Genera of non-algerbaic leaves of polynomial foliations of $\mathbb{C}^2$

Based on a joint work with Yu. Kudryashov

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AMS-EMS-SPM joint meeting, 2015

Outline

1. Motivation and problem statement
   - Polynomial foliations of $\mathbb{C}^2$
   - Monodromy group at infinity
   - Main Theorems
2. A leaf with many handles
3. Leaves with infinitely many handles
4. Result on limit cycles

Class $\mathcal{A}_n$

Notation

Denote by $\mathcal{A}_n$ the set of foliations of $\mathbb{C}^2$ given by

$$\dot{x} = p(x, y),$$
$$\dot{y} = q(x, y),$$

where p, q are polynomials, $\deg p \leq n, \deg q \leq n$.

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Extension to $\mathbb{C}P^2$

Change of coordinates

$$u = \frac{1}{x}; \quad v = \frac{y}{x}; \quad d\tau = -u^{n-1}dt$$

$$\begin{cases} \dot{u} = u^{n+1}P\left(\frac{1}{u}, \frac{v}{u}\right) & =: u\tilde{P}(u, v) \\ \dot{v} = vu^nP\left(\frac{1}{u}, \frac{v}{u}\right) - u^nQ\left(\frac{1}{u}, \frac{v}{u}\right) & =: H(u, v). \end{cases}$$

The leaf at infinity

Let $\{ a_1, \dots, a_{n+1} \}$ be the roots of $H(0, v)$. Generically, $a_i \neq a_j$. $(0, a_j)$ are singularities of the extended foliation, and $L_\infty = \{ u = 0 \} \setminus \{ (0, a_j) \mid 1 \leq j \leq n+1 \}$ is its leaf.

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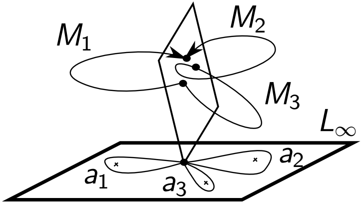
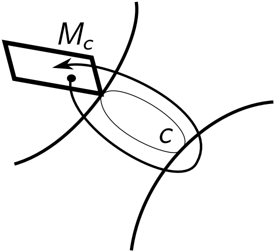
Monodromy group at infinity



Choose $O \in L_\infty$ and take loops $\gamma_j \subset L_\infty$ around a_j starting from O.
The *monodromy (pseudo)group at infinity* is generated by the monodromy maps $M_j := M_{\gamma_j}$ along γ_j.

Limit cycles



Definition

A *limit cycle* is a homotopy class $[c]$ of a closed loop c on a leaf such that $M_c \neq id$

Limit cycles correspond to isolated fixed points of monodromy maps.

Definition

A set of limit cycles of a foliation is called *homologically independent* if for any leaf L all the cycles located on this leaf are linearly independent in $H_1(L)$.

Generic monodromy groups and generic foliations

Generic pseudogroup in $(\mathbb{C}, 0)$

- Orbits are dense in $(\mathbb{C}, 0)$
- Infinite number of isolated fixed points
- Rigidity

Generic foliation from $\mathcal{A}_n$

- Leaves are dense in $\mathbb{C}P^2$
- Infinite number of independent limit cycles
- Rigidity

Generic foliations of $\mathcal{A}_n$

Theorem (M. Khudai-Verenov, 1962; Yu.Ilyashenko, 1978)

For a generic foliation from $\mathcal{A}_n$, all leaves are dense.

Theorem (Yu.Ilyashenko, 1978)

For $n \geq 2$, there exists a full-measure subset of $\mathcal{A}_n$ such that each foliation from this subset possesses infinitely many homologically independent limit cycles.

Theorem (A. Shcherbakov, E. Rosales-Gonzalez, L. Ortiz-Bobadilla, 1998)

For $n \geq 3$, there exists an open dense subset of $\mathcal{A}_n$, such that each foliation from this subset possesses infinitely many homologically independent limit cycles.

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Anosov conjecture

Conjecture (D.Anosov)

The countable set of leaves of a generic foliation are topological cylinders, others are topological discs

This is correct for generic *analytic* foliations in $\mathbb{C}P^2$ (T.Firsova, 2006; T.Golenishcheva-Kutuzova, 2006).

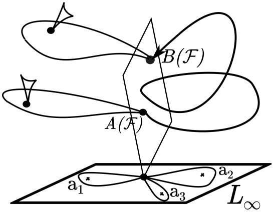
Conjecture (D.Anosov)

Generic foliation from $\mathcal{A}_n$ has no identical cycles.

Definition

An *identical cycle* is a free homotopy class $[c]$ of a closed loop c on a leaf such that $M_c = id$

Density of separatrix connections



Theorem (D.Volk)

In a dense subset of $\mathcal{A}_n$, any foliation has a separatrix connection.

Lemma (D. Volk)

*For any neighborhood $\mathcal{F} \in U \subset$,
for two holomorphic functions $A, B \colon U \to S$, there exists a curve $\gamma \in L_\infty$ such that the condition $M_\gamma(A(\mathcal{F})) = B(\mathcal{F})$ defines a codimention-one analytic submanifold in U.*

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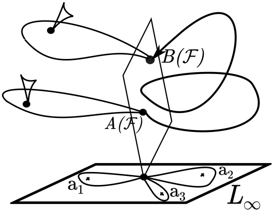

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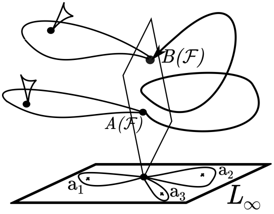

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Lemma (D. Volk; NG, Yu. Kudryashov)

For any neighborhood $\mathcal{F} \in U \subset \mathcal{M}$, $\dim \mathcal{M} > 8$, for two holomorphic functions $A, B \colon U \to S$, there exists a curve $\gamma \in L_\infty$ such that the condition $M_\gamma(A(\mathcal{F})) = B(\mathcal{F})$ defines a codimention-one analytic submanifold in U.

Main theorems

Theorem (NG, Yu. Kudryashov)

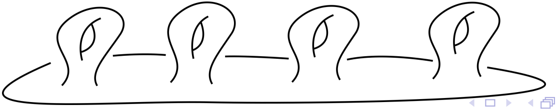
In a dense subset of $\mathcal{A}_n$, any foliation has a leaf with at least $\frac{(n+1)(n+2)}{2} - 2$ handles.

Theorem (NG, Yu. Kudryashov)

Let $\mathcal{A}_n^{sym}$ be the subspace of $\mathcal{A}_n$, $n \geq 2$, given by

$$p(-x, y) = p(x, y), \quad q(-x, y) = -q(x, y).$$

For a foliation $\mathcal{F}$ from some open dense subset of $\mathcal{A}_n^{sym}$, all leaves of $\mathcal{F}$ (except for a finite set of algebraic leaves) have infinite genus.



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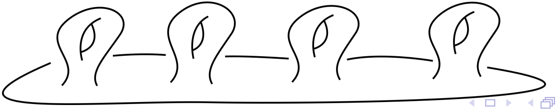

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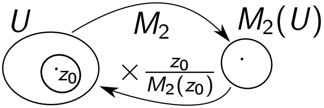
Constructing one limit cycle

Genericity assumptions

- $|M_1'(0)| \neq 1$, hence M_1 is linearizable;
- $M_2 \circ M_1 \neq M_1 \circ M_2$;
- $\langle M_1'(0), M_2'(0) \rangle$ is dense in $\mathbb{C}^*$.

Approximation of a linear map $z \mapsto \tau z$ (if $|M_1'(0)| < 1$)

- Choose k, l s.t. $(M_1^k \circ M_2^l)'(0) \approx \tau$.
- $\Rightarrow M_1^{-N} \circ M_1^k \circ M_2^l \circ M_1^N(z) \rightrightarrows (M_1^k \circ M_2^l)'(0)z \approx \tau z$ as $N \to \infty$.



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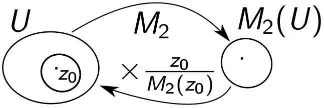

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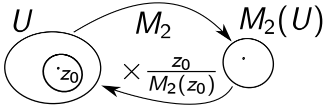

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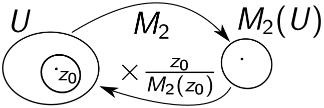

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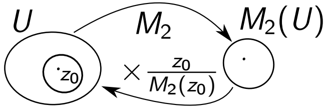

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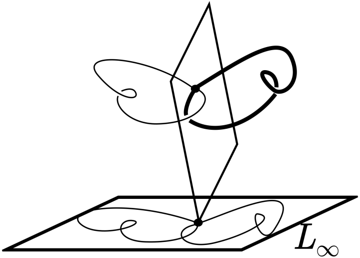
One handle: idea

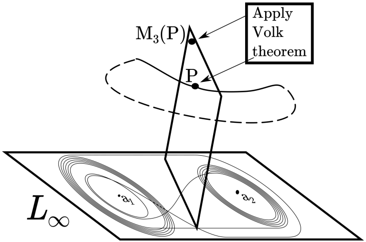
- We construct two monodromy maps with a common fixed point
- and analyze the intersections of corresponding limit cycles.

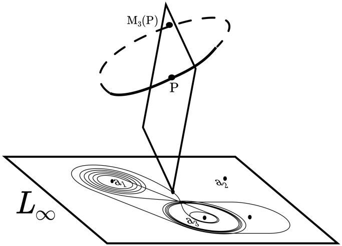


One handle is guaranteed if two limit cycles intersect transversally at one point.

One handle: plan

- Take one limit cycle corresponding to $M_1^{-N} \circ (M_1^k M_2^l) \circ M_1^N \circ M_2(z)$.
- Use Volk theorem to obtain another one in a submanifold of codimension one.





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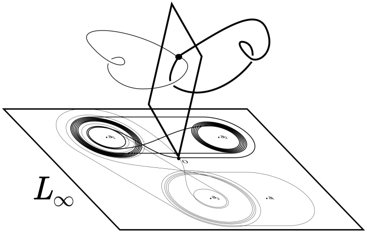


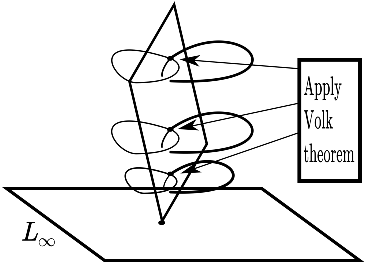
- Analyze intersections.

Result

In $\mathcal{M}_1 \subset \mathcal{A}_n$ with codim $\mathcal{M}_1 = 1$, any $\mathcal{F} \in \mathcal{M}_1$ has one handle.

Many handles

- In $\mathcal{M}_1 \subset \mathcal{A}_n$ with $\operatorname{codim} \mathcal{M}_1 = 1$, we have one handle.
- Apply Volk lemma inside $\mathcal{M}_1$.
- We get $\mathcal{M}_2$, $\operatorname{codim} \mathcal{M}_2 = 2$: each foliation $\mathcal{F} \in \mathcal{M}_1$ has 2 handles on different leaves,



- etc.
- Use Volk lemma to take all handles into one leaf.

The codimension is $2g - 1$, so $g \approx \dim \mathcal{A}_n / 2 = \frac{(n+1)(n+2)}{2}$

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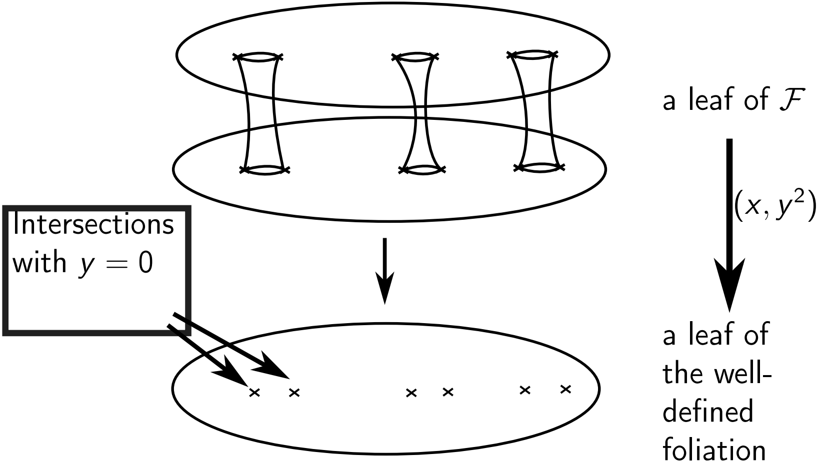
Infinite genus for $p(x,y) = -p(x,-y), q(x,y) = q(x,-y)$



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Limit cycles: improvement of a classical result

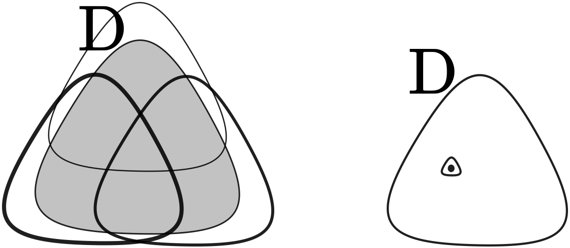
Theorem (NG, YK)

For $n \geqslant 2$, for a dense subset $\mathcal{A}_n^{LC} \subset \mathcal{A}_n$, each $\mathcal{F} \in \mathcal{A}_n^{LC}$ possesses an infinite sequence of limit cycles $[\gamma_j]$ such that:

- *the cycles are homologically independent;*
- *the multipliers of the cycles tend to zero;*
- *the cycles are uniformly bounded, i.e., there exists a ball in $\mathbb{C}^2$ that includes all representatives γ_j;*
- *there exists a cross-section such that γ_j intersect it in a dense subset.*

Limit cycles: main idea

- Find monodromy maps f_j and the domain D detached from zero



such that the images of D under f_j cover D.

- Any long composition has a fixed point which produces a limit cycle
- Multipliers tend to 0.
- Fixed points are dense in D.

Homological independence is not trivial.

Limit cycles: homological independence (hint)

If dependent cycles are simple and disjoint, the dependence is of the form $c_{i_1} \pm \dots \pm c_{i_k} = 0$ in $H_1(L)$.
Thus the multipliers satisfy $\mu_{i_1}^{\pm 1} \cdot \dots \cdot \mu_{i_k}^{\pm 1} = 1$.

This is impossible if the multipliers
• rapidly tend to 1 (as in classical proofs)
• or rapidly tend to zero (as in the our proof).
Now the main difficulty is to avoid intersections and self-intersections of limit cycles.