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Hyperbolic spaces from self-similar group actions

Volodymyr Nekrashevych

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1. Introduction

Self-similar group actions (or self-similar groups) have proved to be interesting mathematical objects from the point of view of group theory and from the point of view of many other fields of mathematics (operator algebras, holomorphic dynamics, automata theory, etc). See the works [BGN02, GNS00, Gri00, Sid98, BG00, Nek02a], where different aspects of self-similar groups are studied.

An important class of self-similar group actions are *contracting actions*. Contracting groups have many nice properties. For example, the word problem is solvable in a contracting group in a polynomial time [Nek]. The author has shown (see [Nek02b]) that a naturally defined topological space, called the *limit space*, is associated with every contracting self-similar action. This topological space is metrizable and finite-dimensional.

It was discovered later (see [Nek02a]) that with many topological dynamical systems (like iterations of a rational function) a self-similar group is associated. This self-similar group (called the *iterated monodromy group*) is often contracting and in fact contains all the essential dynamics of the original dynamical system. In particular, if a map is expanding, then its iterated monodromy group is contracting and the limit space of the iterated monodromy group is homeomorphic to the Julia set of the map.

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In this paper we show how to associate with every self-similar action a graph, which will be Gromov-hyperbolic, if the group action is contracting. The boundary of this graph is then homeomorphic to the limit space of the group action.

In some sense this result (together with the notion of iterated monodromy group) can be interpreted as a new entry to the "Sullivan dictionary" [Sul85], which shows an analogy between the holomorphic dynamics and groups acting on the hyperbolic space.

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2. Self-similar actions

Let X be a finite set (*alphabet*). By X^* we denote the set of all finite words over the alphabet X, including the empty one $\emptyset$. By $X^{-\omega}$ we denote the space of all infinite sequences to the left of the form $\ldots x_2 x_1$, with the topology of the infinite countable direct power of the discrete set X.

Definition 2.1. *A faithful action of a group G on the set X^* is said to be* self-similar *if for every $g \in G$ and every $x \in X$ there exist $h \in G$ such that*

$$g(xw) = g(x)h(w) \tag{1}$$

for every $w \in X^*$.

Iterating equation (1), we get that for every $v \in X^*$ there exists an element h such that $g(vw) = g(v)h(w)$. The element h is uniquely defined and is called *restriction of the element* g in the word v and is denoted $g|_v$.

We easily get the following properties of restrictions:

$$g|_{v_1 v_2} = g|_{v_1}|_{v_2} \qquad (g_1 g_2)|_v = \left(g_1|_{g_2(v)}\right)(g_2|_v). \tag{2}$$

Definition 2.2. *Let $P_n : (X^n)^* \longrightarrow X^*$ be the injective map, which carries the word*

$$(x_1, x_2, \ldots, x_n)(x_{n+1}, x_{n+2}, \ldots, x_{2n}) \ldots \\ \ldots (x_{(k-1)n+1}, x_{(k-1)n+2}, \ldots, x_{kn}) \in (X^n)^*$$

to the word $x_1 x_2 x_3 \ldots x_{kn} \in X^*$.

*The n*th tensor power *of a self-similar action is the action on the set $(X^n)^*$ obtained by conjugating the action on X^* by the map P_n.*

The nth power of a self-similar action obviously is also self-similar.

The adding machine. One of the most classical examples of a self-similar action is the "*adding machine*" defined in the following way. We put $X = \{0, 1\}$ and define a transformation a recurrently by the rules

$$\begin{aligned} a(0w) &= 1w \\ a(1w) &= 0 \cdot a(w) \end{aligned}$$

for all $w \in X^*$.

It is easy then to prove that if $a^n(x_1x_2\ldots x_m) = y_1y_2\ldots y_m$ then

$$y_1 + y_2 \cdot 2 + y_3 \cdot 2^2 + y_4 \cdot 2^3 + \ldots y_m \cdot 2^{m-1} = \left(x_1 + x_2 \cdot 2 + x_3 \cdot 2^2 + x_4 \cdot 2^3 + \ldots x_m \cdot 2^{m-1}\right) + n \pmod{2^m}.$$

3. Contracting actions and their limit spaces

Definition 3.1. *A self-similar action of a group G is called* contracting *if there exists a finite set $\mathcal{N} \subset G$ such that for every $g \in G$ there exists $k \in \mathbb{N}$ such that $g|_v \in \mathcal{N}$ for all the words $v \in X^*$ of length $\geq k$. The minimal set $\mathcal{N}$ with this property is called the* nucleus *of the self-similar action.*

As an example of a contracting action, one can take the adding machine action. Other examples of contracting actions include the Grigorchuk group and the iterated monodromy groups of post-critically finite rational functions.

Let us fix some contracting self-similar action of a group G on the set X^*.

Definition 3.2. *Two sequences $\ldots x_3x_2x_1, \ldots y_3y_2y_1 \in X^{-\omega}$ are said to be* asymptotically equivalent *with respect to the action of the group G, if there exist a finite set $K \subset G$ and a sequence $g_k \in K, k \in \mathbb{N}$ such that*

$$g_k(x_kx_{k-1}\ldots x_2x_1) = y_ky_{k-1}\ldots y_2y_1$$

for every $k \in \mathbb{N}$.

Definition 3.3. *The* limit space *of the self-similar action (denoted $\mathcal{J}_G$) is the quotient of the topological space $X^{-\omega}$ by the asymptotic equivalence relation.*

Example. In the case of the adding machine action of $\mathbb{Z}$ one can prove (see [Nek02b]) that two sequences are asymptotically equivalent if and only if

$$\sum_{n=1}^{\infty} x_n \cdot 2^{-n} = \sum_{n=1}^{\infty} y_n \cdot 2^{-n} \pmod 1.$$

Consequently, the limit space $\mathcal{J}_{\mathbb{Z}}$ is the circle $\mathbb{R}/\mathbb{Z}$.

We have the following properties of the limit spaces, which are proved in [Nek02b].

Proposition 3.1. *The limit space $\mathcal{J}_G$ is metrizable and has topological dimension $\leq |\mathcal{N}| - 1$, where $\mathcal{N}$ is the nucleus of the action.*

4. The limit space as a hyperbolic boundary

Definition 4.1. *Let G be a finitely generated group with a self-similar action on X^*. For a given finite generating system S of the group G we define the* self-similarity graph $\Sigma(G, S)$ *as the graph with the set of vertices X^* and two vertices $v_1, v_2 \in X^*$ belonging to a common edge if and only if either $v_i = xv_j$ for some $x \in X$ (the* vertical *edges) or $s(v_i) = v_j$ for some $s \in S$ (the* horizontal *edges), where $\{i, j\} = \{1, 2\}$.*

As an example, see a part of the self-similarity graph of the adding machine on Figure 1.

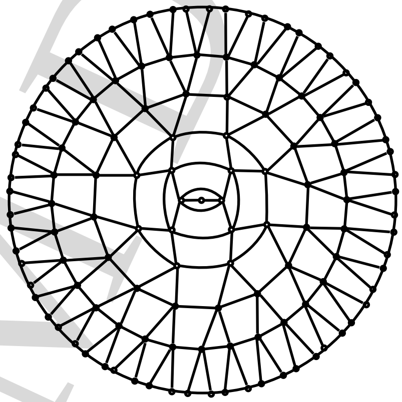

Figure 1: The self-similarity graph of the adding machine

If all restriction of the elements of the generating set S also belong to S, then the self-similarity graph $\Sigma(G, S)$ is an *augmented tree* in sense of V. Kaimanovich (see [Kai03]).

We have a natural metric on the set of the vertices of any graph. The distance between two vertices in this metric is equal to the number of edges in the shortest path connecting them.

The definition of the self-similarity graph depends on the choice of the generating set S. We will use the classical notion of quasi-isometry in order to make it more canonical.

Definition 4.2. *Two metric spaces $\mathfrak{X}$ and $\mathfrak{Y}$ are said to be* quasi-isometric *if there exists a map (which is called then a* quasi-isometry*) $f : \mathfrak{X} \longrightarrow \mathfrak{Y}$ and positive constants L, C such that*

(i)

$$L^{-1}d_{\mathfrak{X}}(x_1, x_2) - C < d_{\mathfrak{Y}}(f(x_1), f(x_2)) < Ld_{\mathfrak{X}}(x_1, x_2) + C,$$

for all $x_1, x_2 \in \mathfrak{X}$ and

(ii) for every $y \in \mathfrak{Y}$ there exists $x \in \mathfrak{X}$ such that $d_{\mathfrak{Y}}(y, f(x)) < C$.

Lemma 4.1. *Let G be a group with a self-similar action, and let ϕ be the associated virtual endomorphism.*

1. *The self-similarity graphs $\Sigma(G, S_1)$ and $\Sigma(G, S_2)$, where S_1, S_2 are two different finite generating sets of the group G, are quasi-isometric.*

2. *The self-similarity graph of the nth power of the self-similar action is quasi-isometric to the self-similarity graph of the original action.*

Proof. 1) The identical map on the set of vertices $\Sigma(G, S_1) \longrightarrow \Sigma(G, S_2)$ is a quasi-isometry. The constant L is any number such that the length of every element of one of the generating sets has length less than L with respect to the other generating set. The constant C can be any positive number.

2) The set of vertices of the nth power is equal to $\{\emptyset\} \cup X^n \cup X^{2n} \cup X^{3n} \cup \ldots$. Let $F : \Sigma_n(G, S) \longrightarrow \Sigma(G, S)$ be the natural inclusion of the vertex sets.

It is easy to see that $d(F(u), F(v)) \leq n \cdot d(u, v)$ and $d(F(u), F(v)) \geq d(u, v)$ for all $u, v \in \Sigma(G, S)$.

For every vertex $v = x_1x_2 \ldots x_m \in X^*$ of the graph $\Sigma(G, S)$ there exists a vertex $x_rx_{r+1} \ldots x_m$ belonging to the vertex set of the graph $\Sigma_n(G, S)$, which is at the distance less than n from v (one must take r to be the minimal number, such that $m - r + 1$ is divisible by n). So the map F satisfies both conditions Definition 4.2. $\square$

Let us recall the definition of Gromov-hyperbolic metric spaces [Gro87].

Let $\mathfrak{X}$ be a metric space with the metric $d(\cdot, \cdot)$. The *Gromov product* of two points $x, y \in \mathfrak{X}$ with respect to the basepoint $x_0 \in \mathfrak{X}$ is the number

$$\langle x \cdot y \rangle = \langle x \cdot y \rangle_{x_0} = \frac{1}{2}\left(d\left(x, x_0\right) + d\left(y, x_0\right) - d\left(x, y\right)\right).$$

Definition 4.3. *A metric space $\mathfrak{X}$ is said to be* Gromov-hyperbolic *(see [Gro87]) if there exists $\delta > 0$ such that the inequality*

$$\langle x \cdot y \rangle \geq \min \left(\langle x \cdot z \rangle , \langle y \cdot z \rangle \right) - \delta \tag{3}$$

holds for all $x, y, z \in \mathfrak{X}$.

If a proper geodesic metric space (for instance a graph) is quasi-isometric to a hyperbolic space, then it is also hyperbolic. For proofs of the mentioned facts and for other properties of the hyperbolic spaces and groups look one of the books [Gro87, CDP90, GH90].

Theorem 4.2. *If the action of a finitely-generated group G is contracting then the self-similarity graph $\Sigma(G, S)$ is a Gromov-hyperbolic space.*

Proof. It is sufficient to prove that some quasi-isometric graph is hyperbolic. Therefore, we can change by statement (1) of Lemma 4.1 the set of generators S so that it will contain all the restrictions of its elements and that there exists $N \in \mathbb{N}$ such that for every element $g \in G$ of length ≤ 4 and any word $x_1 x_2 \ldots x_N \in X^*$, the restriction $g|_{x_1 x_2 \ldots x_N}$ belongs to S. Then the length of any restriction of an element $g \in G$ is not greater then the length of g.

It is sufficient, by statement (3) of Lemma 4.1, to prove that the self-similarity graph $\Sigma_N(G, S)$ of the Nth power of the action is Gromov-hyperbolic.

Let us prove the following lemma.

Lemma 4.3. *Any two vertices w_1, w_2 of the graph $\Sigma_N(G, S)$ can be written in the form $w_1 = a_1 a_2 \ldots a_n w, w_2 = b_1 b_2 \ldots b_m g(w)$, where $a_i, b_i \in X^N$, $w \in \left(X^N\right)^*$, $g \in G$, $l(g) \leq 4$ and $d(w_1, w_2) = n + m + l(g)$.*

Then the Gromov product $\langle w_1 \cdot w_2 \rangle$ with respect to the basepoint $\emptyset$ is equal to $|w| - l(g)/2$.

Proof. Let $v_1 = w_1, v_2, \ldots v_k = w_2$ be the consecutive vertices of the shortest path connecting the vertices w_1 and w_2. Then every v_{i+1} is obtained from v_i by application of one of the following procedures:

1. deletion of the first letter $a \in X^N$ in v_i (*descending edges*);

2. adding a letter $a \in X^N$ to the beginning of v_i (*ascending edges*);

3. application of an element of S to v_i (*horizontal edges*);

If the path has three consecutive vertices v_i, v_{i+1}, v_{i+2} such that $v_{i+1} = a v_i$, $a \in X^N$ and $v_{i+2} = s(v_{i+1})$ for $s \in S$ then $v_{i+2} =$

$bs'(v_i)$, where $b = s(a) \in X^N$ and $s' = s|_a \in S$. We replace the segment $\{v_i, v_{i+1}, v_{i+2}\}$ of the path by the segment $\{v_i, s'(v_i), bs'(v_i) = v_{i+2}\}$.

If the path has three consecutive vertices v_i, v_{i+1}, v_{i+2} such that $v_{i+1} = s(v_i)$ for $s \in S$ and $v_{i+1} = av_{i+2}$ then $v_i = s^{-1}(av_{i+2}) = bs'(v_{i+2})$, where $b = s^{-1}(a) \in X^N$ and $s' = s^{-1}|_a \in S$. Then we replace the segment $\{v_i, v_{i+1}, v_{i+2}\}$ of the path by the segment $\{v_i = bs'(v_{i+2}), s'(v_{i+2}), v_{i+2}\}$.

Let us perform these replacements as many times as possible. Then we will not change the length of the path, so each time we will get a geodesic path connecting the vertices w_1, w_2. Note that a geodesic path can not have a descending edge next after an ascending one. Therefore, eventually after a finite number of replacements we will get a geodesic path in which we have at first only descending, then horizontal and then only ascending edges. Then $w_1 = a_1 a_2 \dots a_n w, w_2 = b_1 b_2 \dots b_m g(w)$, with $a_i, b_i \in X^N$, $w \in \left(X^N\right)^*$, $g \in G$, and $d(w_1, w_2) = n + m + l(g)$.

Suppose that $l(g) > 4$. Let $w = aw'$, $a \in X^N$ and denote $b = g(a)$ and $h = g|_a$. Then by the choice of the number N we have $l\,(h) \leq l(g) - 3$. Since $w_1 = a_1 a_2 \dots a_n a w'$ and $w_2 = b_1 b_2 \dots b_m b h(w')$, we have $d(w_1, w_2) \leq n + 1 + m + 1 + l(h) \leq n + m + l(g) - 1$, which contradicts to the fact that the original path was the shortest one.

We have

$$\langle w_1 \cdot w_2 \rangle = \frac{1}{2}\left(n + |w| + m + |w| - (n + m + l(g))\right) = |w| - \frac{l(g)}{2}.$$

□

Let us take three points w_1, w_2, w_3. We can write them by Lemma 4.3 as

$$w_1 = a_1 a_2 \dots a_n w, \qquad w_2 = b_1 b_2 \dots b_m g_1(w)$$

and

$$w_2 = b_1 b_2 \dots b_p u, \qquad w_3 = c_1 c_2 \dots c_q g_2(u),$$

where $a_i, b_i, c_i \in X^N$, $g_1, g_2 \in G$, $l(g_1), l(g_2) \leq 4$ and

$$\langle w_1 \cdot w_2 \rangle = |w| - l(g_1)/2, \qquad \langle w_2 \cdot w_3 \rangle = |u| - l(g_2)/2.$$

We can assume that $p < m$. Then $|u| > |w|$, so we can write $u = vw$ for some $v \in (X^N)^*$. Then $w_3 = c_1 c_2 \dots c_q g_2(v) h(w)$, where $h = g_2|_v$. We have $l(h) \leq l(g) \leq 4$ and $d(w_1, w_3) \leq n + l(h) + q + |v|$, hence

$$\langle w_1 \cdot w_3 \rangle = \frac{1}{2}(n + |w| + q + |v| + |w| - d(w_1, w_3)) \geq |w| - l(h)/2 \geq |w| - 2.$$

Finally, $\min(\langle w_1 \cdot w_2\rangle, \langle w_2 \cdot w_3\rangle) \le \langle w_1 \cdot w_2\rangle \le |w|$, so

$$\langle w_1 \cdot w_3\rangle \ge \min(\langle w_1 \cdot w_2\rangle, \langle w_2 \cdot w_3\rangle) - 2,$$

and the graph $\Sigma_N(G,S)$ is 2-hyperbolic. $\square$

Let $\mathfrak{X}$ be a hyperbolic space. We say that a sequence $\{x_n\}$ of points of $\mathfrak{X}$ *converges to the infinity* if the Gromov product $\langle x_n \cdot x_m\rangle$ trends to infinity when $m, n \to \infty$. This definition does not depend on the choice of the basepoint. We say that two sequences $\{x_n\}$ and $\{y_n\}$, convergent to the infinity, are *equivalent* if $\lim_{n,m\to\infty}\langle x_n \cdot y_m\rangle = \infty$.

The set of the equivalence classes of the sequences convergent to the infinity in the space $\mathfrak{X}$ is called the *hyperbolic boundary* of the space $\mathfrak{X}$ and is denoted $\partial\mathfrak{X}$. If a sequence $\{x_n\}$ converges to the infinity, then its *limit* is the equivalence class $a \in \partial\mathfrak{X}$, to which belongs $\{x_n\}$ and we say that $\{x_n\}$ *converges to* a.

If $a, b \in \partial\mathfrak{X}$ are two points of the boundary, then their *Gromov product* is defined as

$$\langle a \cdot b\rangle = \sup_{\{x_n\}\in a, \{y_m\}\in b} \liminf_{m,n\to\infty} \langle x_n \cdot y_m\rangle.$$

For every $r > 0$ define

$$V_r = \{(a,b) \in \partial\mathfrak{X} \times \partial\mathfrak{X} : \langle a \cdot b\rangle \ge r\}.$$

Then the set $\{V_r : r \ge 0\}$ is a fundamental neighborhood basis of a uniform structure on $\partial\mathfrak{X}$ (see [Bou71] for the definition of a uniform structure and [GH90] for proofs). We introduce on the boundary $\partial\mathfrak{X}$ the topology, defined by this uniform structure.

Theorem 4.4. *The limit space $\mathcal{J}_G$ of a contracting action of a finitely generated group G is homeomorphic to the hyperbolic boundary $\partial\Sigma(G,S)$ of the self-similarity graph $\Sigma(G,S)$. Moreover, there exists a homeomorphism $F : \mathcal{J}_G \longrightarrow \partial\Sigma(G,S)$, such that $D = F \circ \pi$, were $\pi : X^{-\omega} \longrightarrow \mathcal{J}_G$ is the canonical projection and $D : X^{-\omega} \longrightarrow \partial\Sigma(G,S)$ carries every sequence $\ldots x_2x_1 \in X^{-\omega}$ to its limit*

$$\lim_{n\to\infty} x_n x_{n-1} \ldots x_1 \in \partial\Sigma(G,S).$$

Sketch of the proof. We will need the following well known result (see [CDP90] Theorem 2.2).

Lemma 4.5. *Let $\mathfrak{X}_1, \mathfrak{X}_2$ be proper geodesic hyperbolic spaces and let $f_1 : \mathfrak{X}_1 \longrightarrow \mathfrak{X}_2$ be a quasi-isometry. Then a sequence $\{x_n\}$ of points of $\mathfrak{X}_1$ converges to infinity if and only if the sequence $\{f_1(x_n)\}$ does. The map $\partial f_1 : \{x_n\} \mapsto \{f_1(x_n)\}$ defines a homeomorphism $\partial f_1 : \partial\mathfrak{X}_1 \longrightarrow \partial\mathfrak{X}_2$ of the boundaries.*

We pass, using Lemma 4.5, to an Nth power of the self-similar action in the same way as in the proof of Theorem 4.2, so that for every $g \in G$ such that $l(g) \leq 4$ and for every $a \in X^N$ the restriction $g|_a$ belongs to the generating set.

Suppose that the sequence $\{w_n\}$ converges to the infinity. Choose its convergent subsequence in $X^{-\omega} \cup X^*$. Suppose its limit is $\ldots x_2 x_1 \in X^{-\omega}$. The Gromov product $\langle w_i \cdot w_j \rangle$ with respect to the basepoint $\emptyset$ is equal to $|w| - l(g)/2$, where w and g are as in Lemma 4.3. From this follows that all the accumulation points of $\{w_n\}$ in $X^{-\omega}$ are asymptotically equivalent to $\ldots x_2 x_1$. We also have that the sequence $\{x_n x_{n-1} \ldots x_1\}_{n \geq 1}$ converges to the same point of the hyperbolic boundary as $\{w_n\}$. So the map $D : X^{-\omega} \longrightarrow \partial\Sigma(G, S)$ is surjective and the map $F : \mathcal{J}_G \longrightarrow \partial\Sigma(G, S)$, satisfying the conditions of the theorem, is uniquely defined.

Let $A = \{g \in G : l(g) \leq 4\}$ and for every $n \in \mathbb{N}$ define

$$U_n = \{(w_1 v, w_2 s(v)) : w_1, w_2 \in X^{-\omega}, v \in X^n, s \in A\} \subset X^{-\omega} \times X^{-\omega}.$$

By $\tilde{U}_n$ we denote the image of U_n in $\mathcal{J}_G \times \mathcal{J}_G$.

It is easy to prove now that $\cap_{n \geq 1} U_n$ is equal to the asymptotic equivalence relation on $X^{-\omega}$ and that $\tilde{U}_n$ is the basis of neighborhoods of a uniform structure, defining the topology on $\mathcal{J}_G$.

¿From Lemma 4.3 now follows that

$$V_{n-2} \subseteq D \times D(U_n) \subseteq V_n.$$

This implies that map F is a homeomorphism. □

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Contact information

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