SYMBOLIC DYNAMICS AND SELF-SIMILAR GROUPS

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ABSTRACT. Self-similar groups and permutational bimodules are used to study combinatorics and symbolic dynamics of expanding self-coverings. We describe functors between the category of contracting self-similar groups and the category of expanding self-coverings (with appropriate morphisms). These functors transform some questions in dynamical systems to questions in algebra. As examples we show how some plane filling curves (in particular the original Peano curve) can be interpreted in terms of embeddings of self-similar groups.

1. INTRODUCTION

The paper is an overview of the results and methods in symbolic dynamics via self-similar groups. Our aim is to describe the algebraic structure behind the combinatorics and symbolic dynamics of expanding self-coverings of topological spaces.

Symbolic dynamics encodes a dynamical system $f : \mathcal{X} \longrightarrow \mathcal{X}$ by a shift map on a space of sequences over a finite alphabet using a Markov partition of the space $\mathcal{X}$ and encoding the points of $\mathcal{X}$ by their itineraries with respect to the partition (see the surveys [Adl98] and [Kit98]).

We are interested here in d-fold expanding self-coverings $f : \mathcal{X} \longrightarrow \mathcal{X}$ of topological spaces (possibly of orbispaces), for example post-critically finite rational functions (restricted to their Julia sets) or self-coverings of tori. Then a symbolic presentation of f may be given by a closed covering $\mathcal{X} = \bigcup_{x \in \mathsf{X}} \mathcal{T}_x$ such that $f(\mathcal{T}_x) = \mathcal{X}$, where X is a finite set of indices (an alphabet). The space $\mathcal{X}$ is represented as the quotient of the space $\mathsf{X}^\omega = \{x_1 x_2 \ldots \; : \; x_i \in \mathsf{X}\}$ of all infinite one-sided sequences over the alphabet X and the function f is represented as the image of the shift $x_1 x_2 \ldots \mapsto x_2 x_3 \ldots$ under the quotient map (i.e., f is semiconjugate to the shift).

The aim of the paper is to show an algebraic structure behind such symbolic presentation of the dynamical systems. We show that combinatorics of the dynamical system is described by a self-similarity on a group called the *iterated monodromy group* of the system. The group is generated in some sense by the adjacency between the pieces of the Markov partition and is dual to the equivalence relation on X^ω given by the quotient map $\mathsf{X}^\omega \longrightarrow \mathcal{X}$.

Symbolic presentation of the dynamical system is not unique, since there are different Markov partitions. But we show that if we define self-similarity of a group in a sufficiently "distilled" way, then it will be unique and will depend only on the dynamical system. The choice of a different Markov partition will correspond to a choice of a "coordinate system" (a *basis*) of the self-similarity.

This distilled algebraic structure is the notion of a *permutational bimodule* over a group G, i.e., a set $\mathfrak{M}$ with two commuting (left and right) actions of G on it. We construct two objects naturally associated with such a bimodule. One is an action of G on a rooted tree (which we call the *Fock tree* of the bimodule). The other is an expanding dynamical system $\mathsf{s} : \mathcal{J}_G \longrightarrow \mathcal{J}_G$ (constructed provided $\mathfrak{M}$ is *hyperbolic*), called the *limit dynamical system.*

Both objects are uniquely constructed from the bimodule $\mathfrak{M}$. A choice of a *basis* X of $\mathfrak{M}$ (i.e., an orbit transversal of the right action of the group) gives symbolic presentations of both objects. It labels the vertices of the rooted tree by finite words over the alphabet X and gives a symbolic presentation of the limit dynamical system: the points of $\mathcal{J}_G$ are represented by infinite one-sided sequences over X and s is represented by the shift.

The main feature of self-similar groups and permutational bimodules is the possibility of effective algebraic computations with them. This gives new tools in studying combinatorics of expanding self-coverings. As an example of this computational effectiveness, see the paper [BN06c], where the "twisted rabbit" problem of J. Hubbard and similar questions are answered using the technology of self-similar groups.

Self-similar groups, permutational bimodules and iterated monodromy groups are the subjects of a recent monograph [Nek05] of the author. The main new topic of the present paper is the fact that the two main constructions of [Nek05] (the limit dynamical system and the iterated monodromy group) are mutually inverse functors between the properly defined categories of self-similar groups and of dynamical systems. This categorical approach is the core theme of the paper. This is why the main examples in the paper are maps (semiconjugacies) between spaces and dynamical systems rather than the spaces and the dynamical systems themselves.

An important part of the paper are examples, which illustrate the use of iterated monodromy groups in the study of dynamical systems and semiconjugacies between them. For instance, we interpret the original paper of Peano on his plane-filling curve [Pea90] and a paper of Milnor [Mil04] on matings in terms of self-similar groups.

Structure of the paper. Section "Self-similar groups" describes the category of self-similar groups (permutational bimodules), their action on the Fock tree, the main techniques of self-similar groups (wreath recursions and automata) and shows how to associate a self-similar group (the iterated monodromy group) to a self-covering of a topological space.

The next section "Limit spaces" gives the connection in the other direction: we show how a dynamical system can be constructed from a permutational bimodule, show functoriality of this construction and prove some properties of maps between the limit spaces, which are induced by inclusion of self-similar groups.

Fourth section "Abelian groups" considers some examples of self-similar actions of abelian and virtually abelian groups. There are not many new results in this section, but we compute some examples, which will be used in the last section. Besides that we consider the limit space of a non-self-replicating (non-recurrent) self-similarity on $\mathbb{Z}$ (which is associated to a numeration system with base 1.5).

In fifth section "Complex polynomials" we show how the classical notions from symbolic dynamics of post-critically finite polynomials can be interpreted in terms of self-similar groups.

The last section "Plane-filling curves" considers some examples of plane-filling maps, which are induced by morphisms (embeddings) of self-similar groups. It was amazingly easy to translate the historically first example of a plane-filling curve—the Peano curve [Pea90], into the language of self-similar groups. The paper of Peano was already written in symbolic terms and one just had to rewrite the definitions in a group-theoretic way.

The second example is taken from a paper of J. Milnor [Mil04] on matings of polynomials. We show that the plane-filling dendrite constructed in the paper of J. Milnor is induced by an embedding of self-similar groups and show how the Schreier graphs of the embedded iterated monodromy group can be constructed using the classical paper-folding procedure.

The last example of the paper is the Sierpiński plane-filling curve, which is also constructed as a map induced by an embedding of self-similar groups.

2. Self-similar groups

2.1. **Self-similar actions.** If X is a finite set then by X^* we denote the set of finite words over the alphabet X (the free monoid generated by X).

We consider X^* as a rooted tree with the empty word as the root and where a vertex $v \in \mathsf{X}^*$ is connected to every vertex of the form vx for $x \in \mathsf{X}$.

Definition 1. A *(faithful) self-similar group* (G, X) is a group G together with a faithful action on X^* such that for every $g \in G$ and $x \in \mathsf{X}$ there exist $h \in G$ and $y \in \mathsf{X}$ such that
$$g(xv) = yh(v)$$
for all v.

It is easy to see that every self-similar group acts by automorphisms on the rooted tree X^*.

We will write the equality from the definition formally as
$$g \cdot x = y \cdot h.$$
This formal equality becomes a true equality of compositions of transformations of X^*, if we identify letters with "creation operators"
$$x : v \mapsto xv.$$

2.1.1. *Example: binary adding machine.* Let $\mathsf{X} = \{0, 1\}$. Consider an automorphism of the rooted tree X^* given by the following recurrent definition
$$a(0w) = 1w, \qquad a(1w) = 0a(w),$$
(and the condition that the image of the empty word is the empty word).

It is easy to see that these recurrent relations define the transformation a in a unique way. This transformation is called the *binary adding machine*, or the *binary odometer*. This name comes from the fact that a acts on the binary sequences in the same way as adding 1 acts on the binary numbers. Namely, $a(x_1 x_2 \ldots x_n) = y_1 y_2 \ldots y_n$ if and only if
$$x_1 + 2x_2 + 2^2 x_3 + \cdots 2^{n-1} x_n + 1 = y_1 + 2y_2 + 2^2 y_3 + \cdots 2^{n-1} y_n \pmod{2^n}.$$

The infinite cyclic group $\langle a \rangle$ is a self-similar group (more precisely, it is a self-similar action of the infinite cyclic group). We have
$$a^{2n} \cdot 0 = 0 \cdot a^n, \quad a^{2n+1} \cdot 0 = 1 \cdot a^n$$

and

$$a^{2n} \cdot 1 = 1 \cdot a^n, \quad a^{2n+1} \cdot 1 = 0 \cdot a^{n+1}.$$

2.1.2. *The Basilica group.* Consider the group acting on the tree X^* over the alphabet $\mathsf{X} = \{0, 1\}$ and generated by two transformations a and b, which are given by

$$a(0w) = 1w, \qquad b(0w) = 0w$$
$$a(1w) = 0b(w), \quad b(1w) = 1a(w).$$

We will see later that this group is related to the polynomial $z^2 - 1$, hence the name "Basilica".

It is interesting that this group was defined and studied by R.Grigorchuk and A.Żuk just as a group generated by a three-state automaton, without knowing the relation with dynamics (see [GŻ02a, GŻ02b]).

2.2. **Permutational bimodules.** Let us denote by $\mathfrak{M} = \mathsf{X} \cdot G$ the set of pairs $x \cdot g$, $x \in \mathsf{X}$, $g \in G$, seen as transformations

$$x \cdot g(v) = xg(v)$$

of X^*. The correspondence between the pairs and the transformation is bijective, since we assume that the action of G on X^* is faithful. Then it follows directly from the definition of a self-similar action that G acts on $\mathfrak{M}$ from left and from right by compositions:

(1) $$h \cdot (x \cdot g) = y \cdot h_x g \qquad (x \cdot g) \cdot h = x \cdot gh,$$

where h_x and y are such that $h(xw) = yh_x(w)$, i.e., $h \cdot x = y \cdot h_x$.

The left and the right actions of G on $\mathfrak{M}$ commute and we get an example of a *permutational G-bimodule.*

Definition 2. Let G be a group. A (*permutational*) *G-bimodule* is a set $\mathfrak{M}$ together with commuting left and right actions of G on $\mathfrak{M}$. More explicitly, for every $g \in G$ and $x \in \mathfrak{M}$ we have elements $g \cdot x, x \cdot g \in \mathfrak{M}$ and

- $x \cdot 1 = 1 \cdot x = x$ for all $x \in \mathfrak{M}$,
- $(x \cdot g) \cdot h = x \cdot (gh)$ and $g \cdot (h \cdot x) = (gh) \cdot x$ for all $g, h \in G$ and $x \in \mathfrak{M}$,
- $(g \cdot x) \cdot h = g \cdot (x \cdot h)$ for all $g, h \in G$ and $x \in \mathfrak{M}$.

We will call the bimodule $\mathfrak{M} = \mathsf{X} \cdot G$ defined by (1) the *self-similarity bimodule* of the action (G, X). Self-similarity bimodules have two special properties:

(a) The right action of G on $\mathfrak{M}$ is free, i.e., $x \cdot g = x$ implies $g = 1$.
(b) The number of orbits of the right action of G on $\mathfrak{M}$ is finite.

We call a permutational G-bimodule a *covering bimodule* if it satisfies the properties (a) and (b). We say that a covering bimodule is *d-fold* if the number of orbits of the right action is d.

Definition 3. A *basis* of a covering G-bimodule $\mathfrak{M}$ is a transversal of the orbits of the right action, i.e., a set Y such that every element $x \in \mathfrak{M}$ can be uniquely written in the form $y \cdot g$ for $y \in \mathsf{Y}$ and $g \in G$.

For example, if $\mathfrak{M}$ is the self-similarity bimodule $\mathsf{X} \cdot G$ of a self-similar action (G, X), then the set $\{x \cdot 1 \ : \ x \in \mathsf{X}\}$ is a basis of $\mathfrak{M}$, which we will identify with X (identifying $x \cdot 1$ with x).

On the other hand, if we have a covering G-bimodule $\mathfrak{M}$ and a basis Y, then we get an action of G on Y^* defined recurrently by the condition that

$$g(xw) = yh(w)$$

for all $w \in \mathsf{Y}^*$ if $g \cdot x = y \cdot h$ in $\mathfrak{M}$ (the elements $y \in \mathsf{Y}$ and $h \in G$ are uniquely determined by $g \in G$ and $x \in \mathsf{Y}$, since $g \cdot x \in \mathfrak{M}$ is uniquely written in the form $y \cdot h$, by the definition of a basis). This action of G on Y^* is called the *self-similar action associated with the bimodule $\mathfrak{M}$ and the basis Y*.

The following is more or less straightforward (see Proposition 2.3.4 of [Nek05] for details).

Proposition 2.1. *Let $\mathfrak{M}$ be a covering G-bimodule and let X and Y be two bases of $\mathfrak{M}$. Then the associated actions of G on X^* and Y^* are conjugate and the conjugator $\alpha : \mathsf{X}^* \longrightarrow \mathsf{Y}^*$ is defined by the recurrent formula*

$$\alpha(xw) = y_x h_x(\alpha(w)),$$

where $y_x \in \mathsf{Y}$ and $h_x \in G$ are such that $x = y_x \cdot h_x$ and h_x acts on $\alpha(w) \in \mathsf{Y}^*$ by the action of G on Y^*.

Taking into account the last proposition and additional reasons, which will be more evident below, we adopt the following more general definition.

Definition 4. A *self-similar group* is a triple $(G, \mathfrak{M}, \mathsf{X})$, where G is a group, $\mathfrak{M}$ is a covering G-bimodule and X is a basis of $\mathfrak{M}$.

We will usually omit $\mathfrak{M}$ and write just (G, X), when it is clear what bimodule is used.

In many cases the following technical condition is useful.

Definition 5. A self-similar group $(G, \mathfrak{M}, \mathsf{X})$ is *self-replicating* (or *recurrent*, in terminology of [Nek05]) if the left action of G on $\mathfrak{M}$ is transitive, i.e., if for every $x, y \in \mathsf{X}$ and $h \in G$ there exists $g \in G$ such that $g \cdot x = y \cdot h$.

2.3. **Tensor products and the Fock tree.** Let us go back to the abstract case of a permutational G-bimodule $\mathfrak{M}$ (without whatsoever conditions on the right action).

Definition 6. Let $\mathfrak{M}_1$ and $\mathfrak{M}_2$ be permutational G-bimodules. Their *tensor product* $\mathfrak{M}_1 \otimes \mathfrak{M}_2$ is the quotient of the direct product $\mathfrak{M}_1 \times \mathfrak{M}_2$ by the identifications

$$x_1 \cdot g \otimes x_2 = x_1 \otimes g \cdot x_2$$

for $x_i \in \mathfrak{M}_i$ and $g \in G$. The right and left actions of G on $\mathfrak{M}_1 \otimes \mathfrak{M}_2$ are given by

$$g_1 \cdot (x_1 \otimes x_2) \cdot g_2 = (g_1 \cdot x_1) \otimes (x_2 \cdot g_2).$$

It is not hard to prove that the tensor product of two bimodules is well defined and that tensor product is an associative operation, i.e., that the bimodules $(\mathfrak{M}_1 \otimes \mathfrak{M}_2) \otimes \mathfrak{M}_3$ and $\mathfrak{M}_1 \otimes (\mathfrak{M}_2 \otimes \mathfrak{M}_3)$ are isomorphic, where the isomorphism is given by the map $(x_1 \otimes x_2) \otimes x_3 \mapsto x_1 \otimes (x_2 \otimes x_3)$. For more details see [Nek05].

In particular, if $\mathfrak{M}$ is a G-bimodule, then the bimodules $\mathfrak{M}^{\otimes n}$ are defined for every natural n. We set $\mathfrak{M}^{\otimes 0} = G$ with the natural left and right actions of G on itself. Note that $G \otimes \mathfrak{M}$ is naturally isomorphic to $\mathfrak{M}$ for every G-bimodule $\mathfrak{M}$.

The following is straightforward.

Proposition 2.2. *The disjoint union $\bigsqcup_{n \geq 0} \mathfrak{M}^{\otimes n}$ is a semigroup with respect to the tensor multiplication.*

We call this semigroup the *tensor semigroup* of the bimodule and denote it $\mathfrak{M}^*$. Note that $G = \mathfrak{M}^{\otimes 0}$ is a subgroup of $\mathfrak{M}^*$ and hence $\mathfrak{M}^*$ is a permutational G-bimodule (which can be defined as the *direct sum* of the bimodules $\mathfrak{M}^{\otimes n}$).

We will write in some cases $x_1 x_2 \ldots x_n$ instead of $x_1 \otimes x_2 \otimes \cdots \otimes x_n$.

Lemma 2.3. *Let $\mathfrak{M}_1$ and $\mathfrak{M}_2$ be G-bimodules. If $x_1 \otimes x_2$ and $y_1 \otimes y_2 \in \mathfrak{M}_1 \otimes \mathfrak{M}_2$ belong to one right G-orbit, then x_1 and y_1 belong to one right G-orbit.*

Proof. There exists $g \in G$ such that $x_1 \otimes x_2 = y_1 \otimes y_2 \cdot g$. But this means, by definition of a tensor product, that there exists $h \in G$ such that $x_1 = y_1 \cdot h$ and $h \cdot x_2 = y_2 \cdot g$. Hence, x_1 and y_1 belong to one right orbit. $\square$

Definition 7. Let $\mathfrak{M}$ be a permutational G-bimodule. Then its *Fock tree (of right orbits)* is the set

$$T_{\mathfrak{M}} = \mathfrak{M}^*/G = \bigsqcup_{n \geq 0} \mathfrak{M}^{\otimes n}/G$$

of the right G-orbits of the tensor semigroup of $\mathfrak{M}$. The root of the tree is the unique element of the set $\mathfrak{M}^{\otimes 0}/G = G/G$ and two orbits $A \in \mathfrak{M}^{\otimes n}/G$ and $B \in \mathfrak{M}^{\otimes(n+1)}/G$ are connected by an edge if there exist $m \in A$ and $x \in \mathfrak{M}$ such that $m \otimes x \in B$.

It follows from Lemma 2.3 that the right orbit of $m \in \mathfrak{M}^{\otimes(n-1)}$ is uniquely determined by the right orbit of $m \otimes x \in \mathfrak{M}^{\otimes n}$, hence $T_{\mathfrak{M}}$ is a rooted tree.

Lemma 2.4. *Let $\mathfrak{M}_1$ and $\mathfrak{M}_2$ be G-bimodules and let $A \subset \mathfrak{M}_2$ be an orbit of the right action of G on $\mathfrak{M}_2$. Then for every $y \in \mathfrak{M}_1$ the set $y \otimes A$ is a right orbit of $\mathfrak{M}_1 \otimes \mathfrak{M}_2$.*

In particular (for $\mathfrak{M}_1 = G$) for every $g \in G$ the set $g \cdot A$ is a right orbit.

Proof. For any $x_1, x_2 \in A$ there is $g \in G$ such that $x_1 = x_2 \cdot g$. But then $y \otimes x_1 = y \otimes x_2 \cdot g$, hence $y \otimes x_1$ and $y \otimes x_2$ belong to one right orbit. Consequently, $y \otimes A$ is a subset of a right orbit of $\mathfrak{M}_1 \otimes \mathfrak{M}_2$.

On the other hand, suppose that $y \otimes x_1 \in y \otimes A$ and $y' \otimes x_2 \in \mathfrak{M}_1 \otimes \mathfrak{M}_2$ belong to one right orbit. Then there exists $g \in G$ such that $y \otimes x_1 = y' \otimes x_2 \cdot g$, hence $y' \otimes x_2 = y \otimes x_1 \cdot g^{-1}$, i.e., $y' \otimes x_2 \in y \otimes A$. $\square$

Corollary 2.5. *The left action of G on $\mathfrak{M}^*$ induces an action of G by automorphisms of the Fock tree $T_{\mathfrak{M}}$.*

2.4. **Symbolic labeling of the Fock tree.** In the case of covering bimodules we get a nice symbolic labeling of the vertices of the Fock tree. Let $\mathfrak{M}$ be a covering bimodule and let X be its basis. We have the following property of bases of bimodules (the proof, which is easy, can be found in Proposition 2.3.2 of [Nek05]).

Proposition 2.6. *Let $\mathfrak{M}_1$ and $\mathfrak{M}_2$ be covering G-bimodules with bases X_1 and X_2, respectively. Then $\mathfrak{M}_1 \otimes \mathfrak{M}_2$ is also a covering bimodule and the set $\{x_1 \otimes x_2 : x_i \in \mathsf{X}_i\}$ is its basis.*

In particular, if X is a basis of a bimodule $\mathfrak{M}$, then $\mathsf{X}^n = \{x_1 \otimes \cdots \otimes x_n : x_i \in \mathsf{X}\}$ is a basis of $\mathfrak{M}^{\otimes n}$. Since a basis by definition is a right orbit transversal, we get a bijection $\Lambda : \mathsf{X}^n \longrightarrow \mathfrak{M}^{\otimes n}/G$, mapping an element of X^n to its right orbit. It

follows now from the definition of the Fock tree that the orbit containing $v \in \mathsf{X}^n$ is connected to the orbits containing elements $v \otimes x$ for $x \in \mathsf{X}$. Thus, we obtain an isomorphism $\Lambda : \mathsf{X}^* \longrightarrow T_{\mathfrak{M}}$.

It is also easy to see that the isomorphism Λ conjugates the left action of G on $T_{\mathfrak{M}}$ with the self-similar action $(G, \mathfrak{M}, \mathsf{X})$ associated with the bimodule $\mathfrak{M}$ and the basis X. In particular, we get another proof of the fact that the action $(G, \mathfrak{M}, \mathsf{X})$ does not depend, up to a conjugacy, on the choice of the basis X.

2.5. **Iterated monodromy groups.** Let $\mathcal{M}$ be an arcwise connected and locally arcwise connected topological space and let $\mathcal{M}_1$ be its arcwise connected subset. Suppose that $f : \mathcal{M}_1 \longrightarrow \mathcal{M}$ is a d-fold covering map. We call such maps *partial self-coverings.*

We can iterated the partial defined map f and get d^n-fold coverings $f^n : \mathcal{M}_n \longrightarrow \mathcal{M}$ (possibly defined on smaller subsets).

If we choose a basepoint $t \in \mathcal{M}$, then the fundamental group $\pi_1(\mathcal{M}, t)$ acts naturally (by the *monodromy action*) on the set of preimages $f^{-n}(t)$ of t. In this action the image $\gamma(z)$ of a point $z \in f^{-n}(t)$ under the action of a loop $\gamma \in \pi_1(\mathcal{M}, t)$ is the end of the unique f^n-preimage γ_z of γ which starts in z.

The disjoint union

$$T_f = \bigsqcup_{n \geq 0} f^{-n}(t)$$

has a natural structure of a rooted tree. The root is the basepoint t (which is the unique element of $f^{-0}(t)$) and a vertex of the nth level $z \in f^{-n}(t)$ is connected to the vertex $f(z) \in f^{-(n-1)}(t)$ of the $(n-1)$st level.

It is easy to see that the monodromy action of $\pi_1(\mathcal{M}, t)$ on the levels of the tree T_f preserves the adjacency of the vertices, hence it is an action by automorphisms of the rooted tree T_f. We call it the *iterated monodromy action.*

Definition 8. The *iterated monodromy group* of a partial self-covering $f : \mathcal{M}_1 \longrightarrow \mathcal{M}$, denoted $\mathrm{IMG}(f)$, is the quotient of the fundamental group $\pi_1(\mathcal{M}, t)$ by the kernel of the iterated monodromy action, i.e., it is the group of automorphisms of T_f which are defined by the monodromy action of the elements of the fundamental group.

So far the iterated monodromy group is just an automorphism group of an abstract d-regular rooted tree with no self-similarity. But there is a natural self-similarity on $\mathrm{IMG}(f)$ which is best defined in terms of a covering $\pi_1(\mathcal{M}, t)$-bimodule.

Definition 9. Let $f : \mathcal{M}_1 \longrightarrow \mathcal{M}$ be a partial self-covering and let $t \in \mathcal{M}$ be a point. The $\pi_1(\mathcal{M}, t)$-bimodule $\mathfrak{M}_{f,t}$ *associated with f* is the set of homotopy classes of paths in $\mathcal{M}$ starting in t and ending in a preimage $z \in f^{-1}(t)$ of t under f. The right action is concatenation of the paths

$$\ell \cdot \gamma = \ell \gamma$$

and the left action is concatenation with a lift of γ:

$$\gamma \cdot \ell = \gamma_z \ell,$$

where γ_z is the f-preimage of γ starting in z.

Note that here we compose paths as functions, i.e., in the path $\ell \gamma$ the path γ is passed before the path ℓ.
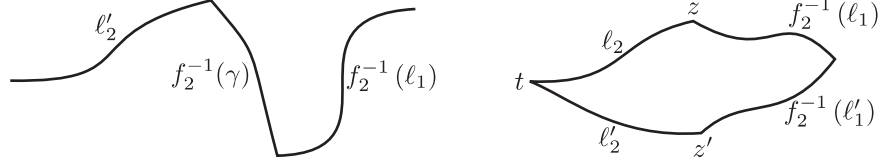
FIGURE 1

Proposition 2.7. *Let* $f_1 : \mathcal{M}_1 \longrightarrow \mathcal{M}$ *and* $f_2 : \mathcal{M}_2 \longrightarrow \mathcal{M}$ *be partial self-coverings of* $\mathcal{M}$. *Then the* $\pi_1(\mathcal{M}, t)$-*bimodule* $\mathfrak{M}_{f_1,t} \otimes \mathfrak{M}_{f_2,t}$ *is isomorphic to the* $\pi_1(\mathcal{M}, t)$-*bimodule* $\mathfrak{M}_{f_1 \circ f_2, t}$ *associated to the composition* $f_1 \circ f_2$ *of the self-coverings. Moreover, the isomorphism is the map*

$$L : \ell_1 \otimes \ell_2 \mapsto f_2^{-1}(\ell_1)_{\ell_2} \ell_2,$$

where $f_2^{-1}(\ell_1)_{\ell_2}$ *is the* f_2-*preimage of* ℓ_1 *starting at the end of the path* ℓ_2.

Proof. We have to show that L is well defined, bijective and agrees with the bi-module structures.

Let us prove that L is well defined. Suppose that $\ell_1 \otimes \ell_2$ and $\ell_1' \otimes \ell_2'$ are equal elements of $\mathfrak{M}_{f_1,t} \otimes \mathfrak{M}_{f_2,t}$. This means that there exists an element $\gamma \in \pi_1(\mathcal{M}, t)$ such that $\ell_1' = \ell_1 \cdot \gamma$ and $\gamma \cdot \ell_2' = \ell_2$. We have $\ell_1 \cdot \gamma = \ell_1 \gamma$ and $\gamma \cdot \ell_2' = f_2^{-1}(\gamma)_{\ell_2'} \ell_2'$. Therefore

$$L(\ell_1 \otimes \ell_2) = f_2^{-1}(\ell_1)_{\ell_2} \ell_2 = f_2^{-1}(\ell_1)_{\ell_2} f_2^{-1}(\gamma)_{\ell_2'} \ell_2' =$$
$$f_2^{-1}(\ell_1 \gamma)_{\ell_2'} \ell_2' = f_2(\ell_1')_{\ell_2'} \ell_2' = L(\ell_1' \otimes \ell_2'),$$

(see the left-hand side part of Figure 1).

Let us show that L is injective. Suppose that $L(\ell_1 \otimes \ell_2) = L(\ell_1' \otimes \ell_2')$. This means that the paths $f_2^{-1}(\ell_1)_{\ell_2} \ell_2$ and $f_2^{-1}(\ell_1')_{\ell_2'} \ell_2'$ are homotopic. In particular the endpoints of the paths are equal. It follows that $\left(f_2^{-1}(\ell_1')_{\ell_2'}^{-1} \right)^{-1} f_2^{-1}(\ell_1)_{\ell_2}$ is a path homotopic to the path $\ell_2' \ell_2^{-1}$ (see the right-hand side part of Figure 1).

Then

$$\gamma = f_2\left(\left(f_2^{-1}(\ell_1')_{\ell_2'}^{-1} \right)^{-1} f_2^{-1}(\ell_1)_{\ell_2} \right) = (\ell_1')^{-1} \ell_1$$

is a loop such that

$$\ell_1' \cdot \gamma = \ell_1$$

and

$$\gamma \cdot \ell_2 = f_2^{-1}(\gamma)_{\ell_2} \ell_2 = \ell_2 \ell_2^{-1} \ell_2 = \ell_2',$$

hence $\ell_1 \otimes \ell_2 = \ell_1' \otimes \ell_2'$.

Let us show that L is surjective. Suppose that $\ell \in \mathfrak{M}_{f_1 \circ f_2, t}$ is an arbitrary element, i.e., a path starting at t and ending in a point $t' \in (f_1 \circ f_2)^{-1}(t)$. Choose a path $\ell_2 \in \mathfrak{M}_{f_2, t}$ starting at t and ending in an f_2-preimage of t. Then $f_2\left(\ell \ell_2^{-1} \right)$ is a path starting in t and ending in an f_1-preimage of t. Let us denote it ℓ_1. Then $\ell_1 \in \mathfrak{M}_{f_1}$ and

$$L(\ell_1 \otimes \ell_2) = f_2^{-1}(\ell_1) \ell_2 = \ell \ell_2^{-1} \ell_2 = \ell.$$

It only remains to show that L agrees with the bimodule structures. The equality

$$L\left(\ell_1 \otimes \ell_2 \cdot \gamma\right) = L\left(\ell_1 \otimes \ell_2\right) \cdot \gamma$$

is obvious.

Let us show that L agrees with the left actions. The path $\gamma \cdot \ell_1$ is by definition the path $f_1^{-1}(\gamma)_{\ell_1} \ell_1$. Then the path $L\left(\gamma \cdot \ell_1 \otimes \ell_2\right)$ is the path of $f_2^{-1}\left(f_1^{-1}(\gamma)_{\ell_1} \ell_1\right)_{\ell_2} \ell_2$. We have therefore

$$L\left(\gamma \cdot \ell_1 \otimes \ell_2\right) = \left(f_1 \circ f_2\right)^{-1}(\gamma)_{f_2^{-1}(\ell_1)_{\ell_2}} f_2^{-1}(\ell_1)_{\ell_2} \ell_2 = \gamma \cdot L\left(\ell_1 \otimes \ell_2\right),$$

i.e., L agrees also with the left action. $\qquad\square$

We get then by induction the following.

Corollary 2.8. *If* $f_i : \mathcal{M}_i \longrightarrow \mathcal{M}$ *for* $i = 1, \ldots, n$ *is a sequence of partial self-covering, then the tensor product bimodule* $\mathfrak{M}_{f_1,t} \otimes \cdots \otimes \mathfrak{M}_{f_n,t}$ *is isomorphic to the bimodule* $\mathfrak{M}_{f_1 \circ \cdots \circ f_n, t}$ *via the map* L *putting into correspondence to* $\ell_1 \otimes \cdots \otimes \ell_n$ *the path of the form*

$$\left(f_2 \circ f_3 \circ \cdots \circ f_n\right)^{-1}(\ell_1) \ldots \left(f_{n-1} \circ f_n\right)^{-1}(\ell_{n-2}) f_n^{-1}(\ell_{n-1}) \ell_n.$$

Corollary 2.9. *Let* $f : \mathcal{M}_1 \longrightarrow \mathcal{M}$ *be a partial self-covering. Then the Fock tree of the bimodule* $\mathfrak{M}_{f,t}$ *is isomorphic to the tree* T_f *of preimages of* t, *where the isomorphism maps an element* $\ell_1 \otimes \cdots \otimes \ell_n \in \mathfrak{M}_{f,t}^{\otimes n}$ *to the end of the path* $L(\ell_1 \otimes \cdots \otimes \ell_n)$.

Proof. It is easy to see that L is a bijection between the vertices of the trees (since two elements $\ell_1, \ell_2 \in \mathfrak{M}_{f,t}$ belong to one right orbit if and only if the ends of the paths ℓ_1 and ℓ_2 coincide).

It is hence sufficient to show that the map L agrees with the vertex adjacency in the Fock tree and in the tree of preimages. But

$$f(L(\ell_1 \otimes \cdots \otimes \ell_n)) = f(f^{-(n-1)}(\ell_1) f^{-(n-2)}(\ell_2) \ldots f^{-1}(\ell_{n-1}) \ell_n) =$$
$$f^{-(n-2)}(\ell_1) f^{-(n-3)}(\ell_2) \ldots \ell_{n-1} f(\ell_n) = L(\ell_1 \otimes \cdots \otimes \ell_{n-1}) f(\ell_n),$$

hence the endpoint of $L(\ell_1 \otimes \cdots \otimes \ell_{n-1})$ is the image under f of the endpoint of $L(\ell_1 \otimes \cdots \otimes \ell_n)$. $\qquad\square$

As a corollary of Proposition 2.7 and Corollary 2.9 we get the following proposition.

Proposition 2.10. *The self-similar action of* $\pi_1(\mathcal{M}, t)$ *associated with the* $\pi_1(\mathcal{M}, t)$*-bimodule* $\mathfrak{M}_{f,t}$ *(and any its basis) is conjugate with the iterated monodromy action of* $\pi_1(\mathcal{M}, t)$ *on the tree of preimages* T_f.

As an example, consider the self-covering $f : z \mapsto z^d$ of the unit complex circle. Take the basepoint $t = 1$. Then the elements of the bimodule $\mathfrak{M}_{f,t}$ are (homotopy classes of) paths from t to the roots of unity of degree d. They are naturally labeled by rotation angle (when we travel along the path). If we measure the angle in full turns, then the angle is any number from the set $\mathbb{Z}/d$. The fundamental group (again via rotation angles) is identified with $\mathbb{Z}$. The right action on the bimodule (concatenation of the path with a loop) is given by $(n/d) \cdot m = n/d + m$, while the left action (concatenation of the path with an f-preimage of the loop) is given by $m \cdot (n/d) = (m + n)/d$.

It is natural hence to identify the bimodule $\mathfrak{M}_{f,t}$ with the set of the linear functions of the form $M_n(x) = (x+n)/d$. If we identify the elements m of $\mathbb{Z}$ with their natural action $S_m(x) = x + m$ on $\mathbb{R}$, then the bimodule operations are just compositions of maps (acting from the right):

$$x \cdot M_n \cdot S_m = (x+n)/d + m,$$
$$x \cdot S_m \cdot M_n = ((x+m)+n)/d.$$

2.6. Another example: Mapping class groups and Thurston maps.
Recall that a *Thurston map* is a post-critically finite branched covering $f : S^2 \longrightarrow S^2$ of the sphere.

Let $f : S^2 \longrightarrow S^2$ be a Thurston map and let P_f be its post-critical set. Denote by G the *mapping class group* of (S^2, P_f), which is defined as the group of homotopy classes relative to P_f of homeomorphisms $g : S^2 \longrightarrow S^2$ acting trivially on P_f.

Denote by $\mathfrak{F}$ the set of homotopy classes of branched coverings of the form $g_1 \circ f \circ g_2$ for $g_1, g_2 \in G$. Every element of $\mathfrak{F}$ is a Thurston map with the post-critical set P_f and with the same dynamics (including the branching indices) on P_f as f.

It is obvious that $\mathfrak{F}$ is a G-bimodule with respect to pre- and post-compositions. This bimodule was used in [BN06c] to solve the "twisted rabbit" problem of J. H. Hubbard, which in a general form asks when to elements f_1, f_2 of the bimodule $\mathfrak{F}$ are conjugate, i.e., when there exists $h \in G$ such that $f_1 = h^{-1} \cdot f_2 \cdot h$.

2.7. Wreath recursion.
Notationally and computationally most convenient way of representing self-similar groups are the associated *wreath recursions*. However, since precise form of the wreath recursion depends on the choice of a basis of the self-similarity bimodule, it is not canonical and we will see later that one has to change the basis of the bimodule (and hence the wreath recursion) rather often.

Let us recall the definition of a (permutational) wreath product.

Definition 10. Let G be a group and let H be a group acting on a finite set X. Then the *permutational wreath product* $H \wr G$ is the semidirect product $H \ltimes G^{\mathsf{X}}$, where H acts on G^{X} by the original action on X.

Let us repeat this definition in a more explicit way. Let $\mathsf{X} = \{1, 2, \ldots, d\}$. Then the elements of $H \ltimes G^{\mathsf{X}}$ are written $h(g_1, g_2, \ldots, g_n)$, where $h \in H$ and $g_i \in G$. The multiplication is given by

$$h(g_1, g_2, \ldots, g_n) \cdot h'(g_1', g_2', \ldots, g_n') = hh'(g_{h'(1)}g_1', \ g_{h'(2)}g_2', \ \ldots, \ g_{h'(n)}g_n').$$

For example, if $\mathsf{X} = \{0, 1\}$, then the elements of $H \wr G$ are written either as (g_0, g_1) or as $\sigma(g_0, g_1)$, where $\sigma = (01)$ is the transposition. (We do not write trivial elements of H or of G^{X}.) Then multiplication in $H \wr G$ is performed using the following simple rules

$$(g_0, g_1) \cdot (g_0', g_1') = (g_0 g_0', g_1 g_1'), \quad (g_0, g_1)\sigma = \sigma(g_1, g_0).$$

For example, $\sigma(1, a)\sigma(1, a) = (a, a)$.

Let us come back to permutational bimodules. Let $\mathfrak{M}$ be a covering G-bimodule. Then $\mathfrak{M}$ as a right G-module $\mathfrak{M}_G$ is free and has a finite number d of orbits. Thus, the right G-space $\mathfrak{M}_G$ is isomorphic to a disjoint union of d copies of G. Namely,

if X is a basis of $\mathfrak{M}$, then every element of $\mathfrak{M}$ can be uniquely written as $x \cdot g$ for $x \in \mathsf{X}$ and $g \in G$, and the right action is given by

$$(x \cdot g) \cdot h = x \cdot (gh).$$

An *automorphism of the right G-space $\mathfrak{M}_G$* is a bijection $F : \mathfrak{M} \longrightarrow \mathfrak{M}$ such that $F(x \cdot g) = F(x) \cdot g$ for all $x \in \mathfrak{M}$ and $g \in G$. Let us identify the automorphism group of the right module $\mathfrak{M}_G$.

The following proposition is straightforward (see Proposition 2.2.1 of [Nek05]). Here $\mathfrak{S}(\mathsf{X})$ denotes the symmetric group of permutations of the set X.

Proposition 2.11. *Let $\mathfrak{M}_G$ be a free right G-space with d orbits. Then the group of automorphisms of $\mathfrak{M}_G$ is isomorphic to $\mathfrak{S}(\mathsf{X}) \wr G$. If we fix an orbit transversal $\mathsf{X} = \{x_1, x_2, \ldots, x_d\}$ then an isomorphism is given by*

$$\psi_{\mathsf{X}} : F \mapsto \pi(g_1, g_2, \ldots, g_d),$$

where $\pi \in \mathfrak{S}(\mathsf{X})$ and g_i are such that

$$F(x_i) = \pi(x_i) \cdot g_i.$$

We will usually order the elements of the alphabet $\mathsf{X} = \{x_1, x_2, \ldots, x_d\}$ and identify in this way the symmetric group $\mathfrak{S}(\mathsf{X})$ with the symmetric group $\mathfrak{S}(d)$ on the set $\{1, 2, \ldots, d\}$. Then the elements of the wreath product are written in the form $\sigma(g_1, g_2, \ldots, g_d)$ for $\sigma \in \mathfrak{S}(d)$ and $g_i \in G$.

Proposition 2.12. *Let $\psi_{\mathsf{X}} : \operatorname{Aut} \mathfrak{M}_G \longrightarrow \mathfrak{S}(d) \wr G$ be the isomorphism defined by a basis $\mathsf{X} = \{x_1, x_2, \ldots, x_d\}$ of $\mathfrak{M}_G$ and let $\mathsf{Y} = \{y_1, y_2, \ldots, y_d\}$ be another basis of $\mathfrak{M}_G$. Let $\pi \in \mathfrak{S}(d)$ and $g_x \in G$ for $x \in \mathsf{X}$ be such that*

$$y_i = x_{\pi(i)} \cdot g_i.$$

Then the isomorphism $\psi_{\mathsf{Y}} = \operatorname{Aut} \mathfrak{M}_G \longrightarrow \mathfrak{S}(d) \wr G$ is equal to the composition of ψ_{X} with conjugation by $\pi(g_1, g_2, \ldots, g_d)$.

Proof. We have $x_{\pi(i)} = y_i \cdot g_i^{-1}$, hence $x_i = y_{\pi^{-1}(i)} \cdot g_{\pi^{-1}(i)}^{-1}$.

Suppose that $\psi_{\mathsf{X}}(F) = \sigma(h_1, h_2, \ldots, h_d)$ for an automorphism F of $\mathfrak{M}_G$. Then for every i we have $F(x_i) = x_{\sigma(i)} \cdot h_i$. Consequently, $F(y_{\pi^{-1}(i)}) \cdot g_{\pi^{-1}(i)}^{-1} = y_{\pi^{-1}\sigma(i)} \cdot g_{\pi^{-1}\sigma(i)}^{-1} h_i$. If we denote $j = \pi^{-1}(i)$, then we get

$$F(y_j) \cdot g_j^{-1} = y_{\pi^{-1}\sigma\pi(j)} \cdot g_{\pi^{-1}\sigma\pi(j)}^{-1} h_{\pi(j)},$$

hence

$$F(y_j) = y_{\pi^{-1}\sigma\pi(j)} \cdot g_{\pi^{-1}\sigma\pi(j)}^{-1} h_{\pi(j)} g_j.$$

Therefore,

$$\psi_{\mathsf{Y}}(F) = \pi^{-1}\sigma\pi \left(g_{\pi^{-1}\sigma\pi(1)}^{-1} h_{\pi(1)} g_1, \ldots, g_{\pi^{-1}\sigma\pi(d)}^{-1} h_{\pi(d)} g_d \right) =$$
$$(g_1^{-1}, \ldots, g_d^{-1}) \pi^{-1} \cdot \sigma(h_1, \ldots, h_d) \cdot \pi(g_1, \ldots, g_d).$$

$\square$

Every element of $g \in G$ defines an automorphism of the right space $\mathfrak{M}_G$ by its left action:

$$g \cdot (x \cdot h) = (g \cdot x) \cdot h.$$

Hence, the left action of G on the bimodule $\mathfrak{M}$ defines a homomorphism from G to the automorphism group of the right module $\mathfrak{M}_G$. By Proposition 2.11, this is a homomorphism from G to $\mathfrak{S}(\mathsf{X}) \wr G$.

Definition 11. Let $\mathfrak{M}$ be a covering G-bimodule and let X be its basis. Then the *associated wreath recursion* is the map $\phi : G \longrightarrow \mathfrak{S}(\mathsf{X}) \wr G$ defining the left action of G on the right module $\mathfrak{M}_G$. It is computed by the rule

$$\phi(g) = \pi(g_1, g_2, \ldots, g_n),$$

where $\pi \in \mathfrak{S}(\mathsf{X})$ and $g_i \in g$ are such that $g \cdot x_i = \pi(x_i) \cdot g_i$.

We will rather routinely change bases of permutation bimodules and change the wreath recursion accordingly, using Proposition 2.12.

Example. Consider the binary adding machine action defined by the wreath recursion

$$\phi(a) = \sigma(1, a)$$

and post-compose the recursion with conjugation by $(1, a)$. We get then the recursion

$$\phi(a) = (1, a^{-1})\sigma(1, a)(1, a) = \sigma(a^{-1}, a^2).$$

This recursion corresponds to the binary numeration system with the set of digits $\{0, 3\}$:

$$1 + (0 + 2x_2 + 2^2 x_3 + \cdots) = 3 + 2(-1 + x_2 + 2x_3 + 2^2 x_4 + \cdots),$$
$$1 + (3 + 2x_2 + 2^2 x_3 + \cdots) = 0 + 2(2 + x_2 + 2x_3 + 2^2 x_4 + \cdots).$$

2.8. **Automata.** Another convenient language used in the theory of self-similar groups is the language of automata theory. We interpret a self-similar group (G, X) as an automaton over alphabet X with set of states G, which being in state $g \in G$ and reading a letter x gives the letter y and goes to the state h, if

$$g \cdot x = y \cdot h$$

in the associated self-similarity bimodule.

In some cases the automaton G may have finite sub-automata, i.e., subsets $A \subset G$ such that $g|_x \in A$ for all $g \in A$ and $x \in \mathsf{X}$. Here and later $g|_x$ is given by the condition $g \cdot x = g(x) \cdot g|_x$.

In this case A is described by its *Moore diagram*. It is a graph with set of vertices A in which we draw for every pair $g \in G$, $x \in \mathsf{X}$ an arrow from g to $g|_x$ labeled by the pair $(x, g(x))$. As an example, see the Moore diagram of the subset $\{a, 1, a^{-1}\}$ of the adding machine action on Figure 2.

2.9. **Category of self-similar groups.** We have seen that in many cases a more canonical object is the self-similarity bimodule, while the choice of a basis is arbitrary. Therefore, we define the category of self-similar groups as the category of covering bimodules.

Definition 12. Objects of the *category of self-similar groups* are pairs $(G, \mathfrak{M})$, where G is a group and $\mathfrak{M}$ is a covering G-bimodule. A morphism $(f, F) : (G_1, \mathfrak{M}_1) \longrightarrow (G_2, \mathfrak{M}_2)$ is a pair consisting of a homomorphism of groups $f : G_1 \longrightarrow G_2$ and a map $F : \mathfrak{M}_1 \longrightarrow \mathfrak{M}_2$ such that

$$F(g_1 \cdot x \cdot g_2) = f(g_1) \cdot F(x) \cdot f(g_2)$$

for all $g_1, g_2 \in G$ and $x \in \mathfrak{M}$. The morphisms are composed in the natural way.

Example. Let $f : \mathcal{M}_1 \longrightarrow \mathcal{M}$ and $f' : \mathcal{M}'_1 \longrightarrow \mathcal{M}'$ be partial self-coverings and let $H : \mathcal{M} \longrightarrow \mathcal{M}'$ be a continuous map such that the diagram

$$
\begin{array}{ccc}
\mathcal{M}_1 & \xrightarrow{H} & \mathcal{M}'_1 \\
\Big\downarrow f & & \Big\downarrow f' \\
\mathcal{M} & \xrightarrow{H} & \mathcal{M}_1
\end{array}
$$

is commutative (in particular $H(\mathcal{M}_1) \subseteq \mathcal{M}'_1$). We call such a map H a *semiconjugacy*.

Let $t \in \mathcal{M}$ and $t' \in \mathcal{M}'$ be basepoints such that $H(t) = t'$. Then the map H naturally defines a morphism of the associated bimodules $H_* : (\pi_1(\mathcal{M}, t), \mathfrak{M}_{f,t}) \longrightarrow (\pi_1(\mathcal{M}', t'), \mathfrak{M}_{f',t'})$, which maps the elements of the fundamental group and the elements of the bimodule to their images under the map H.

It is easy to see that the map $H \mapsto H_*$ is a functor from the category of partial self-coverings and semicojugacies (with fixed basepoints) to the category of self-similar groups.

Similarly, we can replace the fundamental groups $\pi_1(\mathcal{M}, t)$ and $\pi_1(\mathcal{M}', t')$ by the iterated monodromy groups of the maps f and f' respectively, and get functor into the category of faithful self-similar groups, i.e., self-similar groups acting faithfully on their Fock trees.

Lemma 2.13. *Suppose that $(f, F) : (G_1, \mathfrak{M}_1) \longrightarrow (G_2, \mathfrak{M}_2)$ is a morphism. Then the image of a right G_1-orbit of $\mathfrak{M}_1$ under F is a subset of a G_2-orbit of $\mathfrak{M}_2$. Consequently, every morphism induces a map on the sets of right orbits.*

Proof. If $x, y \in \mathfrak{M}$ belong to one right orbit, then there exists $g \in G$ such that $x \cdot g = y$, hence $F(y) = F(x) \cdot f(g)$, i.e., $F(x)$ and $F(y)$ belong to one right orbit. $\qquad\square$

The following is straightforward.

Lemma 2.14. *Let $(f, F) : (G_1, \mathfrak{M}_1) \longrightarrow (G_2, \mathfrak{M}_2)$ be a morphism. Set*

$$F^{\otimes n}(x_1 \otimes x_2 \otimes \cdots \otimes x_n) = F(x_1) \otimes F(x_2) \otimes \cdots \otimes F(x_n).$$

Then $F^{\otimes n}$ is well defined and $(f, F^{\otimes n}) : (G_1, \mathfrak{M}_1^{\otimes n}) \longrightarrow (G_2, \mathfrak{M}_2^{\otimes n})$ is a morphism.

The bimodule $\mathfrak{M}^{\otimes 0}$ is naturally identified with the group G itself, and then $F^{\otimes 0}$ is identified with f. Therefore, in what follows we denote f by $F^{\otimes 0}$ and use one letter F for a morphism of bimodules.

Here is another definition of a morphism of bimodules.

Definition 13. A *morphism* $F : (G_1, \mathfrak{M}_1) \longrightarrow (G_2, \mathfrak{M}_2)$ is a homomorphism $F : \mathfrak{M}_1^* \longrightarrow \mathfrak{M}_2^*$ of the tensor semigroups such that $F(G_1) \subseteq G_2$ and $F(\mathfrak{M}_1) \subseteq \mathfrak{M}_2$.

Let

$$T_{\mathfrak{M}} = \mathfrak{M}^*/G$$

be, as before, the Fock tree of right orbits. Then Lemma 2.14 implies, that every morphism $F : (G_1, \mathfrak{M}_1) \longrightarrow (G_2, \mathfrak{M}_2)$ induces a map of Fock trees $\widetilde{F} : T_{\mathfrak{M}_1} \longrightarrow T_{\mathfrak{M}_2}$ equal to map induced on the right orbits of $\mathfrak{M}^{\otimes n}$ by the morphism $F^{\otimes n}$. The map $\widetilde{F}$ is equivariant, i.e., for any vertex $v \in T_{\mathfrak{M}_1}$ and for any $g \in G$

$$\widetilde{F}(g(v)) = F^{\otimes 0}(g)\left(\widetilde{F}(v)\right).$$

Consequently, we get a functor from the category of covering bimodules into the category of actions on rooted trees (with equivariant maps as morphisms).

2.10. **Faithful self-similar groups and pull-backs of self-coverings.** We say that a permutation bimodule $(G, \mathfrak{M})$ is *faithful* if G acts faithfully on the Fock tree $\mathfrak{M}^*/G$, i.e., if the associated self-similar action is faithful.

Let $(G, \mathfrak{M})$ be an arbitrary covering bimodule and let $K \lhd G$ be the kernel of the action of G on the Fock tree. Let $f : G \longrightarrow G/K$ be the canonical epimorphism.

The next lemma is straightforward.

Lemma 2.15. *The set $\mathfrak{M}/K = \{x \cdot K \ : \ x \in \mathfrak{M}\}$ of the right K-orbits on $\mathfrak{M}$ is a G/K-bimodule in the natural way:*

$$gK \cdot (x \cdot K) = (g \cdot x) \cdot K, \qquad (x \cdot K) \cdot gK = (x \cdot g) \cdot K.$$

The map $(f, F) : (G, \mathfrak{M}) \longrightarrow (G/K, \mathfrak{M}/K)$ mapping $x \in \mathfrak{M}$ to its K-orbit is a morphism.

The bimodule $(G/K, \mathfrak{M}/K)$ is called the *faithful quotient* of $(G, \mathfrak{M})$.

Proposition 2.16. *Let $F : (G_1, \mathfrak{M}_1) \longrightarrow (G_2, \mathfrak{M}_2)$ be a morphism of covering bimodules and suppose that $(G_1, \mathfrak{M}_1)$ is faithful. Suppose that F induces an injective map of the sets of right orbits. Then the group morphism $F^{\otimes 0} : G_1 \longrightarrow G_2$ is injective.*

Proof. The morphism F induces an injective equivariant map $\widetilde{F} : T_{\mathfrak{M}_1} \longrightarrow T_{\mathfrak{M}_2}$ of the Fock trees. Suppose that $g \in G_1$ belongs to the kernel of $F^{\otimes 0}$. Then the action of $F^{\otimes 0}(g)$ on the subtree $\widetilde{F}(T_{\mathfrak{M}_1})$ of $T_{\mathfrak{M}_2}$ is trivial. Hence, by equivariance, the action of g on $T_{\mathfrak{M}_1}$ is trivial. But G_1 is faithful, hence $g = 1$ and $F^{\otimes 0}$ has trivial kernel. $\qquad \square$

Example. Let $f : \mathcal{M}_1 \longrightarrow \mathcal{M}$ be a partial d-fold self-covering and let $H : \mathcal{M}' \longrightarrow \mathcal{M}$ be a continuous map. Then there exists a d-fold covering $f' : \mathcal{M}'_1 \longrightarrow \mathcal{M}'$ (called the *pull-back* of f by H) and a map $H_1 : \mathcal{M}'_1 \longrightarrow \mathcal{M}_1$ such that the diagram

$$
\begin{array}{ccc}
\mathcal{M}'_1 & \xrightarrow{H_1} & \mathcal{M}_1 \\
\downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\
\mathcal{M}' & \xrightarrow{H} & \mathcal{M}
\end{array}
$$

is commutative. Suppose that there also exists an embedding $\mathcal{M}'_1 \hookrightarrow \mathcal{M}'$ making the diagram

$$
\begin{array}{ccc}
\mathcal{M}'_1 & \hookrightarrow & \mathcal{M}' \\
\downarrow{\scriptstyle H_1} & & \downarrow{\scriptstyle H} \\
\mathcal{M}_1 & \hookrightarrow & \mathcal{M}
\end{array}
$$

commutative, where $\mathcal{M}_1 \hookrightarrow \mathcal{M}$ is the identical embedding.

Then $f' : \mathcal{M}'_1 \longrightarrow \mathcal{M}'$ becomes a partial self-covering of $\mathcal{M}'$ by its subset $\mathcal{M}'_1$ (if we identify $\mathcal{M}'_1$ with its image under the embedding) and the map H becomes a morphism of the partial self-coverings.

Hence the morphism H induces a morphism

$$H_* : (\mathrm{IMG}\,(p'), \mathfrak{M}_{p'}) \longrightarrow (\mathrm{IMG}\,(p), \mathfrak{M}_p)$$

of the respective iterated monodromy groups. The corresponding morphism $H_*^{\otimes 0} :$ $\mathrm{IMG}(p') \longrightarrow \mathrm{IMG}(p)$ is an embedding by Proposition 2.16. Note that the morphism induced by H on the fundamental groups is not injective in general. (Consider for example the embedding H of the Julia set of $z^2 - 1$ into $\mathbb{C} \setminus \{0, -1\}$.)

3. Limit spaces

We have seen above that for every permutational G-bimodule $\mathfrak{M}$ we have a naturally defined rooted tree $T_\mathfrak{M}$ on which G acts. A choice of a basis X of $\mathfrak{M}$ defines a labeling of the tree $T_\mathfrak{M}$ by words in the alphabet X such that the obtained action of G on X^* is self-similar. Then the associated wreath recursion gives a convenient way to deal with the action.

Note that the action of G on the boundary of the tree $T_\mathfrak{M}$ can be defined as the action of G on the right module $\mathfrak{M}^{\otimes \omega} = \mathfrak{M} \otimes \mathfrak{M} \otimes \cdots$. This right module is defined as the set of sequences $x_1 \otimes x_2 \otimes \ldots$ of elements of $\mathfrak{M}$ where two sequences $x_1 \otimes x_2 \otimes \ldots$ and $y_1 \otimes y_2 \otimes \ldots$ are identified if for every k the elements $x_1 \otimes x_2 \otimes \cdots \otimes x_k$ and $y_1 \otimes y_2 \otimes \cdots \otimes y_k$ belong to one right orbit.

We go in this section in the other direction and consider the right G-space $\mathfrak{M}^{\otimes -\omega} = \ldots \otimes \mathfrak{M} \otimes \mathfrak{M}$ (defined in an appropriate way). It will have a structure of a metrizable locally compact finite-dimensional topological space. The group G acts on it by a proper co-compact action. We will also define a continuous self-map s on the quotient $\mathcal{J}_G = \mathfrak{M}^{\otimes -\omega}/G$.

Similarly to the case of the action of G on the tree $T_\mathfrak{M}$, a choice of a basis X of $\mathfrak{M}$ gives a symbolic presentation of the spaces $\mathfrak{M}^{\otimes -\omega}$ and $\mathcal{J}_G$ and the map $\mathsf{s} : \mathcal{J}_G \longrightarrow \mathcal{J}_G$. The space $\mathfrak{M}^{\otimes -\omega}$ is represented as a quotient of the space $\mathsf{X}^{-\omega} \times G$ by a closed equivalence relation with finite classes. The space $\mathcal{J}_G$ is also a quotient of the space $\mathsf{X}^{-\omega}$ by a closed relation with finite equivalence classes. The map s is defined in this symbolic representation by the shift.

3.1. **Hyperbolic bimodules.** Let $\mathfrak{M}$ be a covering G-bimodule and let X be its basis. If $g \cdot v = u \cdot h$ in $\mathfrak{M}^{\otimes n}$ for $g, h \in G$ and $v, u \in \mathsf{X}^n$, then we denote $g|_v = h$. Recall that $g \cdot v = u \cdot h$ implies that $g(vw) = uh(w)$ for all $w \in \mathsf{X}^*$, in the associated action of G on X^*. We call therefore $g|_v$ the *restriction of g at v*.

Definition 14. Let X be a basis of a covering G-bimodule $\mathfrak{M}$. We say that the self-similar action (G, X) is *contracting* if there is a finite set $\mathcal{N} \subset G$ such that for every $g \in G$ there exists $n \in \mathbb{N}$ such that $g|_v \in \mathcal{N}$ for all words $v \in \mathsf{X}^*$ of length greater than n. The smallest subset $\mathcal{N}$ of G satisfying this condition is called the *nucleus* of the action.

It is proved in [Nek05] that if the action (G, X) is contracting for some basis X of $\mathfrak{M}$, then the action (G, Y) is contracting for every basis Y. Therefore, being contracting depends only on the bimodule $\mathfrak{M}$ and does not depend on a particular choice of the basis.

Note however that the nucleus $\mathcal{N}$ depends on the choice of the basis.

Definition 15. A covering G-bimodule $\mathfrak{M}$ is said to be *hyperbolic* if the action (G, X) is contracting for some (and hence for every) basis X.

Example. Suppose that $f : \mathcal{M}_1 \longrightarrow \mathcal{M}$ is a partial self-covering and suppose that it is *expanding*, i.e., that there exists $0 < \lambda < 1$ such that every f-preimage of a path γ in $\mathcal{M}$ of length l has length at most $\lambda^{-1}l$. Suppose also that for every $R > 0$

the number of elements of $\pi_1(\mathcal{M}, t)$ represented by loops of length $\leq R$ is finite (this is the case, for instance, when $\mathcal{M}$ is a complete Riemannian manifold). We leave to the reader to prove that then the bimodule $\mathfrak{M}_{f,t}$ is hyperbolic.

It easily follows from the definition that the nucleus $\mathcal{N}$ is an automaton, i.e., that for every $g \in \mathcal{N}$ and every $x \in \mathsf{X}$ we have $g|_x \in \mathcal{N}$. Hence it makes sense to talk about the Moore diagram of the nucleus, which we will use in several cases. See the diagram of the nucleus of the adding machine action on Figure 2.

3.2. **Limit G-space $\mathcal{X}_G$.** Let $\mathcal{X}$ be a locally compact metrizable topological space. A right action of a group G on $\mathcal{X}$ by homeomorphisms is said to be *proper* if for any compact set $K \subset \mathcal{X}$ the set of elements $g \in G$ such that $K \cdot g \cap K \neq \emptyset$ is finite. The action is said to be *co-compact* if there is a compact subset $T \subset \mathcal{X}$ such that $\bigcup_{g \in G} T \cdot g = \mathcal{X}$.

Let $\mathfrak{M}$ be a permutational G-bimodule and let $\mathcal{X}$ be a right G-space. Then the tensor product $\mathcal{X} \otimes \mathfrak{M}$ is defined as the quotient of the set $\mathcal{X} \times \mathfrak{M}$ by the equivalence relation

$$\xi \otimes g \cdot x = \xi \cdot g \otimes x.$$

If $\mathcal{X}$ is a topological space, then $\mathcal{X} \otimes \mathfrak{M}$ is also a topological space with the quotient topology, where the topology on $\mathcal{X} \times \mathfrak{M}$ is the direct product topology of $\mathcal{X}$ with the discrete set $\mathfrak{M}$.

Note that $\mathcal{X} \otimes \mathfrak{M}$ is also a right G-space with respect to the action

$$(\xi \otimes x) \cdot g = \xi \otimes (x \cdot g).$$

Definition 16. A right G-space $\mathcal{X}$ is said to be $\mathfrak{M}$-self-similar if there exists a homeomorphism $\Phi : \mathcal{X} \otimes \mathfrak{M} \longrightarrow \mathcal{X}$ such that

$$\Phi(\xi \cdot g) = \Phi(\xi) \cdot g$$

for all $\xi \in \mathcal{X} \otimes \mathfrak{M}$ and $g \in G$.

If we have fixed a self-similarity $\Phi : \mathcal{X} \otimes \mathfrak{M} \longrightarrow \mathcal{X}$, then we will just right $\xi \otimes x$ instead of $\Phi(\xi \otimes x)$ for $\xi \in \mathcal{X}$ and $x \in \mathfrak{M}$.

We can iterate the self-similarity and define $\xi \otimes v$ for all $\xi \in \mathcal{X}$ and $v \in \mathfrak{M}^{\otimes n}$ inductively by the rule

$$\xi \otimes (v \otimes x) = (\xi \otimes v) \otimes x.$$

This gives a well defined $\mathfrak{M}^{\otimes n}$-self-similarity on $\mathcal{X}$.

Consequently, a self-similarity is an action of the tensor semigroup $\mathfrak{M}^*$ on $\mathcal{X}$ by surjective continuous maps. An arbitrary action will induce a continuous map from $\mathcal{X} \otimes \mathfrak{M}$ to $\mathcal{X}$. If this map is a homeomorphism, then this action is a self-similarity.

We need two more notions to define *contracting* self-similarity.

Definition 17. Let $\mathcal{X}$ be a right G-space. A relation $R \subset \mathcal{X} \times \mathcal{X}$ is called *bounded* if there exists a compact set $K \subset \mathcal{X} \times \mathcal{X}$ such that $R \subset \bigcup_{g \in G} K \cdot g$, where G acts on $\mathcal{X} \times \mathcal{X}$ by the diagonal action.

A neighborhood of the diagonal $U \subset \mathcal{X} \times \mathcal{X}$ is called *uniform* if it contains a G-invariant open neighborhood of the diagonal, where again G acts on $\mathcal{X} \times \mathcal{X}$ by the diagonal action.

The following theorem is proved in [Nek05] (Theorem 3.4.13).

Theorem 3.1. *Let $\mathfrak{M}$ be a hyperbolic G-bimodule. Then there exists a proper co-compact locally compact Hausdorff right G-space $\mathcal{X}_G$ and a contracting self-similarity $\Phi : \mathcal{X} \otimes \mathfrak{M} \longrightarrow \mathcal{X}$, i.e., such self-similarity that for any uniform neighborhood of diagonal U and for any bounded relation R there exists n such that $(\xi_1 \otimes v, \xi_2 \otimes v) \in U$ for all $(\xi_1, \xi_2) \in R$ and all $v \in \mathfrak{M}^{\otimes m}$ for $m \geq n$.*

Moreover, the G-space $\mathcal{X}$ and the self-similarity are unique in the sense that if $\mathcal{X}'$ is another such space with a contracting self-similarity, then there exists a homeomorphism $F : \mathcal{X} \longrightarrow \mathcal{X}'$ such that

$$F(\xi \cdot g) = F(\xi) \cdot g, \quad F(\xi \otimes x) = F(\xi) \otimes x$$

for all $\xi \in \mathcal{X}$, $g \in G$ and $x \in \mathfrak{M}$.

The unique self-similar G-space $\mathcal{X}$ from the theorem is called the *limit G-space* and is denoted $\mathcal{X}_G$.

3.3. Example: the adding machine.

Consider the adding machine action of $\mathbb{Z} \cong \langle a \rangle$ given by the recursion

$$\phi(a) = \sigma(1, a).$$

The self-similarity bimodule $\mathfrak{M} = \{x \cdot a^n \ : \ x = 0, 1, n \in \mathbb{Z}\}$ is defined by the relations

$$a \cdot 0 = 1, \qquad a \cdot 1 = 0 \cdot a.$$

Consider the space $\mathbb{R}$ with the natural (right) action of $\mathbb{Z}$ on it:

$$\xi \cdot a^n = \xi + n.$$

Then $\mathbb{R}$ is a proper co-compact $\langle a \rangle$-space.

Set

$$\xi \otimes 0 = \xi/2, \qquad \xi \otimes 1 = (\xi + 1)/2$$

and extend it to the whole self-similarity bimodule $\mathfrak{M}$ in the only possible way:

$$\xi \otimes 0 \cdot a^n = \xi/2 + n, \qquad \xi \otimes 1 \cdot a^n = (\xi + 1)/2 + n.$$

It is easy to see that this gives an $\mathfrak{M}$-self-similarity structure on $\mathbb{R}$:

$$(\xi \cdot a) \otimes 0 = (\xi + 1)/2 = \xi \otimes 1 \cdot a^0,$$
$$(\xi \cdot a) \otimes 1 = \xi/2 + 1 = \xi \otimes 0 \cdot a.$$

Let $x_1 x_2 \ldots x_k \cdot a^n$ for $x_i \in \{0, 1\}$ and $n \in \mathbb{Z}$ be an arbitrary element of $\mathfrak{M}^{\otimes k}$. Then

$$(2) \quad \xi \otimes x_1 \ldots x_k \cdot a^n = ((\xi + x_1)/2 + \cdots + x_k)/2 + n = \frac{\xi}{2^k} + \frac{x_1}{2^k} + \cdots + \frac{x_k}{2} + n,$$

hence

$$\xi_1 \otimes x_1 \ldots x_k \cdot a^n - \xi_2 \otimes x_1 \ldots x_k \cdot a^n = \frac{\xi_1 - \xi_2}{2^k},$$

and the self-similarity is contracting.

3.4. **Symbolic presentation of $\mathcal{X}_G$.** Theorem 3.1 is proved in [Nek05] by constructing the limit space $\mathcal{X}_G$ explicitly as the infinite tensor product $\mathfrak{M}^{\otimes -\omega} = \ldots \otimes \mathfrak{M} \otimes \mathfrak{M}$ in the following way.

Let us denote, for a set A, by $A^{-\omega}$ the set of sequences $(\ldots, a_2, a_1)$ of elements of A.

Definition 18. Let
$$\Omega = \bigcup_{A \subset \mathfrak{M},\ |A| < \infty} A^{-\omega} \subset \mathfrak{M}^{-\omega}$$
with the topology of the inductive limit of the direct product topologies on $A^{-\omega}$. We say that a sequence $(\ldots, x_2, x_1) \in \Omega$ is *asymptotically equivalent* to $(\ldots, y_2, y_1) \in \Omega$ if there exists a finite set $N \subset G$ and a sequence $g_k \in N$ such that
$$g_k \cdot x_k \otimes \cdots \otimes x_2 \otimes x_1 = y_k \otimes \cdots \otimes y_2 \otimes y_1$$
in $\mathfrak{M}^{\otimes k}$ for all $k \geq 1$.

It is proved in [Nek05] that the limit space $\mathcal{X}_G$ is homeomorphic to the quotient of Ω by the asymptotic equivalence relation. The point of $\mathcal{X}_G$ represented by a sequence $(\ldots, x_2, x_1) \in \Omega$ is denoted $\ldots \otimes x_2 \otimes x_1$ or $\ldots x_2 x_1$. It is uniquely defined as the limit

(3) $$\ldots \otimes x_2 \otimes x_1 = \lim_{n \to \infty} \xi \otimes x_n \otimes \cdots \otimes x_1,$$

where $\xi \in \mathcal{X}_G$ is an arbitrary point.

The action of G and the self-similarity on $\mathcal{X}_G$ are given then by natural rules

(4) $(\ldots \otimes x_2 \otimes x_1) \cdot g = \ldots \otimes x_2 \otimes x_1 \cdot g, \quad (\ldots \otimes x_2 \otimes x_1) \otimes x = \ldots \otimes x_2 \otimes x_1 \otimes x.$

It is also proved that the set $\mathsf{X}^{-\omega} \cdot G \subset \Omega$ of sequences of the form $\ldots \otimes x_2 \otimes x_1 \cdot g$, for $x_i \in \mathsf{X}$ and $g \in G$, intersects every asymptotic equivalence class and hence the space $\mathcal{X}_G$ is homeomorphic to the quotient of $\mathsf{X}^{-\omega} \cdot G$ by the asymptotic equivalence relation. Note that the set $\mathsf{X}^{-\omega} \cdot G$ is invariant under the action of the semigroup $\mathfrak{M}^*$ given in (4).

The asymptotic equivalence relation on $\mathsf{X}^{-\omega} \cdot G$ is described in very simple terms by the following proposition, also proved in [Nek05] (Proposition 3.2.6).

Proposition 3.2. *Let X be a basis of a hyperbolic bimodule $\mathfrak{M}$ and let $\mathcal{N}$ be the nucleus of the self-similar group (G, X).*

Two sequences $\ldots x_2 x_1 \cdot g$ and $\ldots y_2 y_1 \cdot h \in \mathsf{X}^{-\omega} \cdot G$ are asymptotically equivalent if and only if there exists a directed path $\ldots e_2 e_1$ in the Moore diagram of $\mathcal{N}$ such that the arrow e_i is labeled by (x_i, y_i) and the arrow e_1 ends in hg^{-1}.

Example. Consider the case of the adding machine action. We have seen in 3.3 that the limit G-space of the adding machine action is $\mathbb{R}$ with the natural action of $\mathbb{Z} \cong \langle a \rangle$. It follows from (2) and (3) that a sequence $\ldots x_2 x_1 \cdot a^n$ represents the point
$$n + \sum_{k=1}^{\infty} \frac{x_k}{2^k},$$
i.e., the real number which is represented by the binary fraction $n.x_1 x_2 \ldots$.

The form of nucleus of the adding machine action (shown on Figure 2) implies (by Proposition 3.2) that two sequences $\ldots x_2 x_1 \cdot a^n$ and $\ldots y_2 y_1 \cdot a^m$ represent the same point of the limit space if and only if they are either equal, or of the
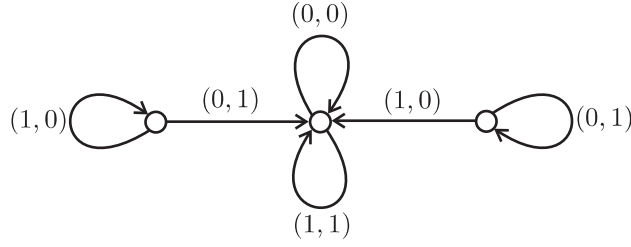
FIGURE 2. Nucleus of the adding machine action

form $\ldots 0001 x_k \ldots x_1.n$ and $\ldots 1110 x_k \ldots x_1.n$, or of the form $\ldots 000.(n+1)$ and $\ldots 111.n$. But this is the usual identification rule of dyadic reals.

3.5. **The limit dynamical system.** The quotient $\mathcal{X}_G/G$ of the limit G-space $\mathcal{X}_G$ by the action of G is called the *limit space* of G and is denoted $\mathcal{J}_G$. The action of G on $\mathcal{X}_G$ is not free (though proper), hence it is natural to consider $\mathcal{J}_G$ as an orbispace.

If we fix a basis X of the self-similarity bimodule $\mathfrak{M} = \mathsf{X} \cdot G$, then we get a symbolic presentation of $\mathcal{J}_G$ defined in the following way (see Proposition 3.2).

Proposition 3.3. *Let us say that sequences $\ldots x_2 x_1, \ldots y_2 y_1 \in \mathsf{X}^{-\omega}$ are asymptotically equivalent if there exists a finite set $N \subset G$ and a sequence $g_k \in N$ such that*

$$g_k(x_k \ldots x_1) = y_k \ldots y_1$$

for all $k \geq 1$.

Then the limit space $\mathcal{J}_G$ is homeomorphic to the quotient of the space $\mathsf{X}^{-\omega}$ by the asymptotic equivalence relation.

Sequences $\ldots x_2 x_1, \ldots y_2 y_1 \in \mathsf{X}^{-\omega}$ are asymptotically equivalent if and only if there exists a sequence h_k of the elements of the nucleus of G such that $h_k \cdot x_k = y_k \cdot h_{k-1}$ for all $k \geq 1$.

The last paragraph of Proposition 3.3 can be formulated in the following way.

Corollary 3.4. *Sequences $\ldots x_2 x_1, \ldots y_2 y_1 \in \mathsf{X}^{-\omega}$ are asymptotically equivalent if and only if there exists a directed path $\ldots e_2 e_1$ in the Moore diagram of the nucleus $\mathcal{N}$ of G such that the arrow e_k is labeled by (x_k, y_k) for every k.*

It is easy to see that the asymptotic equivalence relation on $\mathsf{X}^{-\omega}$ is invariant under the shift

$$\ldots x_2 x_1 \mapsto \ldots x_3 x_2,$$

hence the shift induces a continuous map $\mathsf{s} : \mathcal{J}_G \longrightarrow \mathcal{J}_G$. We call the pair $(\mathcal{J}_G, \mathsf{s})$ the *limit dynamical system* of the group G.

The limit dynamical system $(\mathcal{J}_G, \mathsf{s})$ does not depend on the choice of the basis X and can be defined as the projection under the natural map $\mathcal{X}_G \longrightarrow \mathcal{J}_G$ of the correspondence

$$\xi \otimes x \mapsto \xi$$

on $\mathcal{X}_G$, where $x \in \mathfrak{M}$ and $\xi \in \mathcal{X}_G$.

Let us show that the projection is a well defined map. Suppose that $\xi_1 \otimes x_1$ and $\xi_2 \otimes x_2$ belong to one G-orbit, i.e., that $\xi_1 \otimes x_1 = \xi_2 \otimes x_2 \cdot g$ for some $g \in G$. This

means (by the definition of a tensor product) that there exists $h \in G$ such that $\xi_1 = \xi_2 \cdot h$ and $h \cdot x_1 = x_2 \cdot g$. Hence, ξ_1 and ξ_2 belong to one G-orbit and the correspondence $\xi \otimes x \mapsto \xi$ is projected onto a well defined self-map of $\mathcal{J}_G$.

Example. It follows from the description of the limit space $\mathbb{R}$ of the adding machine action and its self-similarity that the limit dynamical system of the adding machine is the circle $\mathbb{R}/\mathbb{Z}$ together with the double self-covering $\mathsf{s} : x \mapsto 2x \pmod 1$.

3.6. **Functoriality.** Let us show that both constructions of the limit G-space $\mathcal{X}_G$ and of the limit dynamical system $(\mathcal{J}_G, \mathsf{s})$ are functors in the respective categories (of right G-spaces and dynamical systems).

   We say that a map $f : \mathcal{X}_1 \longrightarrow \mathcal{X}_2$ from a G_1-space to a G_2-space is *equivariant* with respect to a homomorphism $\phi : G_1 \longrightarrow G_2$ if $f(\xi \cdot g) = f(\xi) \cdot \phi(g)$ for all $g \in G_1$ and $\xi \in \mathcal{X}_1$.

Proposition 3.5. *Let $F : (G_1, \mathfrak{M}_1) \longrightarrow (G_2, \mathfrak{M}_2)$ be a morphism of hyperbolic bimodules. Then the map*

$$\ldots \otimes x_2 \otimes x_1 \longrightarrow \ldots \otimes F(x_2) \otimes F(x_1)$$

induces a well defined continuous map $\mathcal{X}_F : \mathcal{X}_{G_1} \longrightarrow \mathcal{X}_{G_2}$ equivariant with respect to the homomorphism $F^{\otimes 0} : G_1 \longrightarrow G_2$.

   The map $\mathcal{X}_F$ induces a well defined continuous map $\mathcal{J}_F : \mathcal{J}_{G_1} \longrightarrow \mathcal{J}_{G_2}$ which agrees with the limit dynamical systems $(\mathcal{J}_{G_1}, \mathsf{s})$ and $(\mathcal{J}_{G_2}, \mathsf{s})$, i.e., is such that the diagram

$$
\begin{array}{ccc}
\mathcal{J}_{G_1} & \xrightarrow{\ \mathsf{s}\ } & \mathcal{J}_{G_1} \\
\downarrow{\scriptstyle \mathcal{J}_F} & & \downarrow{\scriptstyle \mathcal{J}_F} \\
\mathcal{J}_{G_2} & \xrightarrow{\ \mathsf{s}\ } & \mathcal{J}_{G_2}
\end{array}
$$

is commutative.

   The maps $F \mapsto \mathcal{X}_F$ and $F \mapsto \mathcal{J}_F$ are functors from the category of contracting self-similar groups to the categories of right G-spaces (and equivariant maps) and dynamical systems (and semiconjugacies), respectively.

   The following is also straightforward.

Proposition 3.6. *If the morphism $F : (G_1, \mathfrak{M}_1) \longrightarrow (G_2, \mathfrak{M}_2)$ induces a surjective map $\mathfrak{M}_1/G_1 \longrightarrow \mathfrak{M}_2/G_2$ of the sets of right orbits, then the map $\mathcal{J}_F : \mathcal{J}_{G_1} \longrightarrow \mathcal{J}_{G_2}$ is surjective.*

Proposition 3.7. *Let $F : (G, \mathfrak{M}) \longrightarrow (G/K, \mathfrak{M}/K)$ be the faithful quotient map. Then $\mathcal{J}_F : \mathcal{J}_G \longrightarrow \mathcal{J}_{G/K}$ is a topological conjugacy.*

   The last statement means that if we restrict ourselves only to faithful contracting groups, then we do not make the class of their limit dynamical systems smaller.

3.7. **Self-similar subgroups.** Particularly interesting and easy to define are the morphisms coming from *self-similar inclusions*

Definition 19. Let (G, X) be a self-similar group. Let $\mathsf{Y} \subseteq \mathsf{X}$ be an arbitrary subset. A subgroup $H \leq G$ is said to be *self-similar on the alphabet* Y if Y is H-invariant and for all $h \in H$ and $y \in \mathsf{Y}$ we have $h|_y \in H$. If H is self-similar on X, then we just call it *self-similar*.

If H is self-similar on Y, then the subset $\mathsf{Y} \cdot H \subset \mathsf{X} \cdot G$ is the self-similarity H-bimodule and the pair of inclusions $H \hookrightarrow G$ and $\mathsf{Y} \cdot H \hookrightarrow \mathsf{X} \cdot G$ is a morphism from $(H, \mathsf{Y} \cdot H)$ to $(G, \mathsf{X} \cdot G)$.

It follows from Proposition 2.16 that this is the form of all morphisms of faithful self-similar groups which induce injective maps of the right orbits (of the alphabets).

The next proposition follows directly from the definition of the asymptotic equivalence relation on $\mathsf{X}^{-\omega}$.

Proposition 3.8. *Let (G, X) be a self-similar group generated by a finite set S and suppose that $H \le G$ is a self-similar subgroup generated by a finite set S_1 with the property that for every $g \in S$ and $v \in \mathsf{X}^*$ there exists $h \in S_1$ such that $g(v) = h(v)$. Then the inclusion $H \hookrightarrow G$ induces a topological conjugacy of the limit dynamical systems $(\mathcal{J}_H, \mathsf{s})$ and $(\mathcal{J}_G, \mathsf{s})$.*

Note, however, that the limit dynamical systems of H and G will be different, if seen as partial self-coverings of orbispaces.

3.8. **Finite extensions.** Let (G, X) be a contracting self-similar group and let H be a self-similar subgroup (i.e., $h|_x \in H$ for every $h \in H$ and every $x \in \mathsf{X}$). Then the embedding $\mathsf{X} \cdot H \hookrightarrow \mathsf{X} \cdot G$ is a morphism of bimodules and we get an H-equivariant continuous map $F : \mathcal{X}_H \longrightarrow \mathcal{X}_G$, as before.

Note that $\mathcal{X}_G$ is also a proper H-space, since H is a subgroup of G. If H has finite index in G, then $\mathcal{X}_G$ is also co-compact with respect to the action of H. The only possibly missing condition of Theorem 3.1 for $\mathcal{X}_G$ to be the limit space of H (and hence for F to be a homeomorphism) is H-self-similarity of $\mathcal{X}_G$.

Just restricting the G-self-similarity of $\mathcal{X}_G$ onto $\mathsf{X} \cdot H$ we get a collection of maps $\xi \mapsto \xi \otimes x$ for $x \in \mathsf{X} \cdot H$ which satisfy the necessary compatibility conditions with the action of H. However, since H is only a subgroup of G, it may happen that the tensor product $\mathcal{X}_G \otimes_H (\mathsf{X} \cdot H)$ is different from $\mathcal{X}_G$.

Sufficient conditions for $\mathcal{X}_G$ to be H-self-similar are given in the following theorem.

Theorem 3.9. *Suppose that a self-similar subgroup H of a contracting group (G, X) has finite index in G, the self-similar groups (G, X) and (H, X) are self-replicating (see Definition 5) and $g \cdot x = x$ for $g \in G$ and $x \in \mathsf{X}$ implies that $g \in H$. Then the map $\mathcal{X}_H \longrightarrow \mathcal{X}_G$ induced by the embedding $H < G$ is a homeomorphism.*

The proof of this theorem is the same as the proof of Theorem 3.7.1 (2) of [Nek05].

The last condition of Theorem 3.9 is satisfied, for instance, if the left action of G on $\mathsf{X} \cdot G$ is free (i.e., if the *associated virtual endomorphism* is injective, see below).

Example. Another example when the conditions of Theorem 3.9 are satisfied is the case of a pull-back $p' : \mathcal{M}'_1 \longrightarrow \mathcal{M}'$ of a partial self-covering $p : \mathcal{M}_1 \longrightarrow \mathcal{M}$ by a finite covering $F : \mathcal{M}' \longrightarrow \mathcal{M}$ of $\mathcal{M}$ by connected space $\mathcal{M}'$. Then F_* maps $\pi_1(\mathcal{M}', t)$ into a subgroup of finite index of $\pi_1(\mathcal{M}, F(t))$. If we choose a basis $\mathsf{X} = \{\ell_i\}$ of $\mathfrak{M}_{p', t}$, then $\{F(\ell_i)\}$ is a basis of $\mathfrak{M}_{p, F(t)}$, which can be identified with X. We get then a finite-index self-similar subgroup $F_*(\pi_1(\mathcal{M}', t))$ of $\pi_1(\mathcal{M}, F(t))$.

Suppose that $\gamma \cdot \ell = \ell$ in $\mathfrak{M}_{p, F(t)}$ for $\gamma \in \pi_1(\mathcal{M}, F(t))$. If γ does not belong to $F_*(\pi_1(\mathcal{M}', t))$, then the preimage $F^{-1}(\gamma)_t$ of γ under F, starting at t does not end in t, hence the end of any p'-preimage of $F^{-1}(\gamma)_t$ does not belong to $p'^{-1}(t)$. But this contradicts to the fact that ℓ and $\gamma \cdot \ell$ are homotopic.
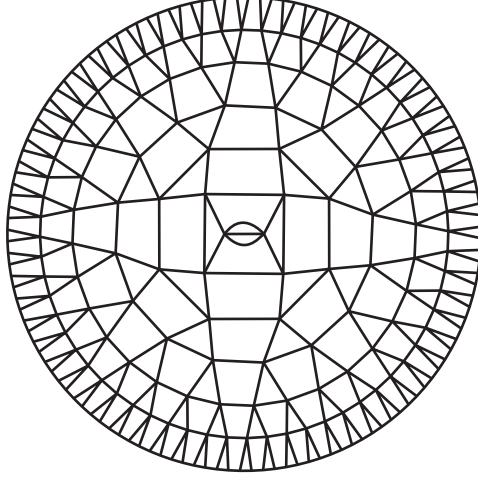
FIGURE 3. Self-similarity graph of the adding machine

3.9. **Limit spaces of the iterated monodromy groups.** It is proved in [Nek05] that if a partial self-covering $f : \mathcal{M}_1 \longrightarrow \mathcal{M}$ is expanding on a neighborhood of its Julia set $\mathcal{J}_f$, then the iterated monodromy group is contracting and the limit dynamical system $(\mathcal{J}_{\mathrm{IMG}(f)}, \mathsf{s})$ is topologically conjugate to the system $(\mathcal{J}_f, f)$.

In particular, if f is a post-critically finite rational function, then we can look at f as at a partial self-covering $f : \widehat{\mathbb{C}} \setminus f^{-1}(P) \longrightarrow \widehat{\mathbb{C}} \setminus P$, where P is the post-critical set of f. Then the mentioned general theorem (in the orbispace setting) implies that $\mathrm{IMG}(f)$ is contracting and that its limit dynamical system is topologically conjugate to the action of f on its Julia set.

3.10. **The limit space $\mathcal{J}_G$ as a hyperbolic boundary.** Let X be a basis of a hyperbolic G-bimodule $\mathfrak{M}$ and suppose that G is generated by a finite set S. Then the tensor semigroup $\mathfrak{M}^*$ is generated by the set $S \cup \mathsf{X}$. Let $\Gamma_{S,\mathsf{X}}$ be the *left* Cayley graph of $\mathfrak{M}^*$, i.e., the graph with the set of vertices $\mathfrak{M}^*$ where a vertex v is connected to the vertices of the form $g \cdot v$ for $g \in S$ (*horizontal edges*) and the vertices $x \cdot v = x \otimes v$ for $x \in \mathsf{X}$ (*vertical edges*).

The group G acts by automorphisms on the left Cayley graph $\Gamma_{S,\mathsf{X}}$ by the right multiplication. The *self-similarity graph* is the quotient $\Gamma_{S,\mathsf{X}}/G$ of the Cayley graph of $\mathfrak{M}^*$ by this action.

It is proved in [Nek05] that the self-similarity graph is Gromov-hyperbolic and that its hyperbolic boundary is homeomorphic to the limit space $\mathcal{J}_G$. See also the paper [Pil05], where another proof of this result is given.

See, for instance, a part of the self-similarity graph for the binary adding machine action on Figure 3.

4. ABELIAN GROUPS

4.1. **Virtual endomorphisms.** Let $(G, \mathfrak{M})$ be a d-fold covering bimodule and suppose that the left action of G on the set of the right orbits $\mathfrak{M}/G$ is transitive (it is equivalent to the condition that for every $x, y \in \mathfrak{M}$ there exist $g, h \in G$ such that $x = g \cdot y \cdot h$). Then the bimodule $\mathfrak{M}$ can be reconstructed from the *associated*

virtual endomorphism, which is defined in the following way. Fix $x \in \mathfrak{M}$ and let G_x be the subgroup of elements $g \in G$ such that $g \cdot x = x \cdot h$ for some $h \in G$. The subgroup G_x has index d in G and the map $\phi : g \mapsto h$ is a homomorphism $\phi : G_x \longrightarrow G$ called the *virtual endomorphism associated to the bimodule* (and the element x).

The bimodule $\mathfrak{M}$ is isomorphic to the bimodule $\phi(G)G$ of formal expressions of the form $\phi(g_1)g_2$, where $\phi(g_1)g_2$ is considered to be equal to $\phi(h_1)h_2$ if $h_1^{-1}g_1$ belongs to the domain G_x of ϕ and

$$\phi(h_1^{-1}g_1) = h_2 g_2^{-1}.$$

The left and right actions of G on the set $\phi(G)G$ are given by

$$h \cdot \phi(g_1)g_2 = \phi(hg_1)g_2, \qquad \phi(g_1)g_2 \cdot h = \phi(g_1)g_2 h.$$

The isomorphism between the bimodules $\mathfrak{M}$ and $\phi(G)G$ is given by the map

$$g_1 \cdot x \cdot g_2 \mapsto \phi(g_1)g_2.$$

A convenient way to interpret the bimodule $\phi(G)G$ is to consider it as a set of partially defined transformations of the group G, where $\phi(g_1)g_2$ is identified with the "affine" transformation

$$g \mapsto \phi(gg_1)g_2.$$

Then the left and the right actions of G on $\phi(G)G$ coincide with the pre- and post-compositions, respectively, with the right action of G on itself.

Let us apply this theory to the case of free abelian groups $\mathbb{Z}^n$. A virtual endomorphism ϕ of $\mathbb{Z}^n$ is a homomorphism from a subgroup of finite index of $\mathbb{Z}^n$ into $\mathbb{Z}^n$, hence it can be extended to a linear operator A on $\mathbb{Q}^n$. Then the associated bimodule is isomorphic to the set of the affine transformations of the form

$$\vec{x} \mapsto A(\vec{x} + g_1) + g_2,$$

for $g_1, g_2 \in \mathbb{Z}^n$, where $\mathbb{Z}^n$ acts on them by pre- and post-composition with translations by integral vectors.

In the case when the action is self-replicating (i.e., if the virtual endomorphism is onto), every element of the bimodule can be written in the form

$$\vec{x} \mapsto A(\vec{x} + g)$$

for some $g \in \mathbb{Z}^n$.

It is proved in [NS04] (see also [Nek05] Proposition 2.9.2) that the self-similar action of $\mathbb{Z}^n$ associated with a virtual endomorphism A is faithful if and only if no eigenvalue of A is an algebraic integer. In particular, A is invertible if the action is faithful.

4.2. *A*-**adic numeration systems.** Let us fix a faithful self-replicating (see Definition 5) action of $\mathbb{Z}^n$. Let A be the matrix of the associated virtual endomorphism. Then A^{-1} is a matrix with integral entries. The domain of the virtual endomorphism is the subgroup $A^{-1}(\mathbb{Z}^n)$, which has index $\det A^{-1}$ in $\mathbb{Z}^n$.

Note that affine maps $A(\vec{x} + g_1)$ and $A(\vec{x} + g_2)$ belong to one right orbit if there exists $g \in \mathbb{Z}^n$ such that $A(\vec{x} + g_1 + A^{-1}(g)) = A(\vec{x} + g_2)$, i.e., when g_1 and g_2 belong to one coset modulo $A^{-1}(\mathbb{Z}^n)$. Consequently, a set $\{A(\vec{x} + g_i)\}_{i=1,\ldots,d}$ is a basis of

the self-similarity bimodule if and only if the set $\{g_i\}_{i=1,\ldots,d}$ is a coset transversal of $\mathbb{Z}^n$ modulo $A^{-1}(\mathbb{Z}^n)$, i.e., if and only if

$$\mathbb{Z}^n = \bigsqcup_{i=1,\ldots,d} A^{-1}(\mathbb{Z}^n) + g_i.$$

We can interpret the set $\{g_i\}_{i=1,\ldots,d}$ as a set of *"digits"* of an *"A-adic"* numeration system on $\mathbb{Z}^n$. Namely, every element of $\mathbb{Z}^n$ can be uniquely written as a formal sum

(5) $$g = g_{i_0} + A^{-1}(g_{i_1}) + A^{-2}(g_{i_2}) + \cdots,$$

where g_{i_0} is uniquely defined by the condition $g - g_{i_0} \in A^{-1}(\mathbb{Z}^n)$, the digit g_{i_1} is determined by the condition $A(g - g_{i_0}) - g_{i_1} \in A^{-1}(\mathbb{Z}^n)$, etc.

The set of all such formal series $\sum_{k=0}^{\infty} A^{-k}(g_{i_k})$ is a profinite abelian group defined as the completion of $\mathbb{Z}^n$ with respect to the chain of finite index subgroups $A^{-k}(\mathbb{Z}^n)$. The group $\mathbb{Z}^n$ acts on it in the natural way and this action is conjugate to the associated action of $\mathbb{Z}^n$ on the boundary of the tree in the natural way (through the homeomorphism mapping the sum (5) to the sequence $x_1 x_2 \ldots$, where x_k is the element $A(\vec{x} + g_{i_k})$ of the self-similarity bimodule). See more detail in [NS04] and [Nek05].

4.3. Example: a numeration system on $\mathbb{Z}[i]$.

Consider $\mathbb{Z}^2$, identified with the additive group $\mathbb{Z}[i]$ of Gaussian integers. Take the linear map $A : z \mapsto z/(1-i) = z(1+i)/2$. It is given by the matrix $\begin{pmatrix} 1/2 & -1/2 \\ 1/2 & 1/2 \end{pmatrix}$. Since the determinant of this matrix (i.e., square of modulus of $(1+i)/2$) is equal to $1/2$, the domain of A has index 2 in $\mathbb{Z}^2$ (it is the subgroup of vectors with even sum of coordinates).

Take the coset transversal $\{0, 1\}$ of $A^{-1}(\mathbb{Z}[i])$ in $\mathbb{Z}[i]$. Let us also denote the corresponding elements $A(z)$ and $A(z+1)$ of the bimodule by *0* and *1*, respectively. Let us denote by u and v the generators 1 and i of $\mathbb{Z}[i]$, seen as affine maps $z \cdot u = z+1$ and $z \cdot v = z + i$. Then we have the following relations in the bimodule of affine transformations $A(z + a)$:

$$z \cdot (u \cdot 0) = A(z+1) = z \cdot 1,$$
$$z \cdot (u \cdot 1) = A(z+2) = A(z) + (1+i) = z \cdot (0 \cdot uv)$$

and

$$z \cdot (v \cdot 0) = A(z+i) = A(z+1) - 1 = z \cdot (1 \cdot u^{-1}),$$
$$z \cdot (v \cdot 1) = A(z+1+i) = A(z) + i = z \cdot (0 \cdot v),$$

hence the self-similarity bimodule is given by the relations

$$u \cdot 0 = 1, \qquad u \cdot 1 = 0 \cdot uv$$
$$v \cdot 0 = 1 \cdot u^{-1}, \quad v \cdot 1 = 0 \cdot v,$$

and the associated action of $\mathbb{Z}[i]$ is the self-similar group generated by

$$u = \sigma(1, uv), \quad v = \sigma(u^{-1}, v).$$

Here and in the sequel we identify the group with its image under the wreath recursion, so that we write $u = \sigma(1, uv)$ instead of $\psi(u) = \sigma(1, uv)$ for the wreath recursion ψ.

Note that if you are given the above recursion, then it is easy to reconstruct the virtual endomorphism A. First, check that u and v commute. Then note that the stabilizer of the first level of the tree is generated by uv and uv^{-1} (since the group is abelian), which are given by

$$uv = (v, v), \quad uv^{-1} = (u, u).$$

We see that restriction of the action of the stabilizer on the subtrees of the first level is a homomorphism acting by (in the additive notation)

$$u + v \mapsto v, \quad u - v \mapsto u.$$

Thus, it is the linear map

$$u \mapsto (u + v)/2, \quad v \mapsto (-u + v)/2.$$

Since we have chosen the coset representatives $\{0, 1\}$, this self-similar action of $\mathbb{Z}[i]$ describes addition in the "binary" base $(1 - i)$ numeration system on $\mathbb{Z}[i]$. This system represents the Gaussian integers as sums $a_0 + a_1(1 - i) + a_2(1 - i)^2 + \cdots$, where $a_i \in \{0, 1\}$.

4.4. **Limit spaces of abelian groups.** It is proved in Proposition 2.11.11 of [Nek05] that a finitely generated self-similar group G acting level-transitively on X^* is contracting if and only if the *spectral radius* of the associated virtual endomorphism ϕ is less than one. Here the spectral radius is defined as

$$r(\phi) = \limsup_{n \to \infty} \sqrt[n]{\limsup_{g \in \text{Dom } \phi^n, l(g) \to \infty} \frac{l(\phi^n(g))}{l(g)}},$$

where $l(g)$ denotes the length of a group element with respect to some fixed finite generating set of G.

This implies that a self-similar action of $\mathbb{Z}^n$ is contracting if and only if the associated virtual endomorphism is a contracting linear map (i.e., its usual spectral radius is less than one).

For instance, the action associated with the binary base $(1 - i)$ numeration system on $\mathbb{Z}[i]$ (see the previous subsection) is contracting.

The following theorem is then straightforward.

Theorem 4.1. *Let $(\mathbb{Z}^n, \mathsf{X})$ be a contracting self-replicating action and let A be the associated virtual endomorphism. Then the limit $\mathbb{Z}^n$-space $\mathcal{X}_{\mathbb{Z}^n}$ is $\mathbb{R}^n$ with the natural action of $\mathbb{Z}^n$ by translations. The self-similarity structure is given by*

$$\xi \otimes (A(z + a)) = A(\xi + a),$$

where $A(z + a)$ is an element of the self-similarity bimodule identified with an affine map, as before.

Let $g_1, \ldots, g_d$ be the set of digits defining the basis $\mathsf{X} = \{x_i = A(\vec{x} + g_i)\}_{i=1,\ldots,d}$ of the self-similarity bimodule, as in 4.2. Then a sequence $\ldots x_{i_2} x_{i_1} \cdot g$ corresponds, by (3) and Theorem 4.1, to the vector

$$g + A(g_{i_1}) + A^2(g_{i_2}) + A^3(g_{i_3}) + \cdots,$$

which is convergent in $\mathbb{R}^n$, since A is contracting.

In this way the A-adic numeration is extended to $\mathbb{R}^n$, i.e., to the limit space $\mathcal{X}_{\mathbb{Z}^n}$, if the self-similar action is contracting (and self-replicating).

4.5. Tiles. Let (G, X) be any contracting group and let $\mathcal{X}_G$ be its limit G-space. We know that $\mathcal{X}_G$ is a quotient of the space $\mathsf{X}^{-\omega} \cdot G$ by the asymptotic equivalence relation.

The image of the set $\mathsf{X}^{-\omega} \cdot 1$ in $\mathcal{X}_G$ is called the *digit tile*, or the *set of fractions* and is denoted $\mathcal{T}$.

For instance, if G is the free abelian group $\mathbb{Z}^n$ and the action is self-replicating, then the set of fractions is equal to the set

$$A(g_{i_1}) + A^2(g_{i_2}) + A^3(g_{i_3}) + \cdots ,$$

where A is the linear map defining the associated virtual endomorphism and g_i are the digits of the associated numeration system. Hence is the term *set of fractions*.

The digit tiles of abelian groups is a classical object studied by many mathematicians (see, for instance [Thu89, Ban91, Ken92, Vin95, Vin00, BJ99] and bibliography therein).

For any contracting action (G, X) we have

$$\mathcal{X}_G = \bigcup_{g \in G} \mathcal{T} \cdot g, \qquad \mathcal{T} = \bigcup_{x \in \mathsf{X}} \mathcal{T} \otimes x.$$

Thus, the images $\mathcal{T} \cdot g$ of the tile by the action of G cover the space $\mathcal{X}_G$ and the tile $\mathcal{T}$ is a union of the "similar" tiles $\mathcal{T} \otimes x$.

If $v \cdot g \in \mathsf{X}^n \cdot G$ is an element of $\mathfrak{M}^{\otimes n}$, then the set $\mathcal{T} \otimes v \cdot g$ is a *tile of nth level*. The image of $\mathcal{T} \otimes v \cdot g$ in $\mathcal{J}_G$ is denoted $\mathcal{T}_v$ (it obviously does not depend on g).

If for every $g \in G$ there exists $v \in \mathsf{X}^*$ such that $g|_v = 1$, then two tiles $\mathcal{T} \cdot g_1$ and $\mathcal{T} \cdot g_2$ have disjoint interiors for $g_1 \neq g_2$ and every tile is the closure of its interior. Then the sets $\mathcal{T}_x$, for $x \in \mathsf{X}$ form a Markov partition of the limit dynamical system, i.e.,

$$\mathsf{s}(\mathcal{T}_x) = \mathcal{J}_G = \bigcup_{x \in \mathsf{X}} \mathcal{T}_x.$$

The next theorem shows a relation between the structure of the nucleus and the tiles. The proof is given in Section 3.3 of [Nek05].

Theorem 4.2. *Let (G, X) be a contracting self-similar action and let $\mathcal{N}$ be its nucleus. Then*

(1) *The tile $\mathcal{T}$ is homeomorphic to the quotient of the space $\mathsf{X}^{-\omega}$ by the equivalence relation which identifies two sequences $\ldots x_2 x_1, \ldots y_2 y_1 \in \mathsf{X}^{-\omega}$ if and only if there exists a direct path $\ldots e_2 e_1$ in the Moore diagram of $\mathcal{N}$ ending in the* trivial state *1 and such that the arrow e_i is labeled by (x_i, y_i).*

(2) *Two tiles $\mathcal{T} \otimes v_1 \cdot g_1$ and $\mathcal{T} \otimes v_2 \cdot g_2$ of nth level have common points if and only if there exists $g \in \mathcal{N}$ such that $g \cdot v_1 \cdot g_1 = v_2 \cdot g_2$.*

(3) *Two tiles $\mathcal{T}_{v_1}$ and $\mathcal{T}_{v_2}$ have common points if and only if there exists $g \in \mathcal{N}$ such that $g(v_1) = v_2$.*

Definition 20. Let G be a group acting on a set M and let $S = S^{-1}$ be a finite generating set of G. Then the *Schreier graph* of the action is the graph with the set of vertices M in which two vertices v_1, v_2 are adjacent if there exists $s \in S$ such that $s(v_1) = v_2$.

Then the last statement of Theorem 4.2 says that adjacency of the nth level tiles $\mathcal{T}_v$ coincides with the Schreier graph of the action of the group $\langle \mathcal{N} \rangle$ on X^n. If the
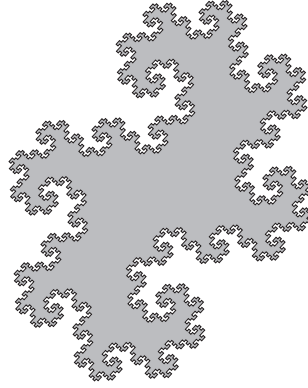
FIGURE 4. Twin dragon

action (G, X) is self-replicating and the group G is level-transitive, then $G = \langle \mathcal{N} \rangle$ and the adjacency graphs of tiles are the Schreier graphs of the group G itself.

Note that the set of horizontal edges of the self-similarity graph constructed in 3.10 is the union of the Schreier graphs of the action of G on the levels X^n of the tree X^*.

4.6. **Twin dragon.** Consider again the numeration system on $\mathbb{Z}[i]$ with the basis $1-i$ and digits 0 and 1, as in 4.3. We have seen that the corresponding self-similar group is generated by $u = \sigma(1, uv)$ and $v = \sigma(u^{-1}, v)$.

The nucleus of the action consists of 7 elements (in the multiplicative notation):

$$\mathcal{N} = \big\{ 1 = (1,1),$$
$$u = \sigma(1, uv), \qquad\qquad v = \sigma(u^{-1}, v),$$
$$u^{-1} = \sigma(u^{-1}v^{-1}, 1), \qquad v^{-1} = \sigma(v^{-1}, u),$$
$$uv = (v, v), \qquad\qquad u^{-1}v^{-1} = (v^{-1}, v^{-1}). \big\}$$

In order to prove this, consider the elements of the set $\mathcal{N} \cdot \{u, v, u^{-1}, v^{-1}\} \setminus \mathcal{N}$:

$$uv^{-1} = (u, u), \qquad vu^{-1} = (u^{-1}, u^{-1}),$$
$$u^2 = (uv, uv), \qquad\qquad v^2 = (vu^{-1}, vu^{-1}),$$
$$uv^2 = \sigma(vu^{-1}, v^2), \qquad vu^2 = \sigma(v, uv^2),$$

and see that their restrictions in words of length 2 belong to $\mathcal{N}$.

By the general theory, the limit space $\mathcal{X}_{\mathbb{Z}[i]}$ is the plane $\mathbb{C}$ with the natural action of $\mathbb{Z}[i]$ on it and the self-similarity

$$z \otimes \mathit{0} = z/(1-i), \quad z \otimes \mathit{1} = (z+1)/(1-i),$$

thus the point of $\mathbb{C}$ corresponding to $\ldots x_2 x_1 \cdot g \in \mathsf{X}^{-\omega} \cdot \mathbb{Z}[i]$ is equal to

$$g + \frac{x_1}{1-i} + \frac{x_2}{(1-i)^2} + \frac{x_3}{(1-i)^2} + \cdots.$$

The set of fraction of this action is considered in [Knu69]. It is called the *twin dragon*, since it is a union of two copies of the "dragon curve" (also called the *Highway dragon*). It is shown on Figure 4.

A part of the tiling of $\mathbb{C}$ by the shifts of the twin dragon is shown on Figure 5.
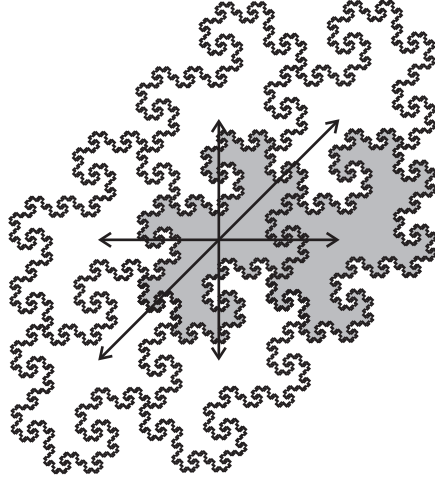
FIGURE 5. Twin dragon tiling

The two shaded tiles are the tiles $\mathcal{T}$ and $\mathcal{T} + 1$. Their union is mapped onto $\mathcal{T}$ by the map $z \mapsto z/(1-i)$ (which illustrates the general fact $\mathcal{T} = \bigcup_{x \in \mathsf{X}} \mathcal{T} \otimes x$). The six arrows shown on the figure show the action of the six non-trivial elements of the nucleus $\mathcal{N}$. Note that they map the tile to all its neighbors (see Theorem 4.2).

4.7. **A non-self-replicating case (base** 1.5 **numeration system).** Consider the virtual endomorphism $n \mapsto 2n/3$ of $\mathbb{Z}$. Its domain is the subgroup $3\mathbb{Z}$ and it is contracting, hence the associated action of $\mathbb{Z}$ is contracting.

The self-similarity bimodule $\mathfrak{M}$ is the set of affine functions of the form $2(x + n)/3 + m$ for $n, m \in \mathbb{Z}$. It has three right orbits $\{2x/3 + m\}, \{2(x+1)/3 + m\}, \{2(x+2)/3\}$ and two left orbits $\{2(x+n)/3\}$ and $\{2(x+n)/3 + 1\}$.

If we choose the coset transversal $0, 1, 2$, then the action of the generator of $\mathbb{Z}$ (denoted here by a) is defined by

(6) $$a \cdot 0 = 1, \quad a \cdot 1 = 2, \quad a \cdot 2 = 0 \cdot a^2,$$

or, in terms of the wreath recursion

$$a = \sigma(1, 1, a^2),$$

where $\sigma = (012)$ is the cycle. Here 0, 1 and 2 are the elements $(2n+0)/3$, $(2n+1)/3$ and $(2n+2)/3$ of the bimodule $\mathfrak{M}$, corresponding to the elements $0, 1, 2$ of the coset transversal.

Since we have more than one left orbit, we can not conclude that the corresponding limit space is $\mathbb{R}$. Actually, it is easy to see that $\mathbb{R}$ (with respect to the natural action of $\mathbb{Z}$) can not be made self-similar, since the space $\mathbb{R} \otimes \mathfrak{M}$ has two connected components corresponding to two left orbits of $\mathfrak{M}$. (If $\xi_1 \otimes x_1$ and $\xi_2 \otimes x_2$ are equal, then x_1 and x_2 belong to one left orbit.)

Proposition 4.3. *The limit space $\mathcal{X}_{\mathbb{Z}}$ of the action* (6) *is the direct product $\mathbb{R} \times \mathbb{Z}_2$ of the real line with the ring of dyadic integers with the diagonal action of $\mathbb{Z}$*

$$(\xi, \zeta) \cdot a^n = (\xi + n, \zeta + n)$$

and the self-similarity

$$(\xi, \zeta) \otimes 0 = (2\xi/3, 2\zeta/3).$$

Recall that $2/3 = 2 + \sum_{k=1}^{\infty} 4^k$ is an element of the ring $\mathbb{Z}_2$, hence we can multiply the elements of $\mathbb{Z}_2$ by $2/3$.

Proof. It is easy to check that the $\mathbb{Z}$-space $\mathbb{R} \times \mathbb{Z}_2$ is proper and co-compact. The quotient $(\mathbb{R} \times \mathbb{Z}_2)/\mathbb{Z}$ is the mapping torus of the adding machine $\zeta \mapsto \zeta + 1$ on $\mathbb{Z}_2$. It remains to prove that it is self-similar and that the self-similarity is contracting.

We have to prove that the map

$$\Phi((\xi, \zeta) \otimes (2(x + n)/3 + m)) = ((2\xi + n)/3 + m, (2\zeta + n)/3 + m)$$

induces a homeomorphism of the $\mathbb{Z}$-space $\mathbb{R} \times \mathbb{Z}_2$ with the tensor product $(\mathbb{R} \times \mathbb{Z}_2) \otimes \mathfrak{M}$. It is sufficient to prove that it is a bijection (since the spaces are locally compact and metrizable). It is obviously well defined and surjective.

Suppose that $((2\xi_1 + n_1)/3 + m_1, (2\zeta_1 + n_1)/3 + m_1) = ((2\xi_2 + n_2)/3 + m_2, (2\zeta_2 + n_2)/3 + m_2)$. We get that $2(\zeta_1 - \zeta_2) = n_2 + 3m_2 - n_1 - 3m_1$ in $\mathbb{Z}_2$, hence $n_2 + 3m_2 - n_1 - 3m_1 \in \mathbb{Z}$ is even. Let $n = (n_2 + 3m_2 - n_1 - 3m_1)/2$. Then $\zeta_1 = \zeta_2 + n$ and $\xi_1 = \xi_2 + n$. We also have

$$\frac{2(x + n) + n_1}{3} + m_1 = \frac{2x + n_2 + 3m_2 - n_1 - 3m_1 + n_1}{3} + m_1 = \frac{2x + n_2}{3} + m_2,$$

hence $(\xi_1, \zeta_1) \otimes ((2x + n_1)/3 + m_1) = (\xi_2, \zeta_2) \otimes ((2x + n_2)/3 + m_2)$ and the map Φ is injective.

The map $\xi \mapsto 2\xi/3$ is contracting on $\mathbb{R}$, since $|2/3| < 1$. The map $\zeta \mapsto 2\zeta/3$ is also contracting on $\mathbb{Z}_2$, since $2/3$ is even. This easily implies that the self-similarity Φ is contracting. $\square$

The space $\mathbb{R} \times \mathbb{Z}_2$ is a direct product of a line by the Cantor set (see Figure 6). The group $\mathbb{Z}$ acts on $\mathbb{R}$ by translation and on $\mathbb{Z}_2$ by the action of the binary adding machine. The fundamental domain of the action is $[0, 1] \times \mathbb{Z}_2$. We get hence the following description of the limit space $\mathcal{J}_{\mathbb{Z}}$ of the "1.5-numeration system".

Proposition 4.4. *The limit space $\mathcal{J}_{\mathbb{Z}}$ of the action* (6) *is the dyadic solenoid, which can be defined as the mapping torus of the binary adding machine, i.e., the direct product $[0, 1] \times \mathbb{Z}_2$ in which every point $(1, \zeta)$ is identified with $(0, \zeta + 1)$. It is homeomorphic to the inverse limit of the double self-coverings of a circle.*

The shift $\mathsf{s} : \mathcal{J}_{\mathbb{Z}} \longrightarrow \mathcal{J}_{\mathbb{Z}}$ *acts by the rule*

$$\mathsf{s}(\xi, \zeta) = \left\{ \begin{array}{ll} \left(\frac{3}{2}\xi, \frac{3}{2}\zeta\right) & \text{if } \zeta \text{ is even} \\ \left(\frac{3}{2}(\xi - 1), \frac{3}{2}(\zeta - 1)\right) & \text{if } \zeta \text{ is odd,} \end{array} \right.$$

where the real coordinate is computed modulo 1.

See Figure 6, where the action of $\mathbb{Z}$ on the limit $\mathbb{Z}$-space $\mathcal{X}_{\mathbb{Z}}$ is shown. Here $\mathbb{Z}$ acts by parallel translation.

4.8. **Expanding endomorphisms of orbifolds.** Let $(\mathbb{Z}^n, \mathsf{X})$ be a contracting self-replicating free abelian group and let A be the associated virtual endomorphism. We assume that the action is faithful, so that A is injective.

Suppose that G is a cocompact proper group of affine transformations of $\mathbb{Z}^n$, whose linear parts commute with A. Then $\mathbb{Z}^n$ has finite index in G and A induces a virtual endomorphism of G mapping an affine transformation $L(\vec{x}) + \vec{v}$ to $L(\vec{x}) + A(\vec{v})$. In this way G is transformed into a self-similar over-group of $\mathbb{Z}^n$, whose
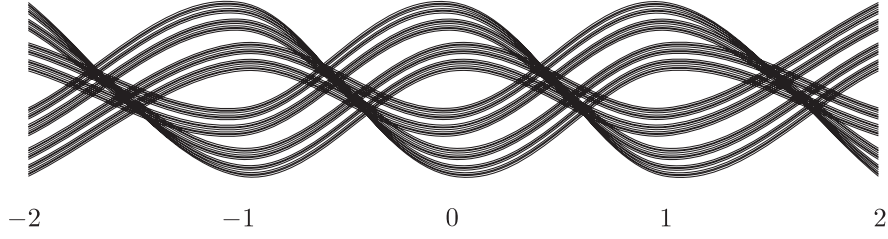
$$-2 \qquad\qquad -1 \qquad\qquad 0 \qquad\qquad 1 \qquad\qquad 2$$

FIGURE 6. The limit $\mathbb{Z}$-space of the numeration system with base 1.5

limit space $\mathcal{X}_G$ is homeomorphic to $\mathbb{R}^n$ with the original action of G on it (by Theorem 3.9).

Let us illustrate this on several examples, which we will also use later.

4.8.1. *Dihedral group.* Consider the group $\mathbb{D}_\infty$ of the affine transformation of $\mathbb{R}^1$ of the form $x \mapsto (-1)^k x + n$, where $n \in \mathbb{Z}$ and $k \in \{0, 1\}$. Consider the virtual endomorphism $n \mapsto n/2$ of $\mathbb{Z}$, i.e., the binary adding machine action of $\mathbb{Z}$. We know that the associated $\mathbb{Z}$-bimodule can be identified with the set of affine transformations of $\mathbb{R}$ of the form $x \mapsto (x + n)/2$ for $n \in \mathbb{Z}$, i.e., with the set of compositions of the elements of $\mathbb{Z}$ with the transformation $x \mapsto x/2$. Let us extend this to a $\mathbb{D}_\infty$-bimodule equal to the set $\{x \mapsto (\pm x + n)/2\}$ of compositions of elements of $\mathbb{D}_\infty$ with $x/2$. It is easy to see that this is a covering $\mathbb{D}_\infty$-bimodule.

We have now the freedom to choose a basis of the $\mathbb{D}_\infty$-bimodule. If we choose the usual basis $\{0, 1\}$ given by $x \otimes 0 = x/2$ and $x \otimes 1 = (x+1)/2$, then we get the usual binary adding machine action of $\mathbb{Z}$ generated by $a = \sigma(1, a)$. Let us compute the recursion defining the element $b : x \mapsto -x$ of $\mathbb{D}_\infty$. We have

$$x \otimes b \cdot 0 = x \cdot b \otimes 0 - x/2 = x \otimes 0 \cdot b$$
$$x \otimes b \cdot 1 = (-x+1)/2 = x \otimes 1 \cdot ba,$$

hence $b = (b, ba)$. Consequently,

$$a^{-1}b = (1, a^{-1})\sigma(b, ba) = \sigma(a^{-1}b, ba),$$

and since $a^{-1}b = ba$, we get that $\mathbb{D}_\infty$ in this action is generated by the adding machine $a = \sigma(1, a)$ and the element $ba = c = \sigma(c, c)$, changing every letter of a word.

On the other hand, if we choose the basis $\{0, 1\}$ given by $x \otimes 0 = x/2$ and $x \otimes 1 = (-x+1)/2$, then $b = (b, c)$ and

$$x \cdot c \cdot 0 = (-x+1)/2 = x \otimes 1$$
$$x \cdot c \cdot 1 = (-(-x+1)+1)/2 = x/2 = x \otimes 0,$$

hence $c = \sigma$. This gives another self-similar action of $\mathbb{D}_\infty$ generated by:

$$b = (b, c), \quad c = \sigma.$$

4.8.2. *A Lattès example.* Consider, as in 4.3, the group $\mathbb{Z}^2$ and the virtual endomorphism

$$A \begin{pmatrix} m \\ n \end{pmatrix} = \begin{pmatrix} (m-n)/2 \\ (m+n)/2 \end{pmatrix}.$$

We again interpret $\mathbb{Z}^2$ as the additive group of the ring of Gaussian integers $\mathbb{Z}[i]$ and A as multiplication by $(1+i)/2$.

Then the limit space of $\mathbb{Z}[i]$ is $\mathbb{C}$ with the natural action. The associated bimodule is the set $\{z \mapsto \frac{1+i}{2}(z+m+ni) \ : \ m+ni \in \mathbb{Z}[i]\}$.

Consider the group G of affine transformations of $\mathbb{C}$ of the form $z \mapsto \pm z + (m+ni)$, and extend the associated bimodule as in the previous example.

If we take the basis given by $z \otimes 0 = z\frac{1+i}{2}$ and $z \otimes 1 = (z+1)\frac{1+i}{2}$, then $\mathbb{Z}[i]$ will be a self-similar subgroup of G. We have computed the respective action in 4.3. Recall that the generators $u : z \mapsto z + 1$ and $v : z \mapsto z + i$ act by the rule

$$u = \sigma(1, uv), \quad v = \sigma(u^{-1}, v).$$

The element $c : z \mapsto -z$ acts by the rule

$$z \otimes c \cdot 0 = -z\frac{1+i}{2} = z \otimes 0 \cdot c$$

$$z \otimes c \cdot 1 = (-z+1)\frac{1+i}{2} = -(z+1)\frac{1+i}{2} + 1 + i = z \otimes 1 \cdot cuv,$$

hence $c = (c, cuv)$.

Let us take now the basis $z \otimes 0 = z\frac{i+1}{2}$ and $z \otimes 1 = (-z-i)\frac{i+1}{2}$ and compute the recursions for $a : z \mapsto -z - i$, $B : z \mapsto -z - i + 1$, $C : z \mapsto -z$. Note that $CaB : z \mapsto -z + 1$.

We have

$$z \otimes a \cdot 0 = (-z-i)\frac{1+i}{2} = z \otimes 1$$

$$z \otimes a \cdot 1 = (-(-z-i)-i)\frac{i+1}{2} = z \otimes 0,$$

hence $a = \sigma$,

$$z \otimes B \cdot 0 = (-z-i+1)\frac{i+1}{2} = -z\frac{i+1}{2} + 1 = z \otimes 0 \cdot CaB$$

$$z \otimes B \cdot 1 = (-(-z-i+1)-i)\frac{i+1}{2} = -(-z-i)\frac{i+1}{2} - i = z \otimes 1 \cdot a,$$

hence $B = (CaB, a)$,

$$z \otimes C \cdot 0 = -z\frac{1+i}{2} = z \otimes 0 \cdot C$$

$$z \otimes C \cdot 1 = (z-i)\frac{i+1}{2} = -(-z-i)\frac{i+1}{2} - i + 1 = z \otimes 1 \cdot B,$$

hence $C = (C, B)$ and the group G is generated by

$$a = \sigma$$
$$B = (CaB, a)$$
$$C = (C, B)$$

The limit space $\mathcal{J}_G$ of the group G is the quotient $\mathbb{C}/G$ of $\mathbb{C}$ by the action of G. It is homeomorphic to the sphere and as an orbispace it has four singular points with isotropy groups of order 2 (the images of the points 0, $1/2$, $i/2$ and $(1+i)/2$). The shift $\mathsf{s} : \mathcal{J}_G \longrightarrow \mathcal{J}_G$ is induced by the map $A^{-1} : \mathbb{C} \longrightarrow \mathbb{C} : z \mapsto (1-i)z$. Hence it is a two-fold branched self-covering of the sphere. It is holomorphic with respect to the holomorphic structure on the quotient space $\mathbb{C}/G$ (which is well defined

since G acts properly by biholomorphic automorphisms of $\mathbb{C}$). Consequently, it is conjugate to a rational function.

The constructed rational function (defined uniquely up to a conjugation by a Möbius transformation) is a *Lattès example* (in a slightly generalized sense), see [Lat18, Mil99].

The quotient map $\mathbb{C} \longrightarrow \mathbb{C}/G$ can be realized as the Weierstrass function

$$\wp(z) = \frac{1}{z^2} + \sum_{\omega \in \mathbb{Z}[i]} \left[\frac{1}{(z+\omega)^2} - \frac{1}{\omega^2} \right]$$

associated with the lattice $\mathbb{Z}[i]$. Consequently, the rational function conjugate to $\mathsf{s} : \mathcal{J}_G \longrightarrow \mathcal{J}_G$ can be defined as the function f with the property

$$\wp((1-i)z) = f(\wp(z)).$$

One can show that f is conjugate to $\frac{i}{2}(z + z^{-1})$ (see, for instance, [Mil04]).

4.8.3. *Triangle orbifold.* Consider the group G of all symmetries of the integral lattice $\mathbb{Z}^2 < \mathbb{R}^2$. It is the group of the affine transformations of the form $\xi \mapsto L(\xi) + \vec{v}$, where $\vec{v} \in \mathbb{Z}^2$ and L is a linear transformation given by a matrix from the set

$$\left\{ \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \pm \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \pm \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \pm \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right\}.$$

Note that this set of matrices forms a group isomorphic to the dihedral group $\mathbb{D}_4$ of order 8.

As before, consider the virtual endomorphism of the group G mapping the affine transformation $L(\xi) + \vec{v}$ to the transformation $L(\xi) + A(\vec{v})$, where A is given by the matrix $\begin{pmatrix} 1/2 & -1/2 \\ 1/2 & 1/2 \end{pmatrix}$. We have seen in 4.6 that if we take the basis given by $\xi \otimes 0 = A(\xi)$ and $\xi \otimes 1 = A\left(\xi + \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right)$, then the digit tiles of the group G will coincide with the twin dragon.

It is easy to see that the group G is generated by the reflections

$$a : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} 1-x \\ y \end{pmatrix}, \quad b : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} y \\ x \end{pmatrix}, \quad c : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x \\ -y \end{pmatrix}$$

with respect to the sides of the fundamental triangle with vertices $\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix}$ and $\begin{pmatrix} 1/2 \\ 0 \end{pmatrix}$, see Figure 7.
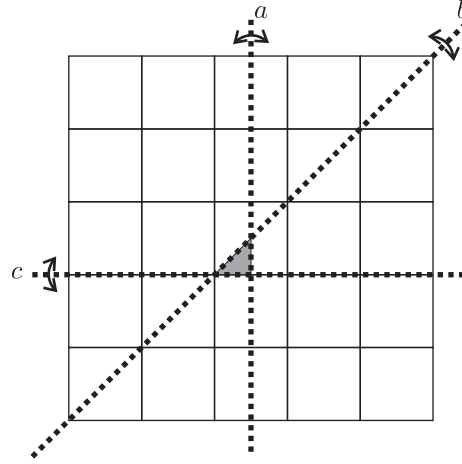
The group G is given by the presentation

$$G = \langle a, b, c \mid a^2 = b^2 = c^2 = (ac)^2 = (bc)^4 = (ab)^4 = 1 \rangle.$$

Take now the basis of the associated G-bimodule given by

$$\begin{pmatrix} x \\ y \end{pmatrix} \otimes 0 = A\left(\begin{pmatrix} x \\ y \end{pmatrix} \cdot a\right) \cdot b = \begin{pmatrix} (1-x+y)/2 \\ (1-x-y)/2 \end{pmatrix},$$

$$\begin{pmatrix} x \\ y \end{pmatrix} \otimes 1 = A\begin{pmatrix} x \\ y \end{pmatrix} \cdot b = \begin{pmatrix} (x+y)/2 \\ (x-y)/2 \end{pmatrix}.$$

Then $\xi \otimes a \cdot 0 = \xi \otimes 1$ and $\xi \otimes a \cdot 1 = \xi \otimes 0$, hence $a = \sigma$.

FIGURE 7. Generators a, b, c

We have

$$\left(\begin{array}{c} x \\ y \end{array}\right) \otimes b \cdot 0 = \left(\begin{array}{c} (1+x-y)/2 \\ (1-x-y)/2 \end{array}\right) = \left(\begin{array}{c} x \\ y \end{array}\right) \otimes 0 \cdot a$$

$$\left(\begin{array}{c} x \\ y \end{array}\right) \otimes b \cdot 1 = \left(\begin{array}{c} (x+y)/2 \\ (-x+y)/2 \end{array}\right) = \left(\begin{array}{c} x \\ y \end{array}\right) \otimes 1 \cdot c,$$

hence $b = (a, c)$,

$$\left(\begin{array}{c} x \\ y \end{array}\right) \otimes c \cdot 0 = \left(\begin{array}{c} (1-x-y)/2 \\ (1-x+y)/2 \end{array}\right) = \left(\begin{array}{c} x \\ y \end{array}\right) \otimes 0 \cdot b,$$

$$\left(\begin{array}{c} x \\ y \end{array}\right) \otimes c \cdot 1 = \left(\begin{array}{c} (x-y)/2 \\ (x+y)/2 \end{array}\right) = \left(\begin{array}{c} x \\ y \end{array}\right) \otimes 1 \cdot b,$$

hence $c = (b, b)$.

We see that the group G is generated by

$$a = \sigma,$$
$$b = (a, c),$$
$$c = (b, b).$$

This recursion was considered in [Nek07]. It is proved there that the nucleus of G is equal to

$$\langle a, b \rangle \cup \langle a, c \rangle \cup \langle b, c \rangle,$$

hence has 15 elements. Analysis of the structure of the nucleus shows that the associated tiling of $\mathbb{C} = \mathcal{X}_G$ is the tiling by the images under the action of G of the fundamental triangle shown on Figure 7.

The limit space $\mathcal{J}_G$ of the group G is the quotient $\mathbb{C}/G$, which is an isosceles rectangular triangle. The boundary points of the triangle are singular: the isotropy groups of the internal points of the sides are of order two (corresponding to reflections), the isotropy group of the vertex of the right angle is isomorphic to $\mathbb{D}_2 \cong C_2 \times C_2$ and the isotropy groups of the vertices of the acute angles are $\mathbb{D}_4$.

The sift map $\mathsf{s} : \mathcal{J}_G \longrightarrow \mathcal{J}_G$ folds the triangle in two along the bisectrix of the right angle, stretches by the linear factor of $\sqrt{2}$ and superimposes it with the original triangle so that (the image of) the point $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is fixed under s.

5. COMPLEX POLYNOMIALS

5.1. **Carateodori loop.** Let $f : \mathbb{C} \longrightarrow \mathbb{C}$ be a degree d polynomial with finite post-critical set P. Then f is a partial self-covering of $\mathbb{C} \setminus P$ and we can compute the iterated monodromy group $\mathrm{IMG}\,(f)$ of this self-covering and the associated bimodule $\mathfrak{M}_{f,t}$.

Let $\mathcal{R}$ be the complement of the filled-in Julia set of f (i.e., the basin of infinity). Then $f : \mathcal{R} \longrightarrow \mathcal{R}$ is a d-fold self-covering. The homotopy type of $\mathcal{R}$ is circle and it is easy to see that the iterated monodromy group of $f|_{\mathcal{R}}$ is isomorphic (as a self-similar group) to the d-adic adding machine.

We get hence a morphism (inclusion) of self-similar groups $\mathrm{IMG}\,(f|_{\mathcal{R}}) \longrightarrow \mathrm{IMG}\,(f)$ induced by the inclusion $\mathcal{R} \hookrightarrow \mathbb{C} \setminus P$, since the last inclusion commutes with the dynamics of f.

On the other hand, the adding machine and $\mathrm{IMG}\,(f)$ are contracting, hence the morphism $\mathrm{IMG}\,(f|_{\mathcal{R}}) \longrightarrow \mathrm{IMG}\,(f)$ induces a semiconjugacy of their limit dynamical systems. The limit dynamical system of the d-adic adding machine is a d-fold self-covering $x \mapsto dx \pmod 1$ of the circle $\mathbb{R}/\mathbb{Z}$. The limit dynamical system of $\mathrm{IMG}\,(f)$ is the action of f on its Julia set $\mathcal{J}_f$.

We get hence a surjective map $R : \mathbb{R}/\mathbb{Z} \longrightarrow \mathcal{J}_f$ such that $f(R(\alpha)) = R(d\alpha)$. This map coincides with the classical *Carateodori map (loop)* from the circle onto the Julia set. This map is the boundary of the biholomorphic isomorphism (uniformization) from the complement of the unit disc to the complement of the filled-in Julia set. See Figure 8, where the Carateodori loop around the Julia set of $z^2 - 1$ is depicted. The curves going from the outer circle to the Julia set on the figure are the images of the rays (of sets of complex numbers having a fixed argument) under the uniformization map.

The embedding of the d-adic adding machine into the iterated monodromy group $\mathrm{IMG}\,(f)$ of a polynomial gives a special choice of a basis of the associated bimodule. For example, one can take the basis of the bimodule associated with the adding machine action, corresponding to the digits $0, 1, \ldots, d-1$, i.e., the affine transformations $x/d, (x+1)/d, \ldots, (x+d-1)/d$. Then the image of this basis in the bimodule associated to $\mathrm{IMG}\,(f)$ is a convenient choice (see, for instance [BN06c]). In terms of the polynomial this basis consists of paths from $+\infty$ to its preimages along the shortest arcs in the positive direction on the circle at infinity. Here we compactify the complex plane by the circle at infinity (for example using the *gnomonic projection*, see [Mil04]) and extend naturally the action of the polynomial to this compactification. This is possible, since a monic polynomial of degree d acts near infinity as z^d.

5.2. **Quadratic polynomials.** Iterated monodromy groups of post-critically finite polynomials are described in terms of automata in [Nek05] Section 6.7. We will not repeat here the construction in full generality, but will only show how the classical theory of kneading sequences for quadratic polynomials is interpreted in terms of self-similar groups. This description is a joint work with Laurent Bartholdi (see [BN06a]).
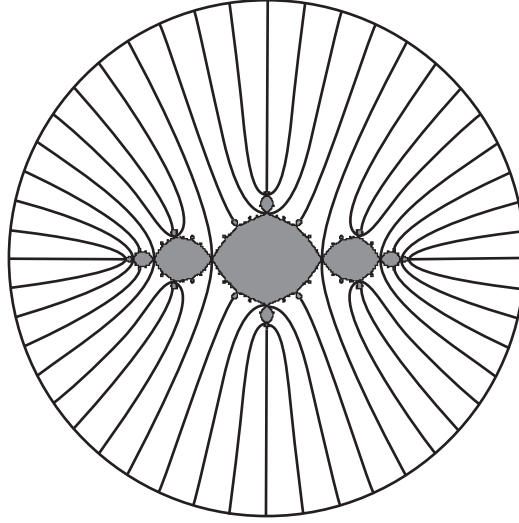
FIGURE 8. Carateodori loop

There are two kinds of self-similar actions (i.e., two choices of a basis of the associated permutational bimodule) of the iterated monodromy groups of polynomials, which are convenient in different situations. The first comes from the canonical choice of the connecting paths along the circle at infinity. The other is related to the kneading sequences and gives a particularly nice recurrent relations for the generators of the group.

5.3. **Groups** $\mathfrak{D}(\theta)$. Fix some point $\theta \in \mathbb{R}/\mathbb{Z}$ of the circle. Consider the doubling map $x \mapsto 2x$ on $\mathbb{R}/\mathbb{Z}$. Let P be the orbit of θ under the doubling map. The set P is finite if and only if the number θ is rational. Every $\alpha \in \mathbb{R}/\mathbb{Z}$ has two preimages under the doubling map: $\alpha/2$ and $(\alpha+1)/2$.

For any subset $A \subset [0, 1)$ (seen as an oriented subset of the circle $\mathbb{R}/\mathbb{Z}$) define recursively an automorphism γ_A of the binary tree $\{0, 1\}^*$ by the following conditions

(1) If $A = A_1 \cup A_2$, where $\theta_1 < \theta_2$ for all $\theta_1 \in A_1$ and $\theta_2 \in A_2$, then

$$\gamma_A = \gamma_{A_1} \gamma_{A_2}.$$

(2) If $\theta \notin A$, then

$$\gamma_A = (\gamma_{A/2}, \gamma_{(A+1)/2}).$$

(3)

$$\gamma_{\{\theta\}} = \sigma \left(\gamma_{(\theta/2,(\theta+1)/2)}^{-1}, \ \gamma_{[\theta/2,(\theta+1)/2]} \right),$$

where $[a, b]$ and (a, b) denote the closed and the open interval, respectively.

If G is a profinite group and $M \subset G$ is a countable set converging to identity, then for any linear ordering $\prec$ of M the product $\prod_{g \in M}^{\prec} g$ taken in the order $\prec$ makes sense, since modulo any normal open subgroup $H \lhd G$ the product has only a finite number of non-trivial factors.

Theorem 5.1. *The conditions (1)–(3) are consistent and uniquely define the automorphisms γ_A of the tree $\{0, 1\}^*$.*
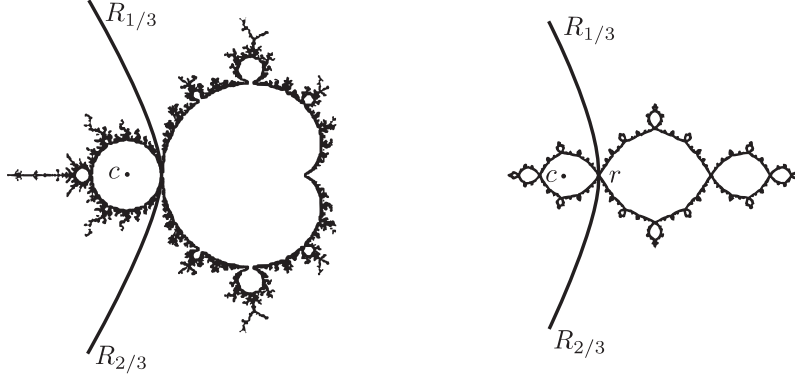
FIGURE 9. External rays

An element $\gamma_{\{\alpha\}}$ is non-trivial if and only if α belongs to the orbit $P = \{2^k\theta :$ $k \geq 0\}$ (mod 1) of θ under the angle doubling map.

For any subset $A \subset [0, 1)$ we have

$$\gamma_A = \prod_{\alpha \in P \cap A}^{<} \gamma_{\{\alpha\}},$$

where the product is taken in the profinite automorphism group of the binary tree with respect to the natural ordering of the set A.

Definition 21. The group $\mathfrak{D}(\theta)$ is the group generated by the proper restrictions of the element $\gamma_{\{\theta\}}$, i.e., by the elements $\gamma_{\{\theta\}}|_v$ for non-empty words v.

Let $\theta \in \mathbb{Q}/\mathbb{Z} \subset \mathbb{R}/\mathbb{Z}$ be a rational point of the circle. Denote by c_θ either the landing point of the parameter ray R_θ, if the denominator of θ is even (i.e., if θ is pre-periodic with respect to the angle doubling), or the center of the hyperbolic component of the Mandelbrot set with the root equal to the landing point of R_θ, if the denominator of θ is odd (if θ belongs to a cycle of the angle doubling map).

Proof of the following theorem is given in [BN06b].

Theorem 5.2. *Suppose that $\theta \in \mathbb{R}/\mathbb{Z}$ is rational. Then the group $\mathfrak{D}(\theta)$ is equal to the self-similar action of* $\mathrm{IMG}\left(z^2 + c_\theta\right)$ *defined by the connecting paths on the circle at infinity.*

Example. Let us compute $\mathfrak{D}(1/3)$. The parameter ray R_θ lands on the root of the hyperbolic component with the center $c_{1/3} = -1$ (see Figure 9).

The orbit of $1/3$ under the angle doubling map is $1/3 \mapsto 2/3 \mapsto 1/3$, hence the only non-trivial elements $\gamma_{\{\alpha\}}$ are $a = \gamma_{\{1/3\}}$ and $b = \gamma_{\{2/3\}}$, so the group $\mathfrak{D}(1/3)$ is generated by two elements.
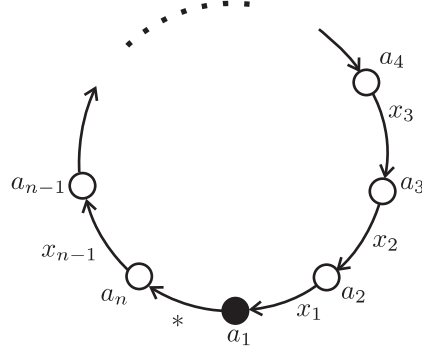
We have

$$a = \gamma_{\{1/3\}} = \sigma\left(\gamma_{(1/6,2/3)}^{-1}, \gamma_{[1/6,2/3]}\right) = \sigma(a^{-1}, ab)$$

and

$$b = \gamma_{\{2/3\}} = \left(\gamma_{\{1/3\}}, \gamma_{\{5/6\}}\right) = (a, 1).$$

We have $\mathfrak{D}(1/3) = \mathrm{IMG}\left(z^2 - 1\right)$. The generator $\gamma_{\{1/3\}}$ is a small loop going in the positive direction around -1 and connected to the circle at infinity along the external (dynamical) ray $R_{1/3}$. Similarly, the element $\gamma_{\{2/3\}}$ is the loop around 0

FIGURE 10. Automaton generating $\mathfrak{K}(v)$

connected to the circle at infinity along the ray $R_{2/3}$. The rays are connected to $+\infty$ along the arcs $[0, 1/3]$ and $[0, 2/3]$, respectively.

5.4. **Kneading sequences and the groups $\mathfrak{K}(v)$ and $\mathfrak{K}(u, v)$.** We denote by $\mathfrak{K}(v)$, for $v = x_1 x_2 \ldots x_{n-1} \in \{0, 1\}^*$, the group generated by the elements $a_1, \ldots, a_n$ given by the recursion

$$a_1 = \sigma(1, a_n), \qquad a_{i+1} = \begin{cases} (a_i, 1) & \text{if } x_i = 0, \\ (1, a_i) & \text{if } x_i = 1, \end{cases} \text{ when } 1 \le i \le n-1.$$

For a pair of words $u = y_1 \ldots y_k$ and $v = x_1 \ldots x_n \in \mathsf{X}^*$ such that $y_k \ne x_n$, we denote by $\mathfrak{K}(u, v)$ the group generated by the elements $b_1, \ldots b_k, a_1, \ldots, a_n$ given by

$$b_1 = \sigma, \qquad b_{i+1} = \begin{cases} (b_i, 1) & \text{if } y_i = 0, \\ (1, b_i) & \text{if } y_i = 1, \end{cases} \text{ when } 1 \le i \le k-1,$$

and

$$a_1 = \begin{cases} (b_k, a_n) & \text{if } y_k = 0 \text{ and } x_n = 1, \\ (a_n, b_k) & \text{if } y_k = 1 \text{ and } x_n = 0, \end{cases} \qquad a_{i+1} = \begin{cases} (a_i, 1) & \text{if } x_i = 0, \\ (1, a_i) & \text{if } x_i = 1, \end{cases}$$

when $1 \le i \le n-1$.

In other terms, the group $\mathfrak{K}(v)$ is generated by the automaton shown on Figure 10, and the group $\mathfrak{K}(u, v)$ is generated by the automaton shown on Figure 11. In those diagrams, only edges leading to non-trivial states are drawn, a label x corresponds to (x, x) and the label $*$ corresponds to $(1, 0)$. Black dots correspond to active states (i.e., to the states acting non-trivially on the first level).

For a given point $\theta \in \mathbb{R}/\mathbb{Z}$ the θ-*itinerary* $I_\theta(\alpha)$ of $\alpha \in \mathbb{R}/\mathbb{Z}$ is the sequence $a_0 a_1 \ldots$, where

$$a_k = \begin{cases} 0 & \text{if } 2^k \alpha \in S_0 \\ 1 & \text{if } 2^k \alpha \in S_1 \\ * & \text{if } 2^k \alpha \in \{\theta/2, (\theta+1)/2\}, \end{cases}$$

where $S_0 \ni 0$ and S_1 are the two semicircles into which $\mathbb{R}/\mathbb{Z}$ is divided by the points $\theta/2$ and $(1 + \theta)/2$.

The itinerary $I_\theta(\theta)$ is called the *kneading sequence* of the point $\theta \in \mathbb{R}/\mathbb{Z}$ and is denoted $\widehat{\theta}$.
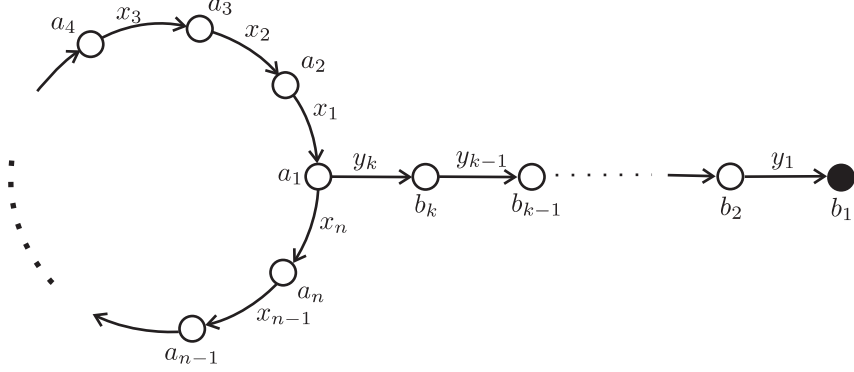
FIGURE 11. Automaton generating $\mathfrak{K}(u,v)$

If θ is rational with odd denominator, then $\widehat{\theta}$ is a sequence of the form $(v*)^\infty$ for $v \in \{0,1\}^*$. Let us denote then $\mathfrak{K}(\widehat{\theta}) = \mathfrak{K}(v)$.

If θ is rational and pre-periodic with respect to the angle doubling map, then $\widehat{\theta}$ is of the form $u(v)^\infty$. Then we denote $\mathfrak{K}(\widehat{\theta}) = \mathfrak{K}(u,v)$.

The following theorem is proved in [Nek05] (Theorem 6.11.1) and in [BN06a].

Theorem 5.3. *Let $\theta \in \mathbb{R}/\mathbb{Z}$ be a rational point. Then the group* $\mathrm{IMG}\left(z^2 + c_\theta\right)$ *is isomorphic to the group $\mathfrak{K}(\widehat{\theta})$ and their actions on the respective rooted trees are conjugate. The dynamical systems $\left(\mathcal{J}_{z^2+c_\theta}, z^2 + c_\theta\right)$ and $\left(\mathcal{J}_{\mathfrak{K}(\widehat{\theta})}, \mathsf{s}\right)$ are conjugate.*

5.5. Abstract tuning. Let $v = x_1 x_2 \ldots x_n \in \{0,1\}^*$ and let $a_1, a_2, \ldots, a_n$ be the generators of the group $\mathfrak{K}(v)$. Let G be an arbitrary self-similar group over the binary alphabet.

The *abstract tuning* $\mathfrak{K}(v,G)$ of $\mathfrak{K}(v)$ by G is the group generated by n copies $G_{(i)} = \{g_{(i)} \ : \ g \in G\}$ of G acting on the binary tree by the rules:

$$g_{(1)} = \begin{cases} (h'_{(n)}, h''_{(n)}) & \text{if } g = (h', h'') \text{ in } G \\ \sigma(h'_{(n)}, h''_{(n)}) & \text{if } g = \sigma(h', h'') \text{ in } G \end{cases}$$

and

$$g_{(i+1)} = \begin{cases} (g_{(i)}, 1) & \text{if } x_i = 0, \\ (1, g_{(i)}) & \text{if } x_i = 1, \end{cases}$$

for $1 \le i \le n-1$.

In other terms, the copy $G_{(i)}$ acts on every beginning of a sequence of the form

$$(x_{i-1} x_{i-2} \ldots x_1) y_1 U y_2 U y_3 U y_4 \ldots,$$

where $U = x_n x_{n-1} \ldots x_1$, changing the subsequence $y_1 y_2 \ldots$ to its image under the original action of G.

Note that the abstract tuning of $\mathfrak{K}(v)$ by the binary adding machine $a = \sigma(1, a)$ coincides with $\mathfrak{K}(v)$. In particular, if G contains the adding machine, then $\mathfrak{K}(v,G)$ contains $\mathfrak{K}(v)$.

Tuning by $\mathfrak{D}(\theta)$ corresponds to the classical ("formal") tuning procedure, as described, for example, in [Mil89, Pil03].
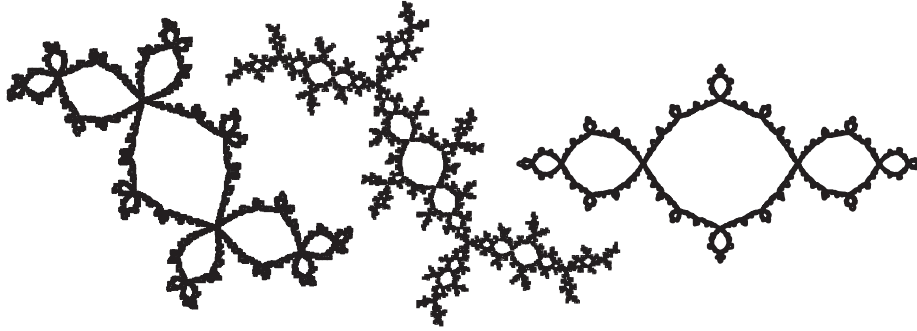
FIGURE 12. Tuning the rabbit by the basilica

As an example, consider tuning of the "Rabbit" polynomial $z^2 - c_{1/7}$ by $z^2 - 1$, i.e., tuning $\mathfrak{K}(11)$ by $\mathfrak{D}(1/3)$. The group $\mathfrak{D}(1/3)$ is generated by two elements a, b given by recursion

$$a = \sigma(a^{-1}, ab), \quad b = (a, 1).$$

Consequently, the group $\mathfrak{K}(11, \mathfrak{D}(1/3))$ is generated by

$$a_1 = \sigma(a_3^{-1}, a_3 b_3), \quad b_1 = (a_3, 1)$$

and

$$a_2 = (1, a_1), \quad b_2 = (1, b_1).$$
$$a_3 = (1, a_2), \quad b_3 = (1, b_2)$$

Let us post-conjugate the wreath recursion by $(a_3, 1)$. We get then the wreath recursion

$$\begin{aligned} a_1 &= \sigma(1, b_3), & b_1 &= (a_3, 1), \\ a_2 &= (1, a_1), & b_2 &= (1, b_1), \\ a_3 &= (1, a_2), & b_3 &= (1, b_2), \end{aligned}$$

thus, $\mathfrak{K}(11, \mathfrak{D}(1/3)) = \mathfrak{K}(11011)$. Figure 12 shows the limit space of $\mathfrak{K}(11011)$ (in the middle). The left hand side picture shows the Julia set of the "rabbit" polynomial (i.e., the limit space of $\mathfrak{K}(11)$) and the right hand side picture shows the "basilica", i.e., the Julia set of $z^2 - 1$, which is homeomorphic to the limit space of $\mathfrak{K}(1) \cong \mathfrak{D}(1/3)$.

To get the middle picture, one has to replace the boundary of every Fatou component of the "rabbit" by a copy of "basilica". On the level of iterated monodromy groups it corresponds to replacing the generators of $\mathfrak{K}(11)$ by products $a_1 b_1, a_2 b_2, a_3 b_3$. Every pair a_i, b_i generates a group isomorphic to IMG $\left(z^2 - 1\right)$. Hence, we have replaced cyclic groups $\langle a_1 b_1 \rangle, \langle a_2 b_2 \rangle, \langle a_3 b_3 \rangle$ (which are self-similar over subsets of X^3) by copies of the Basilica group.

6. PLANE-FILLING CURVES

6.1. **Peano curve.** The original paper of Giuseppe Peano [Pea90] is amazingly close to our approach and can be translated into the language of self-similar groups in a very straightforward way.

Here is a piece of his paper with the definition of a continuous map $T \mapsto (X, Y)$ from the segment $[0, 1]$ onto the square $[0, 1]^2$.

> Adoptons pour base de numération le nombre 3; appelons *chiffre* chacun des nombres $0, 1, 2$; et considèrons une suite illimitée de chiffres $a_1, a_2, \ldots$ que nous écrirons
> $$T = 0, a_1 a_2 a_3 \ldots.$$
> (Pour ce moment, T est seulement une suite de chiffres).
>   Si a est un chiffre, désignons par $\mathbf{k}a$ le chiffre $2 - a$, *complementaire* de a;[...]
>   Faisons correspondre à la suite T les deux suites
> $$X = 0, b_1 b_2 b_3 \ldots, Y = 0, c_1 c_2 c_3 \ldots,$$
> oú les chiffres b et c sont donnés par les rélations
> $$b_1 = a_1, c_1 = \mathbf{k}^{a_1} a_2, b_2 = \mathbf{k}^{a_2} a_3, c_2 = \mathbf{k}^{a_1 + a_3} a_4, b_3 = \mathbf{k}^{a_2 + a_4} a_5, \ldots$$
> $$b_n = \mathbf{k}^{a_2 + a_4 + \cdots + a_{2n-2}} a_{2n-1}, c_n = \mathbf{k}^{a_1 + a_3 + \cdots + a_{2n-1}} a_{2n}.$$
> [...] On peut aussi écrire ces rélations sous la forme:
> $$a_1 = b_1, a_2 = \mathbf{k}^{b_1} c_1, a_3 = \mathbf{k}^{c_1} b_2, a_4 = \mathbf{k}^{b_1 + b_2} c_2, \ldots,$$
> $$a_{2n-1} = \mathbf{k}^{c_1 + c_2 + \cdots + c_{n-1}} b_n, a_{2n} = \mathbf{k}^{b_1 + b_2 + \cdots + b_n} c_n.$$

After this he shows that the defined map $0.a_1 a_2 \ldots \mapsto (0.b_1 b_2 \ldots; 0.c_1 c_2 \ldots)$ agrees with the identifications of ternary expansions of real numbers, from which it follows easily that this map is continuous. Surjectivity follows directly from the construction.

The following theorem is essentially just a reformulation of the Peano's paper. The only new ingredient is looking at it in terms of the dual picture of self-similar contracting groups.

Theorem 6.1. *Let G be the group generated by the following automorphisms of the ternary rooted tree $\{0, 1, 2\}^*$*
$$B = (012)(1, 1, C),$$
$$C = (B, B, B),$$
$$k_0 = (02)(k_1, k_1, k_1),$$
$$k_1 = (k_0, k_0, k_0),$$
$$A = (012)(k_0, k_0, A).$$

Then the limit space of G is homeomorphic to the square $[0, 1] \times [0, 1]$. The cyclic subgroup $\langle A \rangle < G$ is self-similar (after a change of the basis of the self-similarity bimodule) and the inclusion $\langle A \rangle < G$ induces a surjective continuous map from the circle $\mathcal{J}_{\langle A \rangle}$ onto the square $\mathcal{J}_G$.

Proof. Let us identify a sequence $b_1 c_1 b_2 c_2 \ldots \in \{0, 1, 2\}^\omega$ with the pair
$$w = (b_1 b_2 \ldots, c_1 c_2 \ldots) \in \{0, 1, 2\}^\omega \times \{0, 1, 2\}^\omega.$$

It follows from the definitions that B and k_0 act only on the first coordinate of w and C and k_1 act only on the second coordinate. The transformations B and C act as the usual triadic adding machine on the respective coordinates of w. The

actions of k_0 and k_1 on the respective coordinates of w coincide with the action of the transformation

$$\mathbf{k} = (02)(\mathbf{k}, \mathbf{k}, \mathbf{k}),$$

which changes every letter a to $2 - a$.

It follows that the group G_0 generated by B, C, k_0, k_1 is isomorphic to the direct product of two infinite dihedral groups. The map

$$b_1 c_1 b_2 c_2 \ldots \mapsto \left(\sum_{n=1}^{\infty} b_n \cdot 3^{n-1}, \sum_{n=1}^{\infty} c_n \cdot 3^{n-1} \right)$$

is a homeomorphism between the boundary of the rooted tree and the direct square $\mathbb{Z}_3^2$ of the ring of triadic integers. It is easy to see that the homeomorphism conjugates B, C, k_0 and k_1 with the affine transformations

$$(x, y) \mapsto (x + 1, y), (x, y + 1), (-1 - x, y), (x, -1 - y),$$

respectively (since changing every digit a to $2 - a$ is equivalent to subtracting the triadic number from $\ldots 222 = -1$).

It follows now from the general theory that the limit G_0-space $\mathcal{X}_{G_0}$ is homeomorphic to $\mathbb{R}^2$ with the right action of the generators given by the same formulae as their action on $\mathbb{Z}_3^2$. A sequence $\ldots x_2 y_2 x_1 y_1 \in \mathsf{X}^{-\omega}$ encodes the point

$$\left(\sum_{n=1}^{\infty} x_n \cdot 3^{-n}, \sum_{n=1}^{\infty} y_n \cdot 3^{-n} \right) \in \mathbb{R}^2 = \mathcal{X}_{G_0}$$

Consequently, the limit space $\mathcal{J}_{G_0}$ is the quotient of the plane by the affine action of G_0 on $\mathbb{R}^2$, i.e., a square, which can be identified for instance with the fundamental square $[0, 1/2] \times [0, 1/2]$ of the action.

Let us pass now to a different basis of the self-similarity bimodule. Denote

$$\widetilde{0} = 0, \quad \widetilde{1} = 1 \cdot k_0, \quad \widetilde{2} = 2,$$

i.e., we replace every digit a_i by $\widetilde{a}_i = a_i \cdot k_0^{a_i}$ and hence post-conjugate the wreath recursion by $(1, k_0, 1)$.

Then the generator A of G becomes the usual adding machine:

$$A = (\widetilde{012})(1, 1, A).$$

The cyclic group $\langle A \rangle$ is a self-similar subgroup of G, hence the inclusion $\langle A \rangle < G$ induces a surjective continuous map of the limit spaces.

It remains to prove that adding A to the generating set does not change the limit space, i.e., that the inclusion $G_0 < G$ induces a homeomorphism of the limit spaces.

Let us show that for every n and every $v \in \{0, 1, 2\}^n$ the distance between v and $A(v)$ in the Schreier graph of the action of G_1 on $\{0, 1, 2\}^n$ is not more than 2. This will show that the asymptotic equivalences defined by G and G_1 coincide.

But it is easy to see from the recursions

$$Bk_1 = (012)(k_0, k_0, Ck_0)$$
$$Ck_0 = (02)(Bk_1, Bk_1, Bk_1)$$
$$A = (012)(k_0, k_0, A)$$

that if $v = \underbrace{22\ldots2}_{n}\, u$ (for $n \geq 0$) and the first letter of u is not 2, then

$$A(v) = \begin{cases} Bk_1(v) & \text{if } n \text{ is even,} \\ Ck_0(v) & \text{if } n \text{ is odd.} \end{cases}$$

Hence the distance between v and $A(v)$ in the Schreier graph of G_0 is at most 2. $\qquad\square$

Let us look more carefully at the map $F : \mathcal{J}_{\langle A \rangle} \longrightarrow \mathcal{J}_G$ and show that it coincides with the Peano curve. More precisely, the map F is equal to the composition of the Peano curve with the map $[0,1]^2 \longrightarrow [0,1/2]^2$ factoring the square by the action of the Klein's group generated by reflections with respect to the medians of the square.

If $\ldots a_3 a_2 a_1 \in \{0,1,2\}^{-\omega}$ is a sequence representing a point ξ of $\mathcal{J}_{\langle A \rangle}$ (we have $\xi = \sum_{n=1}^{\infty} a_n \cdot 3^{-n} \in \mathbb{R}/\mathbb{Z}$), then the point $F(\xi)$ is represented in $\mathcal{J}_{G_0} = \mathcal{J}_G$ by the sequence

(7) $$\ldots a_3 \cdot k_0^{a_3} \otimes a_2 \cdot k_0^{a_2} \otimes a_1 \cdot k_0^{a_1}.$$

Proposition 6.2. *The sequence* $\ldots a_3 \cdot k_0^{a_3} \otimes a_2 \cdot k_0^{a_2} \otimes a_1 \cdot k_0^{a_1}$ *is equivalent to the sequence*

$$\ldots \mathbf{k}^{a_1+a_3+\cdots+a_{2n-1}}(a_{2n}) \otimes \mathbf{k}^{a_2+a_4+\cdots+a_{2n-2}}(a_{2n-1}) \otimes \cdots$$
$$\otimes \mathbf{k}^{a_2+a_4}(a_5) \otimes \mathbf{k}^{a_1+a_3}(a_4) \otimes \mathbf{k}^{a_2}(a_3) \otimes \mathbf{k}^{a_1}(a_2) \otimes a_1.$$

Proof. We have for every $a \in \{0,1,2\}$ and $n \in \mathbb{N}$

$$a \cdot k_0^n = k_1^n \cdot a \qquad a \cdot k_1^n = k_0^n \cdot \mathbf{k}^n(a).$$

This follows directly from the recurrent definition of k_0 and k_1.

Using this we can move all $k_0^{a_i}$ in the sequence (7) away to the left. Let us take the ending of length $2n$ of the sequence. We have then the following equalities

$$a_{2n} \cdot k_0^{a_{2n}} \otimes a_{2n-1} \cdot k_0^{a_{2n-1}} \otimes \cdots \otimes a_2 \cdot k_0^{a_2} \otimes a_1 \cdot k_0^{a_1}$$
$$= k_1^{a_{2n}} \cdot a_{2n} \cdot k_1^{a_{2n-1}} \otimes a_{2n-1} \cdot k_1^{a_{2n-2}} \otimes \cdots \otimes a_2 \cdot k_1^{a_1} \otimes a_1$$
$$= k_1^{a_{2n}} k_0^{a_{2n-1}} \cdot \mathbf{k}^{a_{2n-1}}(a_{2n}) \cdot k_0^{a_{2n-2}} \otimes \mathbf{k}^{a_{2n-2}}(a_{2n-1}) \cdot k_0^{a_{2n-3}} \otimes \cdots \otimes \mathbf{k}^{a_1}(a_2) \otimes a_1$$
$$= \ldots = k_1^{a_{2n}} k_0^{a_{2n-1}} k_1^{a_{2n-2}} \cdots k_1^{a_2} k_0^{a_1} \cdot$$
$$\mathbf{k}^{a_{2n-1}+a_{2n-3}+\cdots+a_1}(a_{2n}) \otimes \mathbf{k}^{a_{2n-2}+a_{2n-4}+\cdots+a_2}(a_{2n-2}) \otimes \cdots \otimes \mathbf{k}^{a_1}(a_2) \otimes a_1$$

and since the element $k_1^{a_{2n}} k_0^{a_{2n-1}} k_1^{a_{2n-2}} \cdots k_1^{a_2} k_0^{a_1}$ belongs to a finite group $\langle k_0, k_1 \rangle = \langle k_0 \rangle \times \langle k_1 \rangle$, we get the necessary equivalence of sequences. $\qquad\square$

6.2. **Mating and dragon curve.** We are going to give here an interpretation of the results from a paper by John Milnor [Mil04] where he describes and computes a surjective continuous map from a dendrite Julia set onto the complex sphere coming from mating of two polynomials.

Let $f(z) = z^2 + c$, where $c = c_{1/4} \approx -0.2282 + 1.1151i$ be the quadratic polynomial such that $f^2(c)$ is a fixed point of f. The parameter c is a root of the polynomial $x^3 + 2x^2 + 2x + 2$.

Let us take the basepoint $t = +\infty$ and the connecting paths ℓ_0, ℓ_1 to be trivial and the upper semicircle at infinity, respectively. Take the generators of the IMG (f) to be small simple loops α, β, γ going in the positive direction around the points $c, c^2 + c$ and $(c^2 + c)^2 + c$, respectively. We connect the loops to the basepoint by
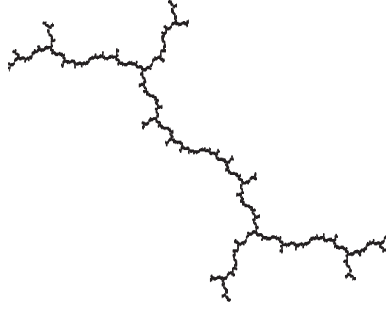
FIGURE 13. Julia set of $z^2 - 0.2282\ldots + 1.1151\ldots i$

external rays at the angles $1/4, 1/2$ and 0, respectively and by the shortest arcs from $+\infty$ to the ray inside the upper semicircle. We get the following wreath recursions (after computing the generators of $\mathfrak{D}\left(1/4\right)$)

$$\alpha = \sigma(\beta^{-1}\alpha^{-1}, \alpha\beta),$$
$$\beta = (\alpha, 1),$$
$$\gamma = (\gamma, \beta).$$

It is easy to see that the elements $\alpha, \beta, \gamma \in \mathrm{IMG}\,(f)$ are of order 2. Note that the loop $\gamma\alpha\beta$ (going in the positive direction around the post-critical set) is the binary adding machine

$$\gamma\alpha\beta = \sigma(1, \gamma\alpha\beta).$$

Let us mate the polynomial f with itself, i.e., take two copies of the plane with the action of f on both of them and glue them together along the circle at infinity identifying the points of the circles which are symmetric with respect to the real axis.

In terms of the category of self-similar groups this mating operation can be interpreted as the amalgam of the embeddings $a \mapsto \gamma\alpha\beta$ and $a \mapsto (\gamma\alpha\beta)^{-1}$ of the adding machine $a = \sigma(1, a)$ into $\mathrm{IMG}\,(f)$, i.e., the universal object H making the diagram

$$
\begin{array}{ccc}
\mathbb{Z} & \longrightarrow & \mathrm{IMG}\,(f) \\
\downarrow & & \downarrow \\
\mathrm{IMG}\,(f) & \longrightarrow & H
\end{array}
$$

commutative (more precisely, we have to define the embedding of the adding machine into $\mathrm{IMG}\,(f)$ as a homomorphism of the permutational bimodules, i.e., also to identify the respective bases).

Let us compute the iterated monodromy group of the mating. Choose in one plane the generators α, β, γ and the basis of the bimodule ℓ_0, ℓ_1, as before. Let α', β', γ' and ℓ_0', ℓ_1' be the paths defined by the same way in the second plane. Note that $\gamma\alpha\beta = (\gamma'\alpha'\beta')^{-1} = \beta'\alpha'\gamma'$. Then we have $\ell_0 = \ell_0'$ and $\ell_1 = \ell_1' \cdot \beta'\alpha'\gamma'$.

The action of α', β', γ' defined with respect to the basis ℓ_0', ℓ_1' is the same as for α, β, γ with respect to ℓ_0, ℓ_1 (with primes added everywhere). Hence, the action of α', β', γ' associated with the basis ℓ_0, ℓ_1 is obtained by pos-conjugating the recursion

by $(1, \beta'\alpha'\gamma')$. Hence, we get

$$\alpha' = \sigma(\gamma', \gamma'),$$
$$\beta' = (\alpha', 1),$$
$$\gamma' = (\gamma', \gamma'\alpha'\beta'\alpha'\gamma').$$

Thus the group $\mathrm{IMG}\,(F)$ is generated by $\alpha, \beta, \gamma, \alpha', \beta', \gamma'$, which are given by the above recursion.

We have $\gamma\alpha\beta\gamma'\alpha'\beta' = 1$, hence we can eliminate $\alpha' = \gamma'\beta\alpha\gamma\beta' = \beta'\gamma\alpha\beta\gamma'$ from the generating set.

Consider now only the paths which do not intersect the external rays at angles $1/2$ and 0 (we want to avoid *Thurston obstructions*, but this is not very important here; we are going just to pass to a smaller subgroup). We change therefore the basis to $\ell_0, \ell_1 \cdot \beta\alpha$, conjugating the recursion by $(1, \beta\alpha)$:

$$\alpha = \sigma,$$
$$\beta = (\alpha, 1), \qquad \beta' = (\gamma'\beta\alpha\gamma\beta', 1),$$
$$\gamma = (\gamma, \alpha\beta\alpha), \quad \gamma' = (\gamma', \gamma\beta'\gamma),$$

then change our generating set to $a = \alpha, b = \alpha\beta\alpha, c = \gamma, b' = \gamma\beta'\gamma, c' = \gamma'$:

$$a = \sigma,$$
$$b = (1, a), \quad b' = (cc'abb', 1),$$
$$c = (c, b), \quad c' = (c', b'),$$

and consider the subgroup generated by $a, B = bb', C = cc'$ which are given then by

$$a = \sigma,$$
$$B = (CaB, a),$$
$$C = (C, B).$$

Proposition 6.3. *The simplicial Schreier graphs of* $\mathrm{IMG}\,(F) = \langle a, b, c, b', c' \rangle$ *and* $G = \langle a, B, C \rangle$ *coincide. Namely, if* $b(v) \neq v$, $b'(v) \neq v$, $c(v) \neq v$, *or* $c'(v) \neq v$, *then* $b(v) = B(v)$, $b'(v) = B(v)$, $c(v) = C(v)$ *and* $c'(v) = C(v)$, *respectively.*

Proof. See a part of the Moore diagram defining a, b, c, b', c' on the top part of Figure 14. The bottom part of the figure shows a smaller part of the Moore diagram. Tracking the paths in these diagrams, one can see that the statement of the proposition is correct. $\square$

Corollary 6.4. *The inclusion* $G < \mathrm{IMG}\,(F)$ *induces a homeomorphism of the limit spaces. The inclusion* $\mathrm{IMG}\,(f) = \langle a, b, c \rangle < \mathrm{IMG}\,(F)$ *induces a surjective continuous map.*

We know that the group generated by a, B and C is isomorphic to the group of affine transformations of $\mathbb{C}$ of the form $z \mapsto \pm z + (m + ni)$, where $m + ni \in \mathbb{Z}[i]$, see 4.8.2. The generators act on the complex plane by the rules $z \cdot a = -z - i$, $z \cdot B = -z - i + 1$ and $z \cdot C = -z$ and the associated virtual endomorphism is induced by the linear map $z \mapsto z/(1 - i)$ on $\mathbb{C}$.

Hence the limit dynamical system of $G = \langle a, B, C \rangle$ and of $\mathrm{IMG}\,(F)$ (as a topological dynamical system) is the Lattès example conjugate to the rational function
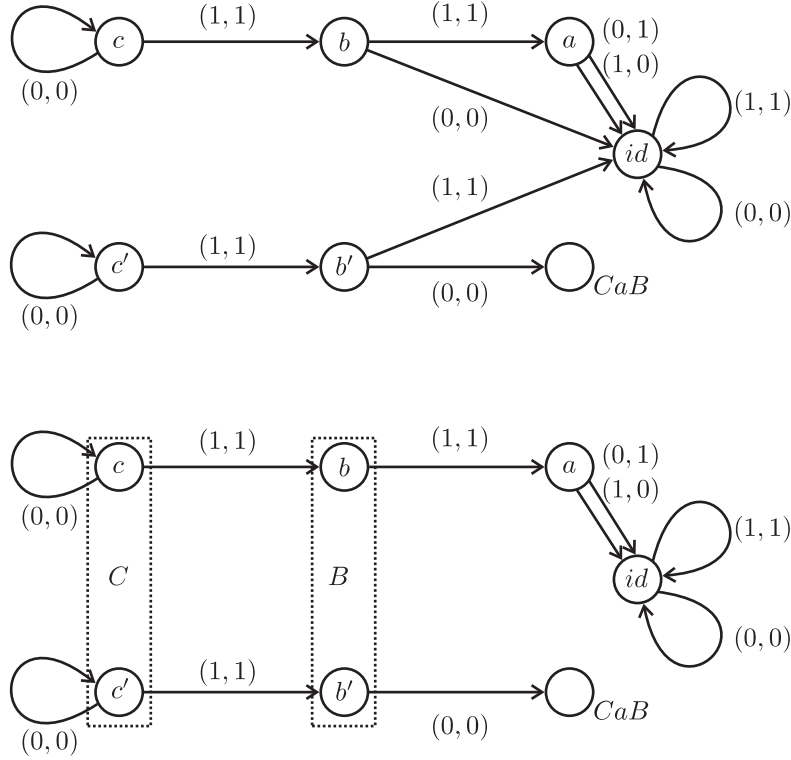
FIGURE 14

$\frac{i}{2}\left(z+z^{-1}\right)$, the limit space $\mathcal{J}_G = \mathcal{J}_{\mathrm{IMG}(F)}$ is the complex sphere and the limit G-space $\mathcal{X}_G$ is the complex plane $\mathbb{C}$ with the described affine action of G (the limit space $\mathcal{X}_{\mathrm{IMG}(F)}$ is more complicated).

Let us take the basepoint $\xi = (1-i)/4$ and take its orbit $\xi \cdot G$. We have $\xi \cdot a = (-1-3i)/4$, $\xi \cdot B = (3-3i)/4$ and $\xi \cdot C = (-1+i)/4$. Connect ξ to its images under the generators. We get three segments. The union Γ of the images of these segments is the left Cayley graph of G. See a part of it on Figure 15 (the highlighted edges will be explained later). The Schreier graph of the left action of G on the self-similarity bimodule coincides with the Cayley graph.

It follows from Proposition 6.3 that the Schreier graph of the action of $\mathrm{IMG}(f) = \langle a, b, c \rangle$ on X^n is a subgraph of the Schreier graph of the action of G. The Schreier graph of G is the quotient of the graph $\Gamma/(1-i)^n$ by the action of G.

It follows from the recurrent definition of a, b and c (see their Moore diagram on Figure 14) that the Schreier graph $\Gamma_n(\mathrm{IMG}(f))$ of the action of $\langle a, b, c \rangle$ on X^n can be constructed using the following recurrent construction (for $n \geq 3$).

Take two copies of the $\Gamma_{n-1}(\mathrm{IMG}(f))$, connect the vertices $00\ldots011 = 0^{n-3}11$ of the two copies by an edge and append 0 to the names of the vertices of the first copy and 1 to the names of the vertices of the second copy (so that the connected vertices are $00\ldots0110$ and $00\ldots0111$). The connecting edge corresponds to generator c. The obtained new graph is $\Gamma_n(\mathrm{IMG}(f))$.
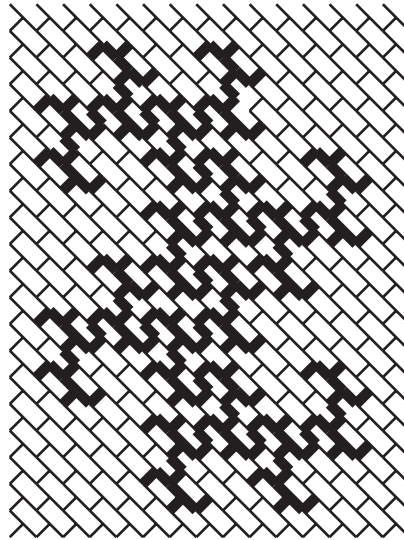
FIGURE 15. Schreier graph of IMG $\left(z^2 - 0.2282\ldots + 1.1151\ldots i\right)$

Note that it follows that the Schreier graphs of IMG (f) are trees. This reflects the fact that the limit space $\mathcal{J}_{\mathrm{IMG}(f)}$ is a dendrite (see Figure 13).

Using this rule one can find the edges of the Schreier graph of G which belong to $\Gamma_n(\mathrm{IMG}\,(f))$. Figure 15 shows these edges for $n = 8$ (one has however, to wrap it around the sphere, taking quotient of the picture by the action of the group G, if we assume that the figure shows the graph $\Gamma/(1-i)^8$).

The Schreier graphs of IMG (f) approximate the limit space $\mathcal{J}_{\mathrm{IMG}(f)}$, i.e., the Julia set of f. The embedding of the Schreier graph into the Schreier graph of G approximates the continuous map from the dendrite $\mathcal{J}_{\mathrm{IMG}(f)}$ onto the sphere $\mathcal{J}_G = \mathcal{J}_{\mathrm{IMG}(F)}$ induced by the embedding IMG $(f) \hookrightarrow$ IMG (F). For more detail see a similar example in the last section of [Nek05].

An interesting observation is the fact that the Schreier graphs of IMG (f) can be constructed using the classical paper folding procedure. We will describe this fact without proofs, which are easy inductive arguments based on the recurrent construction of the Schreier graphs.

Take a strip of paper and fold it n times, each time in the same way (say, put it horizontally and rotate the right half around the middle point by π in the positive direction). Then unfold it so that every bend is a right angle. See the top of Figure 16 where the resulting broken lines for $n = 2, \ldots, 7$ are shown. Properly rescaled lines will converge to the Heighway dragon curve (see [Edg90]).

Take now two copies of such a broken line (for the same n) and put them together in such a way that they have common endpoints and one is rotation by π of the other. We get a closed broken line of 2^{n+1} segments. The internal part of this line will consist of 2^{n-1} squares which are connected by corridors in the same way as the vertices of the Schreier graph of the action of IMG (f) on X^{n-1} are connected by edges. The lower part of Figure 16 shows the corridors and the corresponding Schreier graphs.
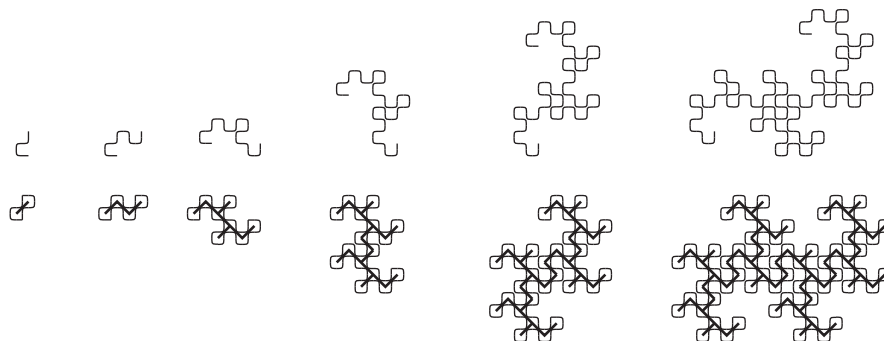
FIGURE 16. Paper folding and IMG (f)

There is a picture of the Schreier graph of IMG (f) on page 66 of the book of B. Mandelbrot [Man82] with the comment

> TWINDRAGON RIVER. After the streams near the source are erased (for legibility), the river tree of a twindragon looks like this.

B. Mandelbrot observes that many plane-filling curves go around plane-filling dendrites (*river trees*), which is particularly evident for the case of the paper-folding curve.

6.3. **Sierpiński curve.** Consider the group G generated by

$$a = \sigma, \quad b = (a, c), \quad c = (b, b).$$

We have seen in 4.8.3 that this group is isomorphic to the group of all symmetries of the square lattice. The orbifold $\mathcal{J}_G = \mathbb{R}^2/G$ is the rectangular isosceles triangle.

The Moore diagram on Figure 17 shows that the generator $x = (a, x)$ of the dihedral group $\langle a, x \rangle$ acts on every word either as a, or as b.

Therefore, the limit space of G coincides with the limit space of $G_1 = \langle a, b, c, x \rangle$ and the inclusion $\mathbb{D}_\infty = \langle a, x \rangle \hookrightarrow G_1$ induces a surjective continuous map of the limit spaces: from the real segment onto the triangle. An approximation of this map is shown on Figure 18. It is called *Sierpiński plane filling curve*, see [Man82].

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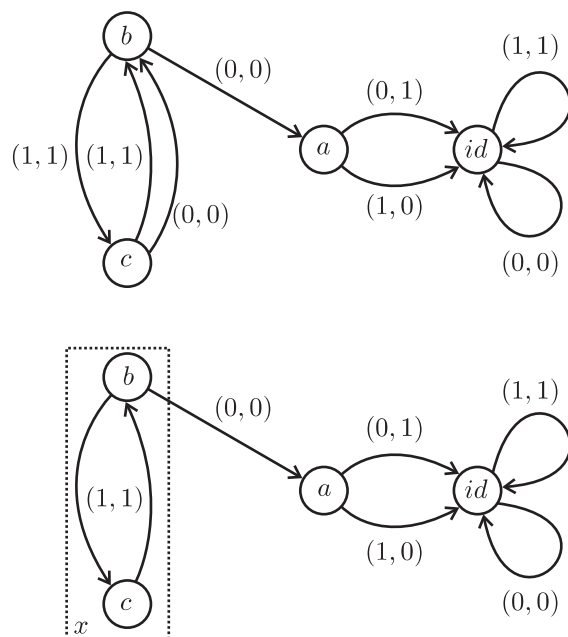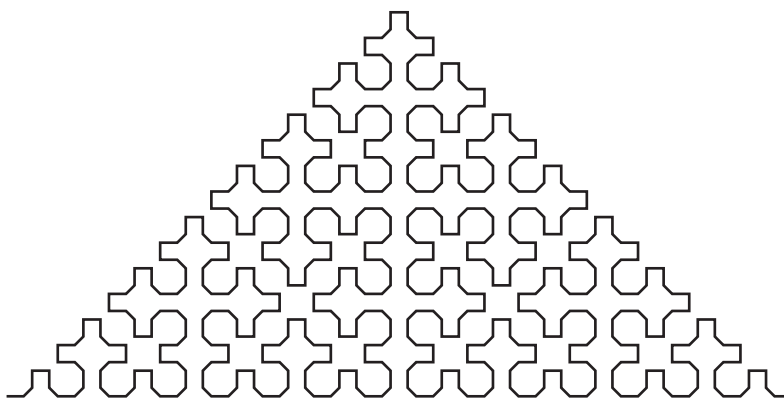
FIGURE 17

FIGURE 18. Sierpiński curve

[GŻ02b] ———, *Spectral properties of a torsion-free weakly branch group defined by a three state automaton*, Computational and Statistical Group Theory (Las Vegas, NV/Hoboken, NJ, 2001), Contemp. Math., vol. 298, Amer. Math. Soc., Providence, RI, 2002, pp. 57–82.

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