Properties of Julia sets.

Reminder: $z \in \bar{\mathbb{C}}$ belongs to Fatou set of f if $\{f^n\}$ is normal on some nbhd $\mathcal{U} \ni z$.

Otherwise z belongs to Julia set.

Repelling and parabolic periodic orbits belong to Julia set. [last time].

Attracting basins are open and belong to Fatou set.

Ex: if f is a polynomial, of degree ≥ 2, ∞ is superattracting. Basin of ∞ belongs to Fatou set. Ex: $z \to z^2$.

Proof. Consider a neighborhood V of $\underline{a}$ where $|f(z)-a| < \lambda |z-a|$, $\lambda < 1$.

If $f^n(z) \to a$, then for some n, for some $\mathcal{U} \ni z$, $f^n(\mathcal{U}) \subset V$.

So $|f^{n+m}(z) - a| < \lambda^m |f^n(z) - a| < \lambda^m \cdot \operatorname{diam}(V)$

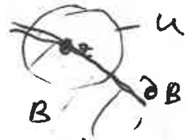
uniform estimate on a distance between f^{n+m} and constant, so $\mathcal{U}$ is in the Fatou set

also proves that $\mathcal{U}$ belongs to the attracting basin so the basin is open. Similar proof works for periodic attr. orbits.

Boundary of the attracting basin belongs to Julia set

[we will later see that the boundary of any attr. basin coincides with the Julia set]

Proof If $z \in \partial B(a)$,



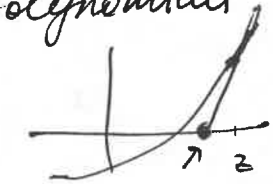
then any its neighborhood contains $z_0 \in B(a)$, $z_0 \subset W \subset B(a)$. Then $f^n \to a$ on W. But it does not

So $\{f^{n_k}|_{\mathcal{U}}\}$ may only converge to $\equiv a$. if it gets into $V \ni a$, it tends to a,

since $f^{n_k}(z)$ cannot approach a (

Ex: Newton method for finding roots of polynomials:

iterate $z \xrightarrow{f} z - \dfrac{p(z)}{p'(z)}$



Ex: $P(z) = z^3 - 1$

$z \to z - \dfrac{z^3-1}{3z^2} =$

$= \frac{2}{3} z + \dfrac{1}{3z^2}$

Is ∞ repelling or attracting?

Roots of p are superattracting points of f. Their Basins of attraction form Fatou set; their common boundary forms Julia set. [showing pictures].

Julia set is forward and backward invariant

Pf. The sequence f^{n_k} converges uniformly on $\mathcal{U}$ if and only if for each $\varepsilon > 0$, there is K: for $m, k > K$, $z \in \mathcal{U}$, $\operatorname{dist}(f^{n_k}(z), f^{n_m}(z)) < \varepsilon$. (Cauchy criterium)

This happens if and only if $\operatorname{dist}(f^{n_k - 1}(\underbrace{f(z)}_{w}), f^{n_m - 1}(\underbrace{f(z)}_{w})) < \varepsilon$

for any $\underbrace{f(z)}_{w} \in f(\mathcal{U})$.

So $\{f^n\}$ is normal on a nbhd $\mathcal{U}$ of z_0 if and only if $\{f^n\}$ is normal on $f(\mathcal{U}) \ni f(z_0)$.

Julia set of f coincides with Julia set of ...

Pf. If $\{f^n\}$ is normal, then its subsequence $\{f^{nm}\}$ is also normal.
If $\{f^{nm}\}$ is normal, ~~any subsequence of $\{f^n\}$~~
then $\{f^{nm+1}\}$, $\{f^{nm+2}\}$, ... $\{f^{nm+m-1}\}$ are also normal
(f^{n_k} converges $\Rightarrow$ $f^s(f^{n_k})$ converges)
and their union is $\{f^n\}$. ☒

Montel-Caratheodori criterion for normality:
if $\{f^n\}$ avoids 3 values on $\bar{\mathbb{C}}$, it is normal.

Corollary: for $z \in J(f)$, for $\mathcal{U} \ni z$, $f^n(\mathcal{U})$ covers all $\bar{\mathbb{C}}$ except 1 or 2 values (exceptional values), if $\deg f \geq 2$.
If there is 1 such value z_0, then $f^{-1}(z_0) = z_0 = f(z_0)$.
Sending z_0 to ∞, we get $f^{-1}(\infty) = \infty \Rightarrow f$ is a polynomial.
If there are 2 such values z_0, w_0, then $f^{-1}(z_0) \in \{z_0, w_0\}$
$f^{-1}(w_0) \in \{z_0, w_0\}$

Send them to $0, \infty$ and get $f(z) = \frac{P(z)}{Q(z)}$
0 has preimages 0 and ∞ only $\Rightarrow P(z) = z^k \cdot c$
∞ has preimages 0 and ∞ only $\Rightarrow Q(z) = z^l \cdot c$
so $f(z) = cz^k$ ($\sim z^2$, $0, \infty$ are superattr), or $f(z) = cz^{-k}$ (see hw
$0, \infty$ are also in Fatou set)

Corollary: For any $z \in J(f)$, $f^{-n}(z)$ are dense in $J(f)$
Indeed, if $f^{-n}(z)$ never visits $\mathcal{U}$, $\mathcal{U} \cap J(f) \neq \emptyset$,
then $f^n(\mathcal{U})$ never contains $z \Rightarrow$ z is exceptional $\Rightarrow$
z is in Fatou set ↯. This gives a way to compute $J(f)$.

Corollary: If $J(f)$ has nonempty interior, then it coincides with $\mathbb{C}$ | next time
$\mathcal{U} \in J(f)$: $\cup f^n(\mathcal{U}) = \mathbb{C} \setminus$ exceptional values
$\cup f^n(\mathcal{U}) \subset J(f)$, $J(f)$ is closed $\Rightarrow$ no exceptional values
$J(f) = \bar{\mathbb{C}}$.

Corollary: The boundary of the ~~Julia~~ attracting basin $\partial B(a)$ belongs to the Julia set.
For $z \in J(f)$, for $\mathcal{U} \ni z$, $f^n(\mathcal{U})$ covers almost all $\bar{\mathbb{C}}$
and thus intersects $B(a)$, so $\mathcal{U}$ intersects $B(a)$ $\Rightarrow$
$J(f) \in \partial B(a)$. The Other inclusion was proved above.