Counting periodic orbits.

Reminder: since every attr. basin contains a critical point, a rational map has finitely many attr. periodic orbits, hence infinitely many repelling or neutral points. We will prove a stronger statement: there are at most finitely many attracting + neutral points. Also, we always have a repellor or a parabolic pt.

· $f(z) = \frac{P(z)}{Q(z)}$ of degree d has $d+1$ fixed pts - solutions of $P(z) = zQ(z)$ (degree $d+1$). If $\deg Q < d$, some of them are at ∞.

Residue fixed pt index of f at a fixed pt: on a local coord. near $\hat{z}$,

$$i(f,\hat{z}) = \frac{1}{2\pi i} \oint_{\text{around } \hat{z}} \frac{dz}{z - f(z)}.$$

· Does not depend on a path.
· Integral stays the same when we perturb f.

[Similar but different notion for real vector fields: # turns of a vector when we go around a zero of a vector field].

For $f'(\hat{z}) = \lambda \neq 1$: $i(f,\hat{z}) = \frac{1}{1-\lambda}$

Q: what if $f(z) = z + z^2$?
- index 0.
Not true for all parab pts.

Th $\sum_{\text{all fixed pts}} i(f, \hat{z}_k) = 1.$

Proof. Assume there is no fixed pt at ∞. Then $\deg Q = d$, $f(\infty)$ has a finite value, and $\frac{1}{2\pi i}\oint_{\text{large disc}} \frac{dz}{z - f(z)} = \frac{1}{2\pi i}\oint_{\text{large disc}} \frac{dz}{z - f(\infty)} = 1$

This is equal to $\sum_{\text{all f.p.}} i(f, \hat{z}_k)$.

What if there is a fixed pt at ∞? Send it to a finite pt using Moeb. m

~~The $i(f,\infty)$ is defined~~

→ $i(f,\hat{z}_k)$ does not change when we perform an analytic change of coordinate. For $f'(\hat{z}) \neq 1$, this is clear from $i(f,\hat{z}) = \frac{1}{1-\lambda}$. For $f'(\hat{z}) = 1$, $i(f,\hat{z})$ equals

$$\frac{1}{2\pi i}\oint_{|z|=c} \frac{dz}{z - (f+\varepsilon)(z)}$$

for small ε, and using ε such that fixed pts of $f+\varepsilon$ are non-repeated, we get a proof

$= 1 - i(f,\infty)$. Compute $i(f,\infty)$:

Ex: for z^2+c, we have $\frac{1}{1-\lambda} + \frac{1}{1-\mu}$ $= 1 - i(f,\infty)$. In the local chart $w = \frac{1}{z}$ at ∞,

$f(z) = z^2 + c$

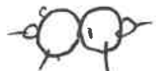
$F(w) = \frac{1}{f(1/w)} = \frac{1}{\frac{1}{w^2} + c} = \frac{w^2}{1 + cw^2}$

has multiplier 0 at 0 $\Rightarrow$ $i(f, \infty) = 1$.

So $\frac{1}{1-\lambda} + \frac{1}{1-\mu} = 0 \rightarrow 1-\lambda+1-\mu = 0 \Rightarrow \lambda+\mu = 2$

z^2+c :  $|\lambda|<1$

Mandelbrot set

$z^2+\lambda z$:  $|\lambda|<1$, $|2-\lambda|<1$

Double Mandelbrot set

alternative param. planes

aleph0.clarku.edu/~djoyce/julia

Differ by conjugacy that takes 0-cr. pt to $(-\lambda/2)$ and fixed pts to fixed pts.

Comment: $|\lambda| \le 1$ $(\lambda \ne 1)$ $\Leftrightarrow$ $\mathrm{Re}\left(\frac{1}{1-\lambda}\right) \ge \frac{1}{2}$ | Moeb map $z \to \frac{1}{1-z}$: 

Corollary: any rational map of degree ≥ 2 has a repelling fixed pt or a parabolic fixed pt with $\lambda = 1$.

Proof: If there is no parab. fixed pt, then we have $d+1$ pts, $\sum_{k=1}^{d+1} \frac{1}{1-\lambda_k} = 1$. Thus $\mathrm{Re}\,\frac{1}{1-\lambda_k} < \frac{1}{d+1} \le \frac{1}{3}$ for at least one of them ☒

Corollary: parabolic pt can split into 2 attractors if $\mathrm{Re}\, i(f, z_0) \ge 1$ only.

Corollary: Julia set of a rational function of deg. ≥ 2 is non-empty. (it contains this pt).

Remark: if a repelling fixed pt has non-real multiplier, Julia set cannot be a smooth curve

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Most periodic orbits repel.

The number of attr + parabolic cycles $\le$

→ The number of critical pts $\le 2d-2$.

$\le$ the number of critical pts

[$f = \frac{P}{Q} \rightarrow f' = \frac{P'Q-PQ'}{Q^2}$ $f'=0 \rightarrow P'Q-PQ' = 0$ - degree $2d-2$ since highest term cancels out.]

The number of neutral periodic cycles $\overset{\lambda \ne 1}{\le}$ $4d-4$

→ The number of neutral cycles become attracting or repelling:

Perturb f so that neutral cycles become attracting or repelling:

$f = \frac{P}{Q} \longrightarrow f_t = \frac{P(z) - tz^d}{Q(z) - t}$

$t = 0$: $f_0 = f$

$t = \infty$: $f_\infty = z^d$

$t \in \mathbb{C} \setminus$ exceptional values where degree drops.

Multipliers of cycles of f_t are not identity.

→ we can extend a solution z_t of $f^q(z_t) = z_t$ analytically in t until multiplier becomes 1, but we cannot approach $t = \infty$ since near z^d, all periodic pts of period q have multipliers ≥ 1 or $\equiv 0$. So we must find a problem, on each ray that avoids exceptional values of f.