|a| < 1$, zero is an attractor and ∞ is a repellor, and similarly, the Julia set coincides with the repellor.

   If $|a| = 1$, the Julia set is empty since the sequence of functions $a^n z$ is normal.

   If $a = 1$, $b \neq 0$, the map f has the form $f(z) = z + 1$, with the parabolic fixed point at infinity. Iterates of this map form a normal family (that tends uniformly to infinity) on any finite open set. On a neighborhood of infinity, the sequence of iterates is not normal. We proved this in class when we proved that the parabolic fixed point belongs to the Julia set. Thus $J(f)$ coincides with the parabolic fixed point.

If $f(z) = \frac{az+b}{cz+d}$ with $c \neq 0$, then ∞ is not a fixed point; in this case, the equation $f(z) = z$ is quadratic, hence it has 2 different roots or 1 repeated root. In any case, sending one of the roots to infinity reduces the problem to one of the cases considered above.

2. (2+1+2 pt) (a) Show that for any polynomial P of degree at least 2 with no repeated roots, the Newton's map $f(z) = z - \frac{P(z)}{P'(z)}$ has the following fixed points: attracting supercritical fixed points at all roots of P, and a repellor at infinity. What happens if P has repeated roots?

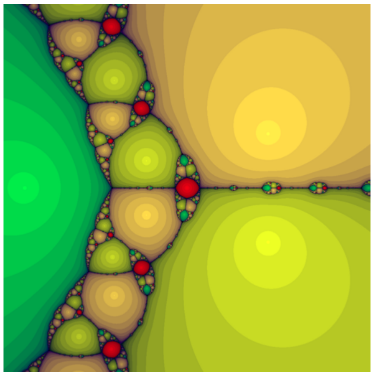
   Solution. Let $\deg P = n$. Since $f(z) = z - \frac{a_0 z^n + \dots}{n a_0 z^{n-1} + \dots}$, the leading term is $z - z/n = z(1 - 1/n)$, thus $f(\infty) = \infty$ and ∞ is repelling.

   The finite fixed points are given by $P(z)/P'(z) = 0$ i.e. coincide with roots of P. If roots are not repeated, we get $f'(z) = 1 - \frac{P'^2 - PP''}{P'^2}|_{z=z_0} = \frac{PP''}{P'^2} = 0$, so these fixed points are superattracting.

   If a root a is repeated, the same computation is valid since f (and thus f') has a removable singularity at a with $f(a) = a$. Let $P(z) = (z - a)^k h(z)$. We get $f'(z) = -\frac{P(z)P''(z)}{P'^2(z)}|_{z=a} = \frac{h \cdot (k(k-1)h + 2(z-a)kh' + (z-a)^2 h'')}{(kh + (z-a)h')^2}|_{z=a} = \frac{k-1}{k}$ which is nonzero. So the point is still attracting but not superattracting (which affects the rate of convergence of the method).

   (b) For the polynomial $P(z) = z^3 - 2z + 2$, observe that the map f has a critical point whose orbit is attracted to the period-2 attracting periodic cycle.

   We have $f(z) = z - \frac{z^3 - 2z + 2}{3z^2 - 2}$ and $f'(z) = -\frac{PP''}{(P')^2} = \frac{(z^3 - 2z + 2)6z}{(3z^2 - 2)^2}$. The map f has critical points at the roots of P that are superattracting, and it also has a critical point 0. The orbit of zero is $0 \to 1 \to 0 \to \dots$, which is a periodic orbit of period 2. This orbit is superattracting since $(f^2)'(0) = f'(0) \cdot f'(1) = 0 \cdot f'(1) = 0$.



Taken from: `https://commons.wikimedia.org/wiki/File:Newton_z3-2z%2B2.png` Author: Henning Makholm [Public domain]

On the picture above, attracting basin of this period-2 orbit is shown in red. Starting in this open domain, the Newton's method does not converge to a root of f.

(c) Let P have no repeated roots. Prove that for any r, there exists $z, \infty > |z| > r$, and n, such that $f^n(z) = \infty$.

Since ∞ is repelling, it belongs to the Julia set. Thus any set $|z| > 2r$ intersects all basins of attraction of the roots of $P(z)$. Since there is more than one basin, any set $|z| > 2r$ must contain points on the boundaries of these basins. Since boundaries of basins belong to the Julia set, we must have a point of a Julia set $z_0 \neq \infty$ with $\infty > |z_0| > r$.

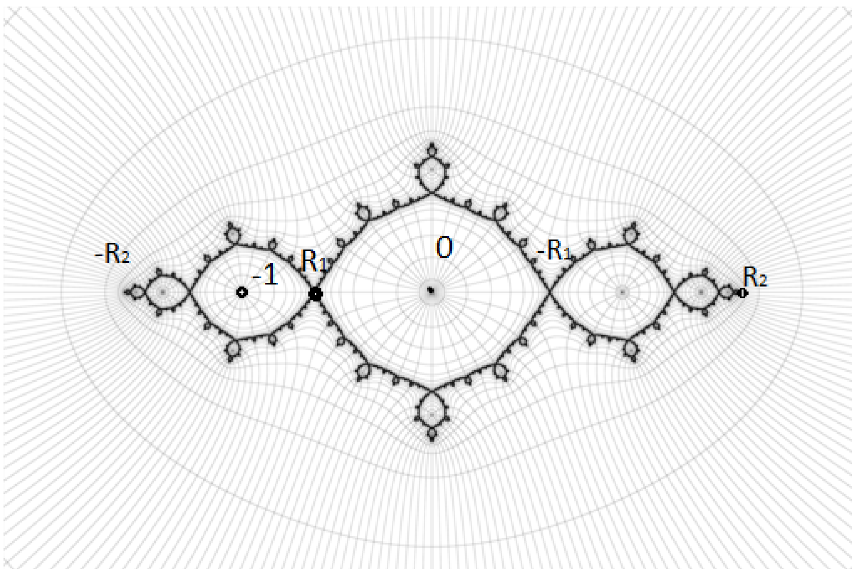
We conclude that images of a neighborhood $B_\varepsilon(z_0)$ of z_0 cover $\overline{\mathbb{C}}$ (except possibly one or two exceptional points in the Fatou set). In particular, we must have a point $z_1 \in B_\varepsilon(z_0)$ with $f^n(z_1) = \infty$. This completes the proof.

Alternatively, one can prove and use a stronger statement: a Julia set cannot have isolated points (See Milnor's book for the proof); since infinity is not isolated, there is a point z_0 with $|z_0| > r$ that belongs to the Julia set, and the statement follows since $f^{-n}(\infty)$ acucumulates to this point.

3. (2+2 pt) The picture below shows *basilica*, the Julia set of $f(z) = z^2 - 1$, corresponding to $c = -1$ in the Mandelbrot set.

(a) Show that $f(z)$ has two real repelling fixed points R_1, R_2, $|R_1| < |R_2|$, and a real attracting periodic orbit of period 2. Show that the real interval $(R_1, -R_1)$ belongs to the attracting basin of this periodic orbit, and the set $(-\infty, -R_2) \cup (R_2, \infty)$ belongs to the basin of ∞.

(b) Show that the intersection of the attracting basin with the real axis is the sequence of adjacent real intervals, with endpoints in $f^{-n}(R_1) \subset J(f)$. Mark R_1, R_2, and their first preimages on the picture.



Solution. (a) The attracting periodic orbit is $0 \to -1 \to 0$; it is superattracting since $f'(0) = 0$. Fixed points, given by $z^2 - 1 = z$, are $R_{1,2} = \frac{\sqrt{5}\pm 1}{2}$. Their multipliers are $|\sqrt{5} \pm 1| > 1$, thus points are repelling.

Note that on the right from R_2, we have $x^2 - 1 > x$. Thus $x < f(x) < f(f(x)) < \dots$; orbits of all points tend to infinity. To the left from $-R_2$, the points are mapped to $(R_2, \infty]$ and then tend to infinity.

The map x^2-1 takes $(R_1, 0)$ to $[-1, R_1)$ and $[-1, R_1)$ to $(R_1, -R_1)$ monotonically, since its derivative does not change sing on these intervals. Thus $f(f(x))$ is monotonic increasing on $(-R_1, 0]$, with the only fixed point at

$f(f(0)) = 0$. Thus the sequence $x < f^2(x) < f^4(x) < \dots$ for $x \in (R_1, 0]$ tends to 0, and $(R_1, 0]$ belongs to the basin of the attracting orbit.

Since f is even, $[0, R_1)$ belongs to this basin as well.

(b) Let U_0 be the connected component of the attracting basin that contains 0, let $U_{-2}, U_{-1}, U_0, U_1, U_2, \dots$ be all connected components that intersect $\mathbb{R}$, numbered left to right. Let x_n be the sequence of real positive preimages of R_1: $x_0 = -R_1 > 0$, and $\sqrt{x_n + 1} = x_{n+1} > x_n$. The first preimage is $x_1 \approx 1.27$. We claim that $U_n \cap \mathbb{R} = (\pm x_n, \pm x_{n+1})$ and $f((x_n, x_{n+1})) = (x_{n-1}, x_n)$.

Indeed, since f is monotonic on $\mathbb{R}^+$, the interval $(-R_1, x_1)$ is mapped to $(R_1, -R_1)$. Positive preimages of x_1 are points $x_2 < x_3 < x_4 \dots < R_2$, and intervals $(x_1, x_2), (x_2, x_3), \dots$ in between them that are mapped to each other right to left. Each interval is eventually mapped to $(R_1, -R_1)$, thus belongs to the attracting basin. Endpoints are eventually mapped to R_1. Thus these intervals are indeed intersections of U_n, $n > 0$ with $\mathbb{R}^+$. Since the picture is symmetric, intersections of U_{-n} with $\mathbb{R}$ are intervals $(-x_n, -x_{n+1})$, i.e. the other preimages of $(R_1, -R_1)$ under f.