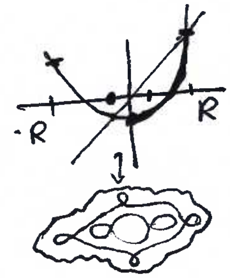
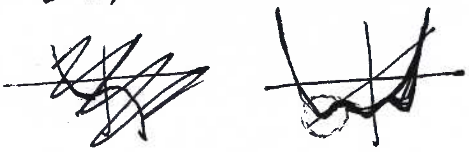
$f_a(0)<0$ since $\frac{\partial f}{\partial a}>0$, $a<0$; → at least 2 f.p.

Since $f_a''(x)>0$ (continuity), f is convex and there are 2 pts, attractor and repellor (A, R).

This proof works for smooth real functions.

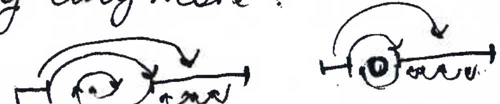
Now, let c decrease further  while the leftmost pt A is attracting, all orbits ~~accumu~~ in [-R, R] accumulate to it.



Graph of $f(f(x))$:  attr basin

At some point, the leftmost pt will not be attracting any more: its multiplier becomes -1.

This is a period-doubling bifurcation.



before after

Next time

$$f_a(x) = \mu_a + \lambda_a x + \nu_a x^2 + \dots, \quad \frac{\partial^2 f}{\partial a \partial x} \neq 0$$

$$f_0(x) = -x + x^2 + q x^3 + \dots$$

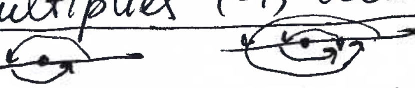
$$f_0(f_0(x)) = -(-x+x^2+qx^3) + (-x+x^2+qx^3)^2 + q(-x+x^2+qx^3)^3 + \dots$$

; assume $q \neq -1$ (non-degeneracy)

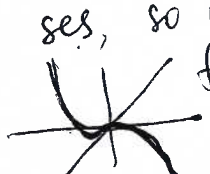
$$= x - 2x^3 - 2qx^3 + \dots$$

with 2 attr. and 2 repel. petals

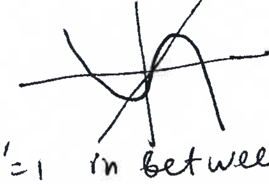
[- more degenerate parabolic pt (n=2)

Then an attracting pt with multiplier (-1) becomes repelling and generates a period-2 cycle. 

→ since $f_0(f_0(x))$ changes sign, $f_a(f_a(x))$ has at least 1 fixed pt. Since f'' increases from positive to negative, f' increases and then decreases, so we can only have 2 pictures:



$f(f(x))$ has 1 fixed pt, attracting



1 ~~attr~~ repellor and 2 attractors around.

Since there is a pt with $f'=1$ in between any 2 fixed pts, and we can only have 2 such pts due to the fact that f' increases and then decreases. We get a basilica 

Keep decreasing c. A new period-2 cycle will become repelling at some point through a period-doubling bifurcation. Looking at a graph of f^4, we can see a new period-4 attracting pt.

→ Cascade of period-doubling bifurcations.

Bifurcation theory: classification of bifurcations with normal forms (see Kuznetsov's textbook on Applied bifurcation theory).

Sample results:

→ Parabolic (saddlenode) bifurcation:

a family $f_\mu(x)$ undergoes a parabolic bifurcation with $p=0$ - if $\frac{\partial f}{\partial \mu}(0) \neq 0$ and $f_0(x) = x \pm a x^2 + \dots$;

a smooth conjugacy turns f_μ into $g_\nu = \underbrace{\nu + x \pm x^2 + O(x^3)}_{\text{normal form}}$.

→ Fold bifurcation: when attractor becomes superattracting [will not discuss in detail]

→ Period-doubling bifurcation:

a family $f_\mu(x)$ undergoes a period-doubling (flip) bifurcation when $\frac{\partial^2 f}{\partial \mu \partial x} \neq 0$ and $f_0(x) = -x + a x^2 + b x^3 + \dots$

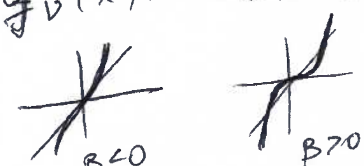
$\uparrow$ derivative changes through 0 $\qquad$ $\uparrow$ mult. -1

with nondegen. condition $\frac{(f_0'')^2}{2} + \frac{f_0'''}{3} \neq 0$ at 0.

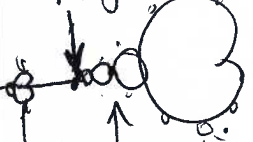
[this condition corresponds to $q \neq -1$ in out previous computation and is needed so that $f_0(f_0(x)) = x + \underset{\neq 0}{A} x^3 + \dots$]

Normal form: $g_\nu(x) = -(1+\beta)x \pm x^3 + O(x^4)$

Graph: $g(g(x))$



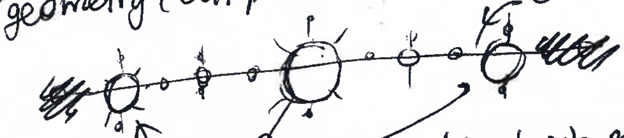
Feigenbaum pt - emergence of chaos in period-doubling cascade



period doubles here

period-3 lives here [show graphs]

Period-3 attr. basin has a different geometry (airplane):

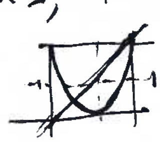


Cantor-like complement, in J.

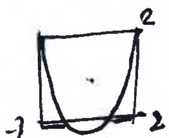
disconnected attracting basin of a period-3 orbit

$M \cap \mathbb{R}$ contains more intersections with hyperbolic components (when c is attracted to a periodic orbit) as well as places where $\{f^n(c)\}$ is aperiodic and accumulates on a Cantor-like set.

While $f([-R,R]) \subset [-R,R]$, the orbit of c is bounded, thus $c \in M$.

leftmost point of M:  $c = -2$.

As soon as $c < -2$, the orbit of 0 leaves to ∞ and J becomes Cantor-like



f_c

middle third removed

f_c^2 . . .

more middle thirds removed . .

since once an orbit leaves $[-R,R]$, it never comes back.

$c = -2$: $z^2 - 2$ has $[-2,2]$ as a Julia set (HW1)
and is semiconjugate to z^2 on $\mathbb{C} \setminus [-2,2]$ via $h(z) = z + 1/z$ which is thus Böttcher coordinate

$c = -2$ is a Misiurewicz point : critical pt is mapped to a repellor.
[Julia sets are dendrite-like in this case]

~~Similar~~

In which order will periodic orbits appear when we move in M towards $c = -2$? [There is an analogue for other Misiurewicz pts].

Sharkovskii's order: $p \succ q$ if existence of period-p orbit for $f: \mathbb{R} \to \mathbb{R}$ continuous implies existence of period-q orbit

$3 \succ 5 \succ 7 \succ \dots \quad 3 \cdot 2 \succ 5 \cdot 2 \succ 7 \cdot 2 \succ \dots \quad 3 \cdot 2^2 \succ 5 \cdot 2^2 \succ 7 \cdot 2^2 \succ \dots \quad \dots \succ \underbrace{2^n \succ 2^{n-1} \succ \dots \succ 1}_{\text{period-doubling}}$

the components will appear in this order till we reach period-3 when all periods are present on $\mathbb{R}$.

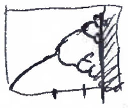
Feigenbaum universality for period-doubling cascades

[similar effects appear for the onset of chaos in many cases, e.g. turbulence models; showing Lorenz's attractor that also appears in period-doubling bifurcations for limit cycles ]

On $x = a$, plot attractors for $ax(1-x)$
↑ sets where all the points go.

$ax(1-x)$ is conjugate to x^2+c for some $c \leftrightarrow a$. ($\lim a_n = a_*$ - chaos)

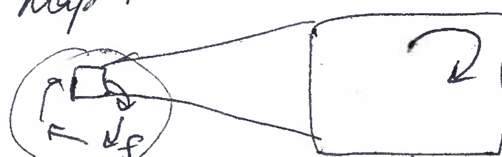
$a_1, a_2, a_3 \dots$ - moments of bifurcations (period-doubling)

We get Bif diagram: 
$a_1\ a_2$

$$\boxed{\lim \frac{a_n - a_{n-1}}{a_{n-1} - a_{n-2}} = \varphi}$$ - Feigenbaum constant - universal, does not depend on a family, same in any family with 1 crit. pt. (non-degenerate) $\cap$.

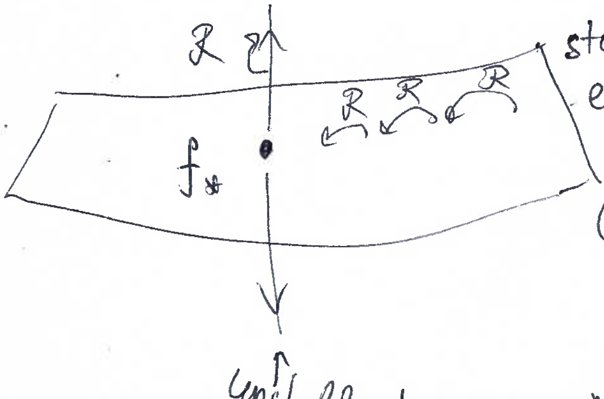
Reason for Feigenbaum universality:

renormalization.

f  consider the 1st-return map to a subset R.
zoom this piece R  $\mathcal{R}f$

New map (1st-return map on a rescaled subset) is called renormalization of f, denoted $\mathcal{R}f$.

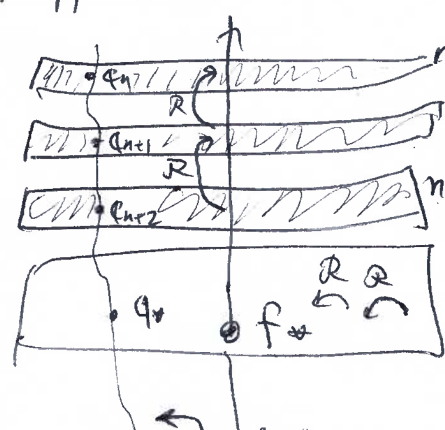
$\mathcal{R}$ is a map in the functional space. With certain choise of R, it has a fixed pt f_* that is hyperbolic with one unstable direction. (f_* is a specific function - Feigenbaum fixed pt - power series are computed)



stable codim-1 manifold: everything converges to f_* here ($\mathcal{R}^n f \to f_*$).

unstable direction: $\mathcal{R}^n f$ goes away from f_*

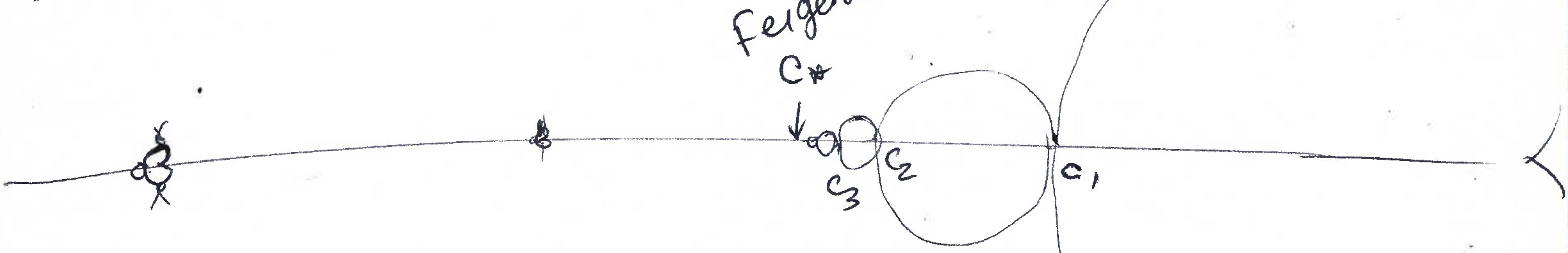
Surfaces that corr. to period-doubling bifurcations are mapped to each other under $\mathcal{R}$:



any family and its a_n

So we get $\lim \frac{a_n - a_{n-1}}{a_{n-1} - a_{n-2}} =$ unstable eigenvalue of $\mathcal{R} = \varphi$

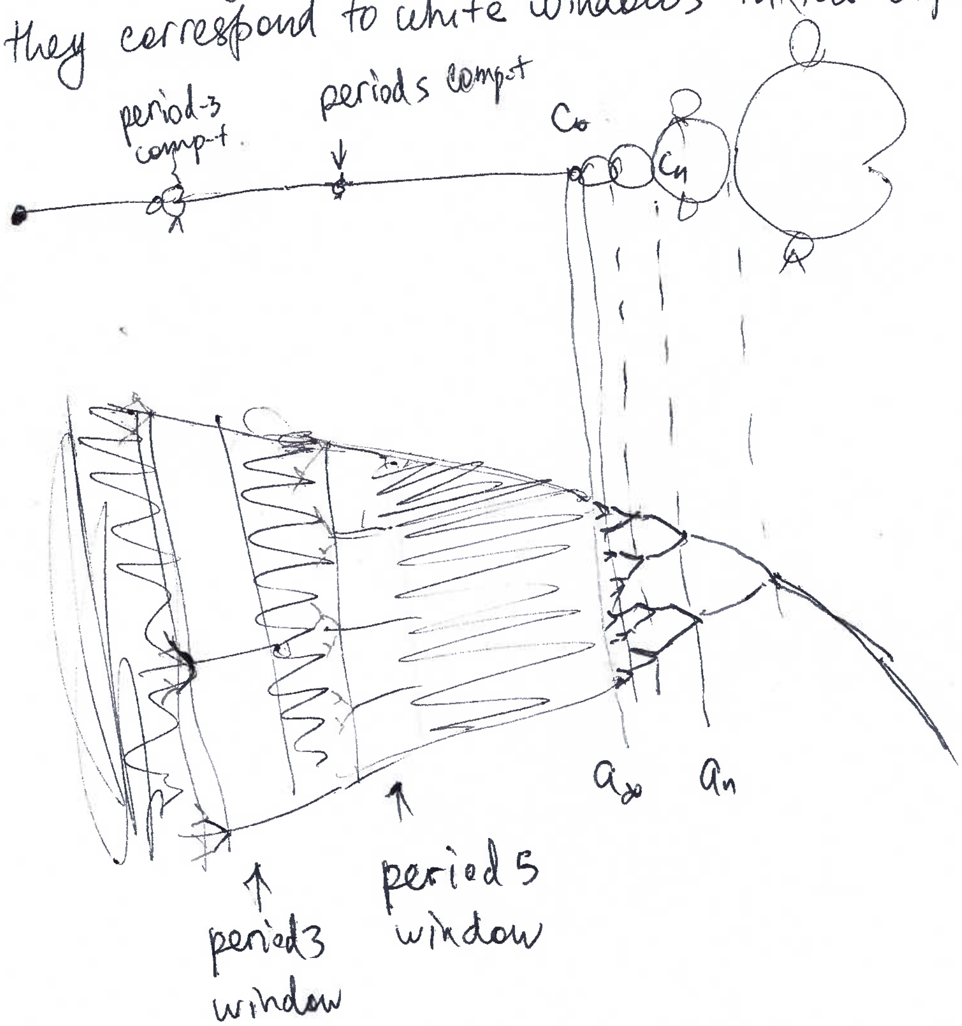
Relation to the Mandelbrot set:



c_n – moments of period-doubling; $\lim \frac{c_{n+1}-c_n}{c_n-c_{n-1}} = \varphi$.

Near c_*, $\mathcal{M}$ is <u>self-similar</u>; $\mathcal{R}$ is responsible for self-similarity. [$\mathcal{R}$ is the main reason for self-sim of the Mandelbrot set in all known cases].

There are many other hyperbolic components of $\mathcal{M}$; they correspond to white windows in the bif. diagram of $ax(1-x)$.



[showing pictures from wikipedia]