Quasiconformal maps.

[L. Ahfors, "Lectures on quasiconformal mappings".

Dillatation. For $f \in C^1$, if $df|_z$ takes a circle into ellipse with (long axis) : (short axis) $= K(z)$, then $K(z)$ is a dilatation of f at z.

The map f is K-quasiconformal if $K \geq \sup K(z)$.

(In particular, $K(z)$ is finite $\Rightarrow$ df is non-degenerate).

Complex dilatation $\mu(z) := \dfrac{f_{\bar z}}{f_z}$, where $f_{\bar z} = \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right) \quad \left(= \frac{\partial f}{\partial \bar z}\right)$

$f_z = \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right) \quad \left(= \frac{\partial f}{\partial z}\right)$

For $f(z) = z$, we have $f_{\bar z} = 0$, $f_z = 1$.

For $f(z) = \bar z$, we have $f_{\bar z} = 1$, $f_z = 0$.

In general, at 0, if $f(z) = c + az + b\bar z +$ (higher-order terms), then $a = \frac{\partial f}{\partial z}$, $b = \frac{\partial f}{\partial \bar z}$.

Ex. $f(x+iy) = Kx + iy \rightarrow \mu(z) = \dfrac{K-1}{K+1}$

Relation of μ to K : we have $|\mu(z)| = \dfrac{K-1}{K+1}$, $K(z) = \dfrac{1+|\mu|}{1-|\mu|}$

Proof: this holds for $f(x+iy) = Kx + iy +$ (higher order terms) as above.

Any f can take this form if we replace $f \to \alpha f(\beta z)$

(pre- and post-compose with rotations and expansions).

$f \to \alpha f$ does not change μ, $\dfrac{f_{\bar z}}{f_z} \mapsto \dfrac{\alpha f_{\bar z}}{\alpha f_z} = \dfrac{f_{\bar z}}{f_z}$.

$f \to f(\beta z)$ changes μ in the following way:

$f(z) = c + az + b\bar z + \dots$ $\quad a = f_z$, $b = f_{\bar z}$

$f(\beta z) = c + a\beta z + b\overline{\beta z} + \dots$ $\quad a\beta = f_z$, $b\bar\beta = f_{\bar z}$

so $\dfrac{f_{\bar z}}{f_z} \rightsquigarrow \dfrac{f_{\bar z}\cdot\bar\beta}{f_z\cdot\beta} = \mu(z)\cdot\dfrac{\bar\beta}{\beta}$

But $|\mu|$ does not change since $\left|\frac{\bar\beta}{\beta}\right| = 1$.

Meaning of μ : it tells you which ellipses are mapped to circles.

$K(z) = \dfrac{1+|\mu|}{1-|\mu|}$ tells you shape of the ellipse; $\arg\mu$ tells you the direction of the longer axis.

Proof: for $Kx+iy$, the statement holds true and axes are vertical / horizontal.

for $f \to \alpha f$, μ does not change and ellipse that is mapped to a circle does not change.

for $f(z) \rightsquigarrow f(\beta z)$, μ is multiplied by $\frac{\bar\beta}{\beta}$ which changes $\arg \mu$, and this ellipse is rotated.

Ahlfors-Bers theorem ~~[illegible]~~ (Measurable Riemann mapping thm)

- For a measurable μ, $\|\mu\|_\infty < k < 1$, on $\mathbb{C}$, there exists a solution of the Beltrami equation:

$$\boxed{\frac{f_{\bar z}}{f_z} = \mu(z)}$$

[Meaning: f takes a given field of ellipses (correspond. to μ) into circles.

- unique if normalized by $f(0)=0$, $f(1)=1$, $f(\infty)=\infty$,
- quasiconformal homeomorphism $f: \mathbb{C} \to \mathbb{C}$.

However, f might be a weak solution: $f_z, f_{\bar z} \in L^2$ exist in the distributions sense and $f_{\bar z} = f_z \mu(z)$ as distributions.

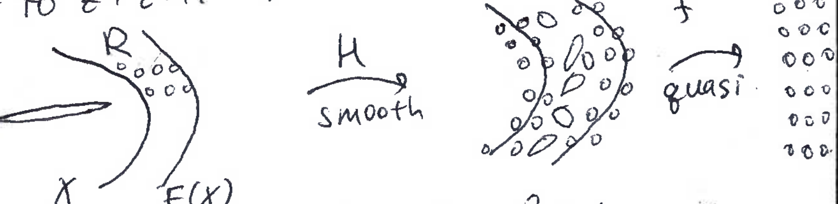
Also, if $\frac{f_{\bar z}}{f_z} = 0$ (and f is a weak solution), then f is a conformal map.

Fatou coordinate for $F(z) = z+1+O(1/z)$. We search for Ψ_{attr} that conjugates F to $z \to z+1$.

$\to$ Use smooth map H to conjugate F to $z \to z+1$: $H: R \to \mathbb{C}$;

$H(\text{left border}) = id$

$H(\text{near right border}) = F^{-1}(z)+1$



Then $H(F(z)) = H(z)+1$.

Define $\mu = \frac{(H^{-1})_{\bar z}}{(H^{-1})_z}$: H takes circles to ellipses, given by μ.

Extend μ periodically, $\mu(z+1) = \mu(z)$ ~~[illegible]~~

Solve ~~[illegible]~~ $\frac{f_{\bar z}}{f_z} = \mu$, $f(0)=0$, $f(1)=1$, $f(\infty)=\infty$ using A.-B. theorem.

- The map f is a weak solution.

However, ~~...~~ $f \circ H$ satisfies $\dfrac{(f\circ H)_{\bar z}}{(f \circ H)_z} = 0$ in weak sense and thus $f \circ H$ is conformal, ~~so~~ since H is smooth in R, f is smooth in $H(R)$, and $\dfrac{f_{\bar z}}{f_z} = \mu$ holds everywhere.

- Now, $f \circ H$ is analytic. H conjugates F to $z \to z+1$.

Prove that f conjugates $z \to z+1$ to $z \to z+1$:
$$f(z)+1 = f(z+1).$$
This follows from 1-periodicity of μ.

Pf. Define $\varphi(z) = f(z+1)-1$.

Then ~~$\varphi_{\bar z}$~~ $\dfrac{\varphi_{\bar z}}{\varphi_z} = \mu(z+1) = \mu(z)$;

$\varphi(0) = f(1) - 1 = 0$

$\varphi(\infty) = f(\infty) - 1 = \infty$

Due to uniqueness in A.-B. theorem, $\varphi(z) = a \cdot f(z)$ where $a = \varphi(1)$.

So $\underline{f(z+1) - 1 = a f(z)}$.

But this is impossible for 1-to-1 maps. if $a \neq 1$? Indeed, select z: $f(z) = \frac{-1}{a-1}$, then $a f(z) + 1 = f(z)$, and $f(z+1) = f(z)$, which is impossible for 1-to-1 maps.

We get $a = 1$, i.e. $f(z+1) - 1 = f(z)$.

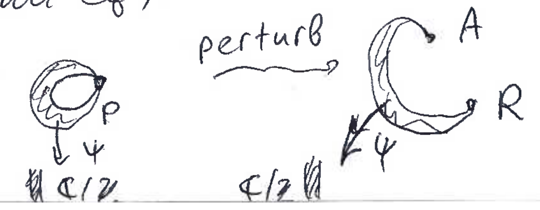
We conclude that $f \circ H$ is a Fatou coordinate: it is analytic on a nbhd of R and conjugates F to $z \to z+1$.

This method of obtaining Fatou coordinates:

(1) implies that Fatou coordinate depends analytically on additional parameters, since this is true for solutions of Beltrami eq.

(2) provides sharp estimates if we know how close F is to id. (via estimates on solutions of Beltrami eq)

(3) works for perturbed Fatou coordinates when parabolic pt gets destroyed.



Another application of Ahlfors-Bers thm: classification of complex manifolds.

Any complex manifold that is a sphere topologically (homeomorphic to a sphere) is a sphere $\overline{\mathbb{C}}$.

Proof: find a smooth mapping h to $\overline{\mathbb{C}}$; use $\frac{(h^{-1})_{\bar z}}{(h^{-1})_z} = \frac{f_{\bar z}}{f_z}$ to find f so that $f \circ h$ is analytic.

Another application of quasiconformal mappings: Teichmuller space.

~~Di~~ Different complex tori: $\mathbb{C}/\mathbb{Z}+\tau\mathbb{Z}$ for different τ.

Teichmuller Distance between T_1 and T_2: smallest dilatation K of the quasiconformal mapping between T_1 and T_2 → $\underline{\mathrm{dist}(T_1, T_2) = \frac{1}{2}\log K}$.

For tori, it coincides with distance $\frac{1}{2}\mathrm{dist}_{\Pi}(\tau_1, \tau_2)$ in $\mathbb{H}$. [explain]

For other complex manifolds, we get different Teichmuller spaces.

Comment. Tori $\mathbb{C}/\mathbb{Z}+\tau\mathbb{Z}$ and $\mathbb{C}/\mathbb{Z}+(\tau+1)\mathbb{Z}$ are the same torus, but with different generators.

Moduli space: $\tau \in \mathbb{H}$ up to changes $\tau \to \tau+1$, $\tau \to -1/\tau$ 

Teichmuller space: fix two generators of the torus. ~~Only~~ Consider them marked. Now you get $\mathbb{H} \ni \tau$.