External rays - properties. f - polynomial of degree d.

→ $\psi: \mathbb{C} \to \mathbb{C}$; $\psi(B(\infty)) = \{|z|>1\}$ - Bottcher coordinate, 1-to-1
We assume that no critical pts escape, and thus $J(f)$ is connected.
→ $\psi^{-1}\{\{\arg z = 2\pi\varphi\}\} = R_\varphi$ - external ray.
$\{\arg z = 2\pi\varphi\} =: \tilde{R}_\varphi$ - straight ray.

We have: $f(R_\varphi) = R_{d\cdot\varphi}$, since $(z^d)(\tilde{R}_\varphi) = \tilde{R}_{d\cdot\varphi}$
Ray lands if there exists a landing pt $p = \lim_{r\to 1} \psi^{-1}(re^{2\pi i\varphi})$,
$p \subset \partial B(\infty) = J(f)$.
If $J(f)$ is loc. connected, all rays land (Caratheodori).

→ If R_φ lands at p then $R_{d\cdot\varphi}$ also lands at $f(p)$: since f is analytic at p and takes curves that have limits to curves that have limits.

→ If R_φ lands at p then $R_{\varphi/d + k/d}$ lands at a pt of $f^{-1}(p)$ since:

If p is not crit: d branches of f^{-1} at p are conformal maps that take R_φ to $R_{\varphi/d + k/d}$ and all of them hence have limits. Limits belong to $f^{-1}(p)$ due to the above.

- If p is crit: some branches of f^{-1} act as $\sqrt{\ }$:

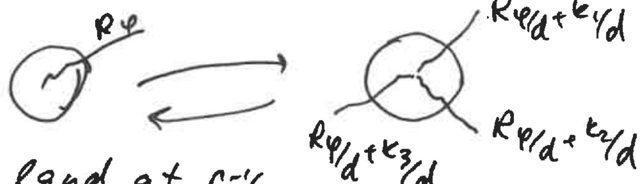


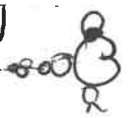
image of R_φ is several curves that land at $f^{-1}(p)$.

→ If R_φ lands at p, then all $f^{-1}(p)$ are landing pts of some rays.
(same argument as above).

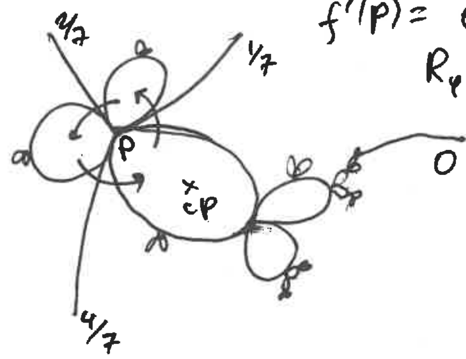
Periodic rays: $R_{d^n\varphi} = R_\varphi$. Preperiodic rays: $R_{d^n\varphi} = R_{d^m\varphi}$.
All rational rays are periodic or preperiodic.

Th Periodic ray lands. [Snail lemma]
Landing point is repelling or parabolic, [fixed ray lands at a fixed pt, inverse is incorrect]
periodic point.
Preperiodic ray lands at a preperiodic pt.
Every repelling or periodic pt is a landing pt of ≥ 1 ray.
For parabolic pts, rays land in each repelling petal.

Ex: Douady rabbit, parabolic case (fat rabbit)



$f'(p) = e^{+2\pi i\, 1/3}$ – 3 repelling petals



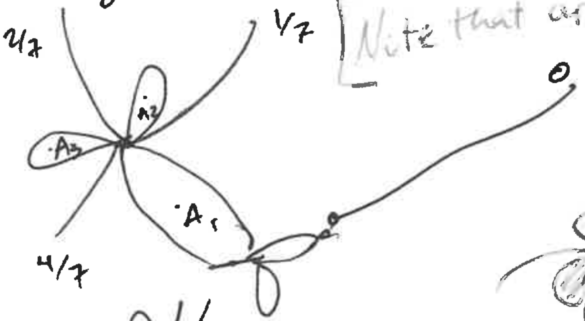
$R_\varphi \to R_{2\cdot\varphi} \to R_{4\varphi} \to R_{8\varphi} = R_\varphi \Rightarrow \varphi = 1/7,\ 2/7,\ 3/7,\ 4/7,\ 5/7,\ 6/7$

different order of $\varphi, 2\varphi, 4\varphi$ on the circle.

We need $1/7 < 2/7 < 4/7$ clockwise.

Now we can compute the whole lamination (images & preim of these rays).

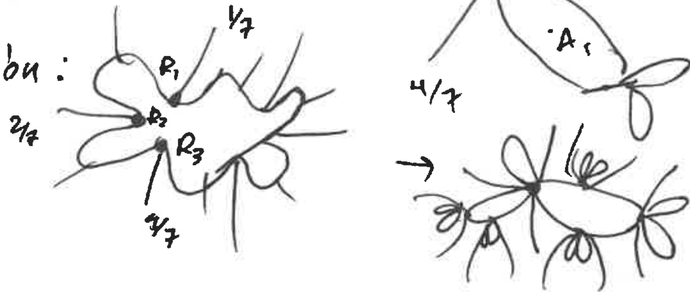
f has 2 fixed points. One is P. finding another one. Discussing images of ears.



[Note that arg $f'(p) \neq e^{2\pi i \ldots}$ results in spiraling ears]

Skinny rabbit: same lamination.

Parabolic Bifurcation:



Green's function of M

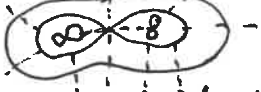
→ Bottcher coordinate extends to $B(\infty)$ univalently only if cr. pts do not escap

via $\varphi(z) = \sqrt[d]{\varphi(f(z))}$.

But Green's function always extends to $B(\infty)$ [harmonic function].

$G(z) = \log|\varphi(z)|$ ignores arg $\varphi(z)$, so we can extend via $G(z) := \frac{1}{d} G(f(z))$.

[We can still define equipotentials as $\{G(z) = \text{const}\}$ and external rays as perpendicular to equipotentials, but rays will not be rays if critical pts escape].



We always define $G(z) = 0$ outside $B(\infty)$. Near $\partial B(\infty)$, we have $G \to 0$.

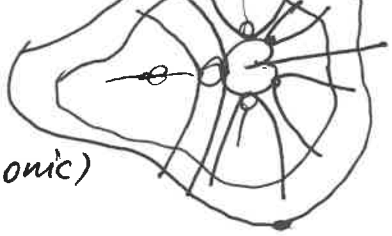
For $z^2 + c$, let G_c be the Green's function. ~~Then~~ Let φ_c be Bottcher coordinate, defined up to the critical pt c

Let $\Phi(c) = \varphi_c(c)$.
(like Bottcher's for M)

→ analytic on $\mathbb{C}$
→ defined outside M
→ $\varphi_c(c) \in \{|z| > 1\}$
→ tends to 1 on ∂M

$\boxed{\text{thus assumes every value once}}$

Th. $\Phi(c) = \varphi_c(c)$ is analytic and takes $\mathbb{C}\setminus\overline{M}$ biholomorphically to $\mathbb{C}\setminus\overline{\mathbb{D}}$.



Equipotentials for M: $\Phi^{-1}\{|z|=r\}$

Green function for M: $\log|\Phi|$ (harmonic)

External rays for M: $\Phi^{-1}\{\arg z = 2\pi\varphi\}$.

Open question: is M locally connected?

If yes, all rays land. We have a full description of rays that land together → this would be a complete top. model for M. (← lamination).

Proof of the thm (sketch).

→ $G_c(z)$ is continuous on c, z.

(nontrivial since $J(f)$ does not move continuously)

But partial limits of $G_{c_n}(z)$, $c_n \to c$, are harmonic with zeros on $J(f)$

(see details in [CG].

$\Phi(c) \in \{|z|>1\}$ by definition.

~~Thus $\Phi(c)$ is continuous near~~

$\Phi(c)$ is analytic in $\mathbb{C}\setminus M$: φ_c is analytic wrt c (by construc

- well-defined when c escapes since $c \in B(\infty)$
- as $c \to M$, $G_c(z)$ is continuous and $G_c(c) = \log|\Phi(c)|$ tends to 0, thus $|\Phi(c)| \to 1$.
- has simple pole at ∞, $\varphi_c(c) \sim c$ since $\varphi_c \sim \mathrm{id}$ near ∞.

Next t

~~Apply argume~~

This already implies that $\mathbb{C}\setminus M$ is connected: cannot have bded components due to max principle.

Also, applying argument principle to $1/\Phi(c)$, we get that $\Phi(c)$ is injective and onto:

~~→ 1-to-1 near ∞ since it has a simple pole at ∞.~~

~~→ For $\Phi(c_1) = \Phi(c_2)$, take a curve outside M that has ∞, c_1, c_2 outsid~~

~~Then $\#\{\Phi(z)=k\} \geq 2$, thus $\Phi^{-1}(\gamma)$ makes 2 turns aro~~

~~Taking γ near M, we guarantee that $\Phi^{-1}(\gamma)$ is close to $|z|=1$.~~

~~But then it makes ≥ 2 turns around 0 as well and $\#\{\Phi^{-1}(z)=0\}$~~

~~≥ 2 preimages.~~