COMBINATORICS OF POLYNOMIAL ITERATIONS

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ABSTRACT. A complete description of the iterated monodromy groups of post-critically finite backward polynomial iterations is given in terms of their actions on rooted trees and automata generating them. We describe an iterative algorithm for finding kneading automata associated with post-critically finite topological polynomials and discuss some open questions about iterated monodromy groups of polynomials.

1. INTRODUCTION

The topics of these notes are group-theoretical and combinatorial aspects of iterations of post-critically finite polynomials, including topological polynomials and post-critically finite non-autonomous backward iterations.

As the main object encoding the combinatorics of iterations we use the associated iterated monodromy groups and permutational bimodules. These algebraic structures encode in a condensed form all topological information about the corresponding dynamical systems. More on iterated monodromy groups and their applications in symbolic dynamics see [Nek05, Nek07b, BGN03].

In the context of polynomial iterations the iterated monodromy groups are analogs or generalizations of the classical tools of symbolic dynamics of quadratic and higher degree polynomials: kneading sequences, internal addresses, Hubbard trees, critical portraits etc. In some cases the transition from the classical objects to iterated monodromy groups and permutational bimodules are very straightforward, but in some cases they are more involved. For more on symbolic dynamics of post-critical polynomials, see the works [BFH92, BS02, Kel00, Poi93a, Poi93b]. For relations of kneading sequences and iterated monodromy groups of quadratic polynomials, see [BN06a].

We answer some basic questions about iterated monodromy groups of polynomials and formulate some problems for further investigations. This area is fresh and many questions are open, even though they might be not so hard to answer.

We give in our paper a complete description of the iterated monodromy groups of *post-critically finite backward iterations* of topological polynomials. Here a post-critically finite backward iteration is a sequence $f_1, f_2, \ldots$ of complex polynomials (or orientation preserving branched coverings of planes) such that there exists a finite set P such that all critical values of $f_1 \circ f_2 \circ \cdots \circ f_n$ belong to P for every n.

The *iterated monodromy group* of such a sequence is the automorphism group of the *tree of preimages*

$$T_t = \bigsqcup_{n\geq 0} (f_1 \circ f_2 \circ \cdots \circ f_n)^{-1}(t),$$

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induced by the monodromy actions of the fundamental group $\pi_1(\mathbb{C} \setminus P, t)$. Here t is an arbitrary basepoint.

We prove that a group of automorphisms of a rooted tree is the iterated monodromy groups of a backward polynomial iteration if and only if it is generated by a set of automorphisms satisfying a simple planarity condition.

Namely, if $A \subset \mathfrak{S}(\mathsf{X})$ is a set of permutations, then its *cycle diagram* $D(A)$ is an oriented 2-dimensional CW-complex in which for every cycle of each permutation we have a 2-cell such that the corresponding cycle is read on the boundary of the cell along the orientation. Two cycles of different permutations are not allowed to have common edges in $D(A)$.

We say that a set $A \subset \mathfrak{S}(\mathsf{X})$ is *dendroid* (*tree-like in* [Nek05]) if the diagram $D(A)$ is contractible. A set of automorphisms $A = \{a_1, a_2, \ldots, a_n\}$ of a rooted tree T is said to be *dendroid* if A acts on every level of the tree T as a dendroid set of permutations.

The main result of Section 5 is the following complete description of the iterated monodromy groups of polynomial iterations (given in Propositions 5.1 and 5.2, which contain more details).

Theorem 1.1. *An automorphism group of a rooted tree T is an iterated monodromy group of a post-critically finite backward iteration of polynomials if and only if it is generated by a dendroid set of automorphisms of T.*

Very little is known about the class of groups generated by dendroid sets of automorphisms. This class contains many interesting examples of groups (especially in relation with questions of amenability and growth of groups see [GŻ02, BV05, BP06, Nek07a, BKNV06]), and further study of such groups is of great interest for group theory and dynamics.

We also give in our paper a general receipt for construction of dendroid sets of automorphisms of rooted trees using automata. We define a class of *dendroid automata* and prove in Theorem 4.4 that dendroid sets of automorphisms of a rooted tree are defined using sequences of dendroid automata. General definitions of automata are given in Subsection 3.5; dendroid automata are described in Definition 4.4. See paper [Nek07a], where three dendroid automata were used to construct an uncountable set of groups with unusual properties.

We see that combinatorics of post-critically finite backward iterations is described by the corresponding sequences of dendroid automata and that composition of polynomials corresponds to a certain composition of automata.

It is natural in the case of iterations of a single post-critically finite polynomial f to ask if we are able to describe the iterated monodromy group by one automaton of some special kind (i.e., by a special constant sequence of automata). It was shown in [Nek05] that for any post-critically finite polynomial f there exists n such that the iterated monodromy group *of the nth iteration* of f is described by a particularly simple *kneading automaton* (one can take $n = 1$ if f is hyperbolic). The kneading automaton is found using the technique of *external angles* to the Julia set, i.e., using analytic techniques.

The last section of the paper deals with the problem of finding the kneading automaton associated with a post-critically finite polynomial using purely topological information and group-theoretical techniques. We propose an iterative algorithm, which seems to work in many cases, which we call a *combinatorial spider algorithm.* It takes as input the dendroid automaton associated with the polynomial (called *twisted kneading automaton*), which can be easily found from the action of the topological polynomial on the fundamental group of the punctured plane, and simplifies it until we get the associated kneading automaton, or a twisted kneading automaton close to it. Note that associated automata together with a cyclic ordering of its states uniquely determines the Thurston combinatorial class of the polynomial (hence determines the complex polynomial uniquely, up to an affine conjugation), see Proposition 6.1.

The algorithm seems to work in most examples, but the general question of convergence of the combinatorial spider algorithm is still not very clear. We show that this algorithm is equivalent to some simple computations in a permutational bimodule over a subgroup of the outer automorphisms of the free group. Convergence problem of the combinatorial spider algorithm is closely related to the (sub)-hyperbolicity of this bimodule, which is also open.

The structure of the paper is as follows. Section 2 studies elementary properties of dendroid subsets of the symmetric group. Section 3 is an overview of notions and basic results of the theory of groups acting on rooted trees, including permutational bimodules, wreath recursions and automata. The techniques and language of this section is used in all subsequent sections of the paper.

Section 4 gives a complete description of dendroid sets of automorphisms of a rooted tree. We introduce the notion of a dendroid automaton (Definition 4.4) and prove that dendroid sets of automorphisms of a rooted tree are exactly automorphisms defined by sequences of dendroid automata (Theorem 4.4).

In Section 5 we apply the developed techniques and give a complete description of iterated monodromy groups of post-critically finite backward polynomial iterations, as described in Theorem 1.1 above. In particular, we give an explicit description of the associated sequence of dendroid automata (Proposition 5.5), and discuss cyclic ordering of states of a dendroid automaton and its relation with topology (Subsection 5.3).

The last section "Iterations of a single polynomial" deals with iterations of post-critically finite polynomials. We describe the *twisted kneading automata*, which are the automata, which we obtain, if apply the general techniques of dendroid automata to the case of iterations of a single polynomial. This automaton, in principle, already describes the iterated monodromy group of the polynomial, but it may be too complicated. So, a natural question is to transform it to the simplest possible form. This can be done analytically (using external angles and invariant spiders), as in Chapter 6 of [Nek05], but this approach may be not available, if the polynomial is given by purely topological information.

The suggested algorithm for simplification of the twisted kneading automata is described in Subsection 6.3. The underlying algebraic structure of a permutational bimodule over a subgroup of the outer automorphism group of the free group is described in Subsection 6.4. In the last subsection we give some simple criterion of absence of obstructions (i.e., realizability as complex polynomials) for topological polynomials given by their kneading automata.

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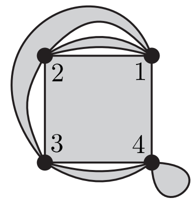

FIGURE 1. A cycle diagram

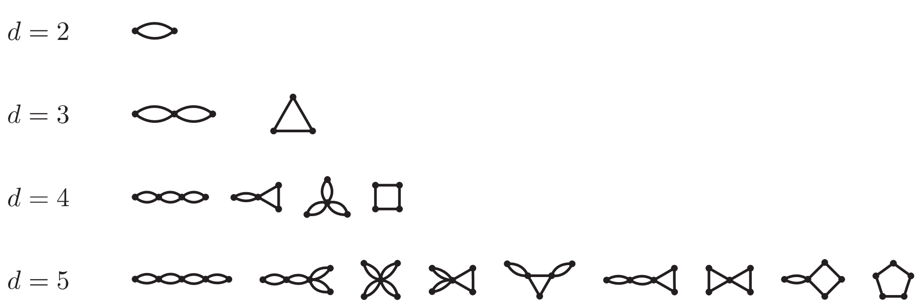

FIGURE 2. Dendroid sets of permutations

2. DENDROID SETS OF PERMUTATIONS

Let X be a finite set and denote by $\mathfrak{S}(\mathsf{X})$ the symmetric group of all permutations of X. Let $A = (a_i)_{i \in I}$ be a sequence of elements of $\mathfrak{S}(\mathsf{X})$. Draw an oriented 2-dimensional CW-complex with the set of 0-cells X in which for every cycle $(x_1, x_2, \ldots, x_n)$ of every permutation a_i we have a 2-cell with the vertices $x_1, x_2, \ldots, x_n$ so that their order on the boundary of the cell and in the cycle of the permutation coincide. We label each cycle by the corresponding permutation. Two different 2-cells are not allowed to have common 1-cells.

The constructed CW-complex is called the *cycle diagram* of the sequence A and is denoted $D(A)$. For example, Figure 1 shows the cycle diagram for $\mathsf{X} = \{1, 2, 3, 4\}$ and $A = \{(12)(34), (1234), (123)\}$.

Definition 2.1. A sequence A of elements of $\mathfrak{S}(\mathsf{X})$ is said to be *dendroid* if its cycle diagram $D(A)$ is contractible.

Note that if A is dendroid, then only trivial cycles can appear twice as cycles of elements of A. In particular, only the trivial permutation can appear more than once in the sequence A. Moreover, any two cycles of A are either disjoint (hence commute) or have only one common element.

See Figure 2 for all possible types of cycle diagrams of dendroid sets of permutations of X for $|\mathsf{X}| \leq 5$. We do not show there the trivial cycles.

Proposition 2.1. *Let A be a sequence of elements of $\mathfrak{S}(\mathsf{X})$ generating a transitive subgroup. Denote by N the total number of cycles of the elements of A (including the trivial ones).*

Then the sequence A is dendroid if and only if

$$N - 1 = |\mathsf{X}| \cdot (|A| - 1).$$

Here and later $|A|$ denotes the length of the sequence A (the size of the index set).

Proof. Choose a point in each face of the cycle diagram $D(A)$ and replace each face by a star, i.e., by the graph connecting the chosen point with the vertices of the face. The obtained one-dimensional complex Γ is homotopically equivalent to the diagram $D(A)$, hence the sequence is dendroid if and only if the graph Γ is a tree. The graph Γ has $|\mathsf{X}|+N$ vertices and $|\mathsf{X}|\cdot|A|$ edges (every permutation contributes $|\mathsf{X}|$ edges). It is well known that a graph is a tree if and only if it is connected and the number of vertices minus one is equal to the number of edges. Hence, A is dendroid if and only if $D(A)$ is connected and $|\mathsf{X}|+N-1=|\mathsf{X}|\cdot|A|$. $\square$

Corollary 2.2. *Let $A=(a_i\in\mathfrak{S}(\mathsf{X}))_{i\in I}$ be a dendroid sequence. Suppose that the sequence $B=(b_i\in\mathfrak{S}(\mathsf{X}))_{i\in I}$ is such that b_i is conjugate to a_i or to a_i^{-1} in $\mathfrak{S}(\mathsf{X})$ for every $i\in I$ and that the permutations b_i generate a transitive subgroup of $\mathfrak{S}(\mathsf{X})$. Then B is also dendroid.*

Proof. The numbers of cycles in A and B are the same and $|A|=|B|$. $\square$

Corollary 2.3. *Let $A=(a_1,a_2,\ldots,a_m)$ be a dendroid sequence. Then the sequences $(a_1^{-1},a_2,\ldots,a_m)$ and $(a_1^g,a_2,\ldots,a_m)$ for $g\in\langle a_2,\ldots,a_m\rangle$ are dendroid.*

Proposition 2.4. *Suppose that $a_1,a_2,\ldots,a_m$ is a dendroid sequence of elements of $\mathfrak{S}(\mathsf{X})$. Then the product $a_1a_2\cdots a_m$ is a transitive cycle.*

Proof. It is sufficient to prove the proposition for the case when each a_i is a cycle, since we can replace a_i by the sequence of its cycles without changing the cycle diagram.

Suppose that two cycles $(x_1,x_2,\ldots,x_n)\cdot(y_1,y_2,\ldots,y_m)$ have only one common point, i.e., $|\{x_1,x_2,\ldots,x_m\}\cap\{y_1,y_2,\ldots,y_m\}|=1$. Without loss of generality we may assume that $y_m=x_1$. Then

$$(y_m,x_2,\ldots,x_n)\cdot(y_1,y_2,\ldots,y_m)=(y_1,y_2,\ldots y_{m-1},x_2,\ldots,x_n,y_m).$$

Thus, product of two cycles having one common point is a cycle involving the union of the points moved by the cycle. This finishes the proof, since any two cycle in a dendroid sequence of permutations have at most one common point. If two cycles are disjoint, then they commute; if we replace two cycles having a common point by their product, which is a cycle on their union, then we will not change the homotopy type of the cycle diagram. $\square$

Corollary 2.5. *Suppose that $A=(a_1,a_2,\ldots,a_m)$ is a dendroid sequence and let*

$$(a_{i_{1,1}},\ldots a_{i_{1,k_1}}),\quad (a_{i_{2,1}},\ldots,a_{i_{2,k_2}}),\quad \ldots,\quad (a_{i_{l,1}},\ldots,a_{i_{l,k_l}})$$

be any partition of the sequence A into disjoint sub-sequences. Then the sequence of products

$$(a_{i_{1,1}}\cdots a_{i_{1,k_1}},\quad a_{i_{2,1}}\cdots a_{i_{2,k_2}},\quad \ldots,\quad a_{i_{l,1}}\cdots a_{i_{l,k_l}})$$

is dendroid.

Proof. By Proposition 2.4 the sets of vertices of the connected components of the cycle diagram of $(a_{i_{j,1}},\ldots,a_{i_{j,k_j}})$ are exactly the sets of vertices of the cycles of the product $a_{i_{j,1}}\cdots a_{i_{j,k_j}}$, hence the cycle diagrams of A and of the sequence of products are homotopically equivalent. $\square$

3. Automorphisms of rooted trees and bimodules

3.1. **Rooted trees.** A *rooted tree* is a tree T with a fixed vertex, called the *root* of the tree. We consider only locally finite trees.

The *level number* n (or the *nth level*) of a rooted tree T is the set of vertices on distance n from the root. The set of vertices of T is then a disjoint union of its levels $L_0, L_1, \ldots$, where the 0th level contains only the root. Two vertices may be connected by an edge only if they belong to consecutive levels.

We say that a vertex u is *below* a vertex v if the path from the root to u passes through v. The set of vertices below v together with v as a root form a *rooted subtree* T_v.

An *automorphism* of a rooted tree T is an automorphism of the tree T fixing the root. Every automorphism of a rooted tree preserves the levels.

A group acting on a rooted tree is said to be *level transitive* if it is transitive on the levels of the tree. A *level-homogeneous rooted tree* is a rooted tree admitting a level transitive automorphism group.

A level-homogeneous rooted tree is uniquely determined, up to an isomorphism of rooted trees, by its *spherical index* $(d_1, d_2, \ldots)$, where d_1 is degree of the root and $d_k + 1$ is the degree of a vertex of the kth level for $k \geq 2$. In other words, d_k is the number of vertices of the kth level adjacent to a common vertex of the $(k-1)$st level.

For a given sequence $\kappa = (d_1, d_2, \ldots)$ a rooted tree with spherical index κ can be constructed as a *tree of words* in the following way. Choose a sequence of finite sets (*alphabets*) $\mathsf{X} = (X_1, X_2, \ldots)$ such that $|X_k| = d_k$. The nth level of the tree of words X^* is equal to

$$\mathsf{X}^n = X_1 \times X_2 \times \cdots \times X_n = \{x_1 x_2 \ldots x_n \ : \ x_k \in X_k\},$$

so that $\mathsf{X}^* = \bigsqcup_{k\geq 0} \mathsf{X}^k$. Here X^0 consists of a single *empty word* $\varnothing$, which will be the root of the tree X^*.

We connect two words $v \in \mathsf{X}^n$ and $u \in \mathsf{X}^{n+1}$ if and only if u is a continuation of v.

If X is a finite alphabet, then the *regular tree* X^* is defined as above for the constant sequence $(\mathsf{X}, \mathsf{X}, \ldots)$, i.e., X^* is the set of all finite words over the alphabet X (in other terms, it is the free monoid generated by X).

For an arbitrary rooted tree T the *boundary* ∂T is the set of simple infinite paths in T starting in the root. By ∂T_v we denote the subset of paths passing through a given vertex v. The set of subsets ∂T_v for all vertices v of T is a basis of a natural topology on ∂T.

If T is a tree of words X^* over a sequence of alphabets $(X_1, X_2, \ldots)$, then the boundary $\partial\mathsf{X}^*$ is naturally identified with the direct product

$$\mathsf{X}^\omega = \mathsf{X}_1 \times \mathsf{X}_2 \times \cdots .$$

For more on groups acting on rooted trees and related notions, see [GNS00, Sid98, Nek05].

3.2. **Permutational bimodules.** Let G be a level-transitive automorphism group of a rooted tree T. Let v be a vertex of the nth level L_n of T and let G_v be the stabilizer of v in G. The subtree T_v is invariant under G_v. Denote by $G|_v$ the automorphism group of T_v equal to the restriction of the action of G_v onto T_v.

Since we assume that G is level transitive, the conjugacy class of G_v in G, and hence the isomorphism class of the action of $G|_v$ on T_v depend only on the level of the vertex v and do not depend on the choice of v.

Denote by $\mathfrak{M}_{\varnothing,v}$ the set of isomorphisms $\phi : T_v \longrightarrow T_u$ for $u \in L_n$ induced by the action of an element $g \in G$ (different u give different elements of $\mathfrak{M}_{\varnothing,v}$).

Note that for every $\phi \in \mathfrak{M}_{\varnothing,v}$ and $g \in G|_v$ the composition $\phi \cdot g$ belongs to $\mathfrak{M}_v$, since g is an automorphism of T_v induced by the action of an element of G.

Similarly, for every $g \in G$ the composition $g \cdot \phi$ (restricted to T_v) is an element of $\mathfrak{M}_{\varnothing,v}$. It is easy to see that in this way we get a right action of $G|_v$ and a left action of G on $\mathfrak{M}_{\varnothing,v}$ and that these actions commute.

Note that the right action of $G|_v$ on $\mathfrak{M}_{\varnothing,v}$ is *free*, i.e., that $\phi \cdot g = \phi$ implies that g is trivial, and that two elements $\phi_1 : T_v \longrightarrow T_{u_1}$ and $\phi_2 : T_v \longrightarrow T_{u_2}$ belong to one $G|_v$-orbit if and only if $u_1 = u_2$.

Let us formalize the obtained structure in the following definition.

Definition 3.1. Let G and H be groups. A *permutational* $(G - H)$*-bimodule* is a set $\mathfrak{M}$ with a left action of G and right action of H, which commute. More explicitly, we have maps $G \times \mathfrak{M} \longrightarrow \mathfrak{M} : (g, x) \mapsto g \cdot x$ and $\mathfrak{M} \times H \longrightarrow \mathfrak{M} : (x, h) \mapsto x \cdot h$ satisfying the following conditions:

(1) $1 \cdot x = x \cdot 1 = x$ for all $x \in \mathfrak{M}$;
(2) $g_1 \cdot (g_2 \cdot x) = (g_1 g_2) \cdot x$ and $(x \cdot h_1) \cdot h_2 = x \cdot (h_1 h_2)$ for all $x \in \mathfrak{M}$, $g_1, g_2 \in G$ and $h_1, h_2 \in H$;
(3) $(g \cdot x) \cdot h = g \cdot (x \cdot h)$ for all $x \in \mathfrak{M}$, $g \in G$ and $h \in H$.

A *covering* bimodule is a bimodule in which the right group action is free and has a finite number of orbits.

We have seen that $\mathfrak{M}_{\varnothing,v}$ is a *covering permutational* $(G - G|_v)$*-bimodule*. In general, if v_1, v_2 are vertices of T such that v_2 is below v_1, then $\mathfrak{M}_{v_1,v_2}$ is the $(G|_{v_1} - G|_{v_2})$-bimodule consisting of the isomorphisms $T_{v_2} \longrightarrow T_u$ induced by elements of $G|_{v_1}$ (or, equivalently, of G_{v_1}).

We will say that $\mathfrak{M}$ is a *G-bimodule* if it is a $(G - G)$-bimodule.

Definition 3.2. Two $(G - H)$-bimodules $\mathfrak{M}_1$ and $\mathfrak{M}_2$ are *isomorphic* if there exists a bijection $F : \mathfrak{M}_1 \longrightarrow \mathfrak{M}_2$ which agrees with the actions, i.e., such that

$$F(g \cdot x \cdot h) = g \cdot F(x) \cdot h$$

for all $g \in G$, $x \in \mathfrak{M}$ and $h \in H$.

3.3. **Tensor products and bases.**

Definition 3.3. Let $\mathfrak{M}_1$ and $\mathfrak{M}_2$ be a $(G_1 - G_2)$-bimodule and a $(G_2 - G_3)$-bimodule, respectively. Then the *tensor product* $\mathfrak{M}_1 \otimes \mathfrak{M}_2 = \mathfrak{M}_1 \otimes_{G_2} \mathfrak{M}_2$ is the $(G_1 - G_3)$-bimodule equal as a set to the quotient of $\mathfrak{M}_1 \times \mathfrak{M}_2$ by the identification

$$x_1 \otimes g \cdot x_2 = x_1 \cdot g \otimes x_2$$

for $x_1 \in \mathfrak{M}_1$, $x_2 \in \mathfrak{M}_2$, $g \in G_2$. The actions are defined by the rule

$$g_1 \cdot (x_1 \otimes x_2) \cdot g_3 = (g_1 \cdot x_1) \otimes (x_2 \cdot g_3)$$

for $g_i \in G_i$ and $x_i \in \mathfrak{M}_i$.

It is an easy exercise to prove that $\mathfrak{M}_1 \otimes \mathfrak{M}_2$ is a well defined $(G_1 - G_3)$-bimodule. Moreover, if $\mathfrak{M}_1$ and $\mathfrak{M}_2$ are covering bimodules, then $\mathfrak{M}_1 \otimes \mathfrak{M}_2$ is also a covering bimodule. It is also not hard to prove that tensor product of bimodules is an associative operation, i.e., that $(\mathfrak{M}_1 \otimes \mathfrak{M}_2) \otimes \mathfrak{M}_3$ is isomorphic to $\mathfrak{M}_1 \otimes (\mathfrak{M}_2 \otimes \mathfrak{M}_3)$, where the isomorphism is induced by the natural identification of the direct products of sets.

Proposition 3.1. *Let v_1 and v_2 be vertices of T such that v_2 is below v_1. Then the bimodule $\mathfrak{M}_{\varnothing,v_2}$ is isomorphic to $\mathfrak{M}_{\varnothing,v_1} \otimes \mathfrak{M}_{v_1,v_2}$.*

Proof. Let $\phi_1 : T_{v_1} \longrightarrow T_{u_1}$ and $\phi_2 : T_{v_2} \longrightarrow T_{u_2}$ be elements of $\mathfrak{M}_{\varnothing,v_1}$ and $\mathfrak{M}_{v_1,v_2}$. Then u_2 is below v_1, since ϕ_2 is restriction of an element of G_{v_1} onto $T_{v_2} \subset T_{v_1}$. Define

$$F(\phi_1 \otimes \phi_2) = \phi_1 \circ \phi_2 : T_{v_2} \longrightarrow T_{\phi_1(u_2)}.$$

Since both ϕ_1 and ϕ_2 are restrictions of elements of G, the isomorphism $F(\phi_1 \otimes \phi_2)$ belongs to $\mathfrak{M}_{\varnothing,v_2}$. We leave to the reader to prove that F is a well defined bijection preserving the actions. □

Corollary 3.2. *Let $\varnothing, v_1, v_2, \ldots$ be a path in the tree T starting at the root, such that v_n belongs to the nth level of T. Denote $G_n = G|_{v_n}$ (called* nth upper companion group *in* [Gri00]*). Then the $(G-G_n)$-bimodule $\mathfrak{M}_{\varnothing,v_n}$ is isomorphic to the tensor product*

$$\mathfrak{M}_{\varnothing,v_1} \otimes \mathfrak{M}_{v_1,v_2} \otimes \cdots \otimes \mathfrak{M}_{v_{n-1},v_n}.$$

In particular, the action of G on the nth level of the tree T is conjugate with the action of G on the right G_n-orbits of this tensor product.

Covering bimodules can be encoded symbolically using the notion of a basis of the bimodule.

Definition 3.4. Let $\mathfrak{M}$ be a covering $(G-H)$-bimodule. A *basis* of $\mathfrak{M}$ is an orbit transversal for the right H-action, i.e., a set $\mathsf{X} \subset \mathfrak{M}$ such that every H-orbit of $\mathfrak{M}$ contains exactly one element of X.

If X is a basis of $\mathfrak{M}$, then every element of $\mathfrak{M}$ can be written uniquely as $x \cdot h$ for some $x \in \mathsf{X}$ and $h \in H$.

Then for every $g \in G$ and $x \in \mathsf{X}$ there exists a unique pair $y \in \mathsf{X}$, $h \in H$ such that

$$g \cdot x = y \cdot h.$$

If X_1 and X_2 are bases of the bimodules $\mathfrak{M}_1$ and $\mathfrak{M}_2$, respectively, then the set $\mathsf{X}_1 \otimes \mathsf{X}_2 = \{x_1 \otimes x_2 \ : \ x_1 \in \mathsf{X}_1, x_2 \in \mathsf{X}_2\}$ is a basis of the bimodule $\mathfrak{M}_1 \otimes \mathfrak{M}_2$ (see Proposition 2.3.2 of [Nek05]).

By induction, if $\mathfrak{M}_i$ for $i = 1, 2, \ldots$ is a $(G_{i-1} - G_i)$-bimodule, and X_i is its basis, then

$$\mathsf{X}_1 \otimes \mathsf{X}_2 \otimes \cdots \otimes \mathsf{X}_n \subset \mathfrak{M}_1 \otimes \mathfrak{M}_2 \otimes \cdots \otimes \mathfrak{M}_n$$

is also a basis of the $(G_0 - G_n)$-bimodule $\mathfrak{M}_1 \otimes \cdots \otimes \mathfrak{M}_n$.

If the sequence $\mathsf{X} = (\mathsf{X}_1, \mathsf{X}_2, \ldots)$ of bases is fixed, then we denote by X^n the basis $\mathsf{X}_1 \otimes \cdots \otimes \mathsf{X}_n$ and by X^* the disjoint union of X^n for $n \geq 0$. Here X^0 contains only one element $\varnothing$ with the property $\varnothing \otimes v = v$ for all $v \in \mathsf{X}^*$. We will often write elements $x_1 \otimes x_2 \otimes \cdots \otimes x_n \in \mathsf{X}^n$ just as words $x_1 x_2 \ldots x_n$.

Then every element of the bimodule $\mathfrak{M}_1 \otimes \cdots \otimes \mathfrak{M}_n$ can be uniquely written in the form $v \cdot h$ for $v \in \mathsf{X}^n$ and $h \in G_n$. In particular, for any $v \in \mathsf{X}^n$ and every $g \in G_0$ there exist a unique $u \in \mathsf{X}^n$ and $h \in G_n$ such that

$$g \cdot v = u \cdot h.$$

We denote $u = g(v)$ and $h = g|_v$. We have then the following properties, which are easy corollaries of the definitions:

$$g_1(g_2(u)) = (g_1g_2)(u), \quad 1(u) = u,$$
$$g(u \otimes v) = g(u) \otimes g|_u(v).$$

Consequently, we get in this way a natural action of G_0 on the rooted tree X^*. The natural action of G_0 on X^* does not depend, up to a conjugacy of the actions, on the choice of the bases X_i. The most direct way to see this is to note that this action coincides with the natural action of G_0 on the set $\bigsqcup_{n\ge 0} \mathfrak{M}_1 \otimes \cdots \otimes \mathfrak{M}_n / G_n$ of the right orbits of the tensor product bimodules. An orbit is mapped by the conjugacy to the unique element of X^* contained in it. For more on the tree of right orbits see [Nek07b].

3.4. **Wreath recursions.** Let $\mathfrak{M}$ be a covering $(G - H)$-bimodule and let X be its basis. Then for every $g \in G$ and $x \in \mathsf{X}$ there exist unique elements $h \in H$ and $y \in \mathsf{X}$ such that

$$g \cdot x = y \cdot h.$$

Recall that we denote $y = g(x)$ and $h = g|_x$.

For a fixed $g \in G$ we get then a permutation $\sigma_g : x \mapsto g(x)$ of X induced by g and a sequence of *sections* $(g|_x)_{x\in\mathsf{X}}$. For $g_1, g_2 \in G$ we have $\sigma_{g_1g_2} = \sigma_{g_1}\sigma_{g_2}$ and

$$(g_1g_2)|_x = g_1|_{\sigma_{g_2}(x)} g_2|_x.$$

Hence we get a homomorphism from G to the wreath product $\mathfrak{S}(\mathsf{X}) \wr H = \mathfrak{S}(\mathsf{X}) \ltimes H^{\mathsf{X}}$:

$$\psi : g \mapsto \sigma_g (g|_x)_{x\in\mathsf{X}},$$

since the elements of the wreath product are multiplied by the rule

$$\sigma_1(g_x)_{x\in\mathsf{X}} \cdot \sigma_2(h_x)_{x\in\mathsf{X}} = \sigma_1\sigma_2(g_{\sigma_2(x)}h_x)_{x\in\mathsf{X}}.$$

We call the homomorphism ψ the *wreath recursion* associated with the bimodule $\mathfrak{M}$ and the basis X.

We will usually order the basis $\mathsf{X} = \{x_1, x_2, \ldots, x_d\}$ and write the elements of the wreath product as sequences

$$\sigma(g_x)_{x\in\mathsf{X}} = \sigma(g_{x_1}, g_{x_2}, \ldots, g_{x_d}),$$

thus implicitly identifying the wreath product $\mathfrak{S}(\mathsf{X}) \wr H$ with $\mathfrak{S}(d) \wr H$ (where $\mathfrak{S}(d) = \mathfrak{S}(\{1, 2, \ldots, d\})$).

If we change the basis X, then we compose the wreath recursion with an inner automorphism of the group $\mathfrak{S}(d) \wr H$. More explicitly, if $\mathsf{Y} = \{y_1, \ldots, y_d\}$ is another basis of $\mathfrak{M}$, then $y_i = x_{\pi(i)} \cdot h_i$ for some $\pi \in \mathfrak{S}(d)$ and $h_i \in H$. Then the wreath recursions ψ_{X} and ψ_{Y} are related by

$$\psi_{\mathsf{Y}}(g) = \alpha^{-1}\psi_{\mathsf{X}}(g)\alpha,$$

where $\alpha = \pi(h_1, h_2, \ldots, h_d)$, see Proposition 2.3.4 of [Nek05] and Proposition 2.12 of [Nek07b].

In particular, conjugation of all coordinates of H^{X} in the wreath product by an element of the group H is equivalent to a change of the basis of the bimodule.

If G is a finitely generated group, then the wreath recursion, and hence the bimodule, are determined by a finite number of equations of the form

$$g_i = \sigma_i(h_{1,i}, h_{2,i}, \ldots, h_{d,i}),$$

where $\{g_i\}$ is a finite generating set of G and $h_{j,i}$ are elements of H. Equations of this form is the main computational tool in the study of coverings using iterated monodromy groups.

3.5. **Automata.** Permutation bimodules and wreath recursions can be encoded by *automata*. We interpret equalities

$$g \cdot x = y \cdot h$$

in a bimodule as a work of an automaton, which being in a state g and reading an input letter x, gives on the output the letter y and goes to the state h, ready to process further letters.

More formally, we adopt the following definition.

Definition 3.5. An *automaton* over the alphabet X is a set of *internal states* A together with a map

$$\tau : A \times \mathsf{X} \longrightarrow \mathsf{X} \times A.$$

For $a \in A$ and $x \in \mathsf{X}$, the first and second coordinates of $\tau(a, x)$ as functions from $A \times \mathsf{X}$ to X and A are called the *output* and the *transition* functions, respectively.

If $\mathfrak{M}$ is a covering G-bimodule and X is a basis of $\mathfrak{M}$, then the *associated automaton* $\mathcal{A}(G, \mathsf{X}, \mathfrak{M})$ is the automaton with the set of internal states G, defined by

$$\tau(g, x) = (y, h), \text{ iff } g \cdot x = y \cdot h,$$

i.e., by

$$\tau(g, x) = (g(x), g|_x).$$

We will use similar notation for all automata, so that for a state q and a letter x we have

$$\tau(q, x) = (q(x), q|_x),$$

i.e., $q(x)$ denotes the value of the output function and $q|_x$ denotes the value of the transition function.

We will also usually write

$$g \cdot x = y \cdot h$$

instead of

$$\tau(g, x) = (y, h).$$

Automata are conveniently described by their *Moore diagrams*. It is an oriented graph with the set of vertices equal to the set A of internal states of the automaton. For every pair $q \in A$ and $x \in \mathsf{X}$ there is an arrow going from q to $q|_x$ labeled by the pair $(x, q(x))$.

We will need to deal also with the $(G - H)$-bimodules for different groups G and H. Therefore, we will also use the following generalized notion.

Definition 3.6. An *automaton* over alphabet X is given by its *input set* A_1, *output set* A_2 and a map

$$\tau : A_1 \times \mathsf{X} \longrightarrow \mathsf{X} \times A_2.$$

We also use the notation $q(x)$ for the first coordinate of $\tau(q, x)$ and $q|_x$ for the second coordinate of $\tau(q, x)$.

Definition 3.7. An automaton is called a *group automaton* if for every element a of the input set the transformation $x \mapsto a(x)$ of the alphabet is a permutation.

We assume implicitly that the input and the output sets of a group automaton contain special *trivial states* denoted 1 with the property that

$$1 \cdot x = x \cdot 1$$

for all letters x of the alphabet.

Perhaps it would be less confusing to dualize Definition 3.6 and say that X, A_1 and A_2 are the set of internal states, input and output alphabets, respectively. But since we think of the groups acting on words, and not words acting on groups, we stick to terminology of Definition 3.6.

Moore diagrams is not an appropriate way of describing such generalized automata. Therefore, we will usually describe them using *dual Moore diagrams*. It is a directed graph with the set of vertices X in which for every $x \in \mathsf{X}$ and q in the input set there is an arrow from x to $q(x)$ labeled by $(q, q|_x)$.

Products of automata correspond to tensor products of bimodules and are described in the following way.

Definition 3.8. Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be automata over the alphabets X_1 and X_2, input sets A_1 and A_2, output sets A_2 and A_3, respectively. Their *product* $\mathcal{A}_1 \otimes \mathcal{A}_2$ is the automaton over the alphabet $\mathsf{X}_1 \times \mathsf{X}_2$ with the input set A_1 and the output set A_2, with the output and transition functions given by the rules

$$q_1(x_1, x_2) = (q_1(x_1), q_1|_{x_1}(x_2))$$

and

$$q_1|_{(x_1,x_2)} = (q_1|_{x_1})\,|_{x_2}.$$

Product of automata is different from the dual notion of *composition*, which is defined only for automata with coinciding input and output sets. Namely, if $\mathcal{A}_1$ and $\mathcal{A}_2$ are automata over the alphabet X with the sets of internal states A_1 and A_2, respectively, then their *composition* is the automaton with the set of internal states $A_1 \times A_2$ over the alphabet X in which the output and transition functions are defined by the rules

$$(q_1, q_2)(x) = q_1\left(q_2(x)\right)$$

and

$$(q_1, q_2)|_x = \left(q_1|_{q_2(x)}, q_2|_x\right).$$

Definition 3.9. Let $\mathcal{A}_1, \mathcal{A}_2, \ldots$ be a sequence of group automata over alphabets $X_1, X_2, \ldots$, respectively. Suppose that $\mathcal{A}_i$ has input set A_{i-1} and output set A_i. Then the *action of* A_0 *on* X^*, for $\mathsf{X} = (X_1, X_2, \ldots)$, is the action defined on the nth level $\mathsf{X}^n = \prod_{i=1}^n X_i$ of X^* as the action of the input set A_0 of the product automaton $\mathcal{A}_1 \otimes \mathcal{A}_2 \otimes \cdots \otimes \mathcal{A}_n$ on its alphabet X^n.

It is not hard to see that the action defined by a sequence of automata is an action by automorphisms of the rooted tree X^*.

3.6. **Hyperbolic and sub-hyperbolic bimodules.**

Definition 3.10. Let $\mathfrak{M}$ be a covering G-bimodule and let X be its basis. We say that the bimodule $\mathfrak{M}$ is *hyperbolic* if there exists a finite set $\mathcal{N} \subset G$ such that for every $g \in G$ there exists $n_0 \in \mathbb{N}$ such that $g|_v \in \mathcal{N}$ whenever $v \in \mathsf{X}^n$ and $n \geq n_0$.

It is proved in [Nek05] (Corollary 2.11.7) that the property of being hyperbolic does not depend on the choice of the basis X (though the set $\mathcal{N}$ does).

If the group G is finitely generated, then hyperbolicity can be expressed in terms of a uniform contraction of the length of the group elements under restriction.

Definition 3.11. Let G be a finitely generated group and let $l(g)$ denote the word length of $g \in G$ with respect to some fixed finite generating set of G (i.e., the minimal length of a representation of g as a product of the generators and their inverses).

Then the number

$$\rho = \limsup_{n\to\infty} \sqrt[n]{\limsup_{l(g)\to\infty} \max_{v\in\mathsf{X}^n} \frac{l(g|_v)}{l(g)}}$$

is called the *contraction coefficient* of the bimodule $\mathfrak{M}$ (with respect to the basis X).

It is not hard to prove that the contraction coefficient does not depend on the choice of the generating set. It is also proved in [Nek05] (Proposition 2.11.11) that it does not depend on the basis X, if $\rho < 1$. Moreover, the following holds

Proposition 3.3. *The bimodule $\mathfrak{M}$ is hyperbolic if and only if its contraction coefficient is less than 1.*

It is possible that the action of G on X^* associated with $\mathfrak{M}$ is not faithful. Then the kernel K of the action is uniquely determined as the maximal subgroup with the property that it is normal and if $g\cdot x = y\cdot h$ for $x, y \in \mathfrak{M}$ and $g \in K$, then $h \in K$. Then the set $\mathfrak{M}/K$ of the right K-orbits of $\mathfrak{M}$ is naturally a G/K-bimodule such that the action of G/K on the tree X^* is faithful (and coincides with the action of G/K on X^* induced by G). The G/K-bimodule $\mathfrak{M}/K$ is called the *faithful quotient* of the bimodule $\mathfrak{M}$.

Definition 3.12. A bimodule is said to be *sub-hyperbolic* if its faithful quotient is hyperbolic.

In general, we say that a normal subgroup $N \lhd G$ is $\mathfrak{M}$-invariant, if $g|_x \in N$ for all $g \in N$ and $x \in \mathsf{X}$. This property does not depend on the choice of X. The kernel of the induced action on X^* is the maximal $\mathfrak{M}$-invariant normal subgroup. If N is a normal $\mathfrak{M}$-invariant subgroup, then the set $\mathfrak{M}/N$ of the right N-orbits is naturally a G/N-bimodule. It is easy to see that if $\mathfrak{M}$ is hyperbolic, then $\mathfrak{M}/N$ is also hyperbolic and that the faithful quotient of $\mathfrak{M}/N$ coincides with the faithful quotient of $\mathfrak{M}$. Consequently, we have the following version of the definition of a sub-hyperbolic bimodule.

Proposition 3.4. *A G-bimodule $\mathfrak{M}$ is sub-hyperbolic if and only if there exists a $\mathfrak{M}$-invariant normal subgroup $N \lhd G$ such that the G/N-bimodule $\mathfrak{M}/N$ is hyperbolic.*

Examples. If f is a hyperbolic post-critically finite rational function, then the bimodule $\mathfrak{M}_f$ over the fundamental group of the sphere minus the post-critical set is hyperbolic. For the definition of $\mathfrak{M}_f$ see Subsection 5.2. It follows from uniform expansion of f on a neighborhood of the Julia set of f, which does not contain the post-critical points.

On the other hand, if f is sub-hyperbolic, then $\mathfrak{M}_f$ is hyperbolic only as a bimodule over the fundamental group of the associated *orbifold*. The bimodule over the fundamental group of the punctured sphere is *sub-hyperbolic*. For more on the bimodules associated with post-critically finite rational functions, see Section 6.4 of [Nek05].

4. Dendroid automata

4.1. **Dendroid sets of automorphisms of a rooted tree.**

Definition 4.1. A sequence $(a_1, a_2, \ldots, a_m)$ of automorphisms of a rooted tree T is said to be *dendroid* if for every n the sequence of permutations defined by $(a_1, a_2, \ldots, a_m)$ on the nth level of T is dendroid.

If $A = (a_1, a_2, \ldots, a_m)$ is a dendroid sequence of automorphisms of a rooted tree T, then it generates a level-transitive subgroup of $\mathrm{Aut}\,(T)$. Consequently, the tree T is spherically homogeneous.

We will assume that A does not contain trivial automorphisms of T. Then all elements of A are different, and we may consider it as a set.

Denote by D_n the cycle diagram of the action of the sequence A on the nth level L_n of the tree T. We say that an oriented edge e of D_n *corresponds to* $a_i \in A$ if it is an edge of a 2-cell corresponding to a cycle of a_i.

If $\gamma = (e_1, e_2, \ldots, e_k)$ is a path in the 1-skeleton of D_n (we are allowed to go against the orientation), then we denote by $\pi(\gamma)$ the element $g_k g_{k-1} \ldots g_1$, where g_i is the element of A corresponding to the edge e_i, if we pass it along the orientation; and its inverse, if we go against the orientation. In particular, if e is an oriented edge of D_n, then $\pi(e)$ is the corresponding element of A.

Proposition 4.1. *Let $A = \{a_1, a_2, \ldots, a_m\}$ be a dendroid set of automorphisms of a rooted tree T generating a group $G < \mathrm{Aut}\,(T)$. Then for every vertex v of T the group $G|_v$ is also generated by a dendroid set.*

Recall, that $G|_v$ is restriction of the action of the stabilizer G_v on the sub-tree T_v.

Proof. The stabilizer G_v of v in G is generated by $\pi(\gamma)$, where γ runs through a generating set of the fundamental group of the 1-skeleton of the cycle diagram D_n of the action of A on the nth level L_n of T.

Choose one edge in each 2-cell of D_n. Let E be the set of chosen arrows (called *marked edges*). Let D_n' be the union of non-marked edges. It is a spanning tree of the 1-skeleton of D_n.

For every edge $e \in E$ let γ_e be the path going from v to the beginning of e through edges belonging to D_n', then along e and then back to v using only the edges from D_n'. Since D_n' is a tree, this description gives a unique automorphism $\pi(\gamma_e)$ of T.

It is well known that then the set of the loops $\{\gamma_e\}_{e\in E}$ is a free generating set of the fundamental group of the 1-skeleton of D_n. Hence the elements of the form $\pi(\gamma_e)$ give a generating set of G_v.

Let us denote by b_e the restriction of $\pi(\gamma_e)$ onto the sub-tree T_v. Then $\{b_e\}_{e\in E}$ is a generating set of $G|_v$. Let us prove that it is dendroid.

Let L_m be a level of T below the level L_n of v. Denote by D_m the cycle diagram of the action of A on L_m.

The 1-skeleton of D_m covers the 1-skeleton of D_n by the natural map $L_m \longrightarrow L_n$ sending a vertex v to its ancestor (i.e., to the vertex of L_n below which v is). Let

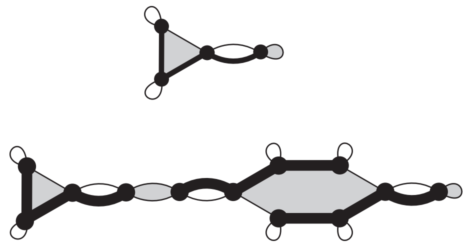

Figure 3.

D''_m be the inverse image of D'_n under this covering map. Since D'_n is a tree, the graph D''_m is a disjoint union of $|L_m|/|L_n|$ trees, which are mapped isomorphically onto D'_n by the covering. In particular, every connected component of D''_m contains exactly one vertex, which is below v. We get hence a natural bijection between the set of connected components of D''_m and $T_v \cap L_m$.

For every $e \in E$ and every connected component C of D''_m there exists exactly one preimage of e in D_m starting in a vertex v_1 of C and exactly one preimage ending in a vertex v_2 of C, since the projection $D_m \longrightarrow D_n$ is a covering of the 1-skeletons. The vertex v_2 is connected to the vertex v_1 by a directed chain of edges belonging to C and corresponding to the same element of A as e does, since e must belong to a cycle.

It follows from the definition of the generators γ_e that the inverse images of the edges $e \in E$ connect the components of D''_m exactly in the same way as the generators b_e act on $T_v \cap L_m$ (if we identify the components with the vertices of $T_v \cap L_m$ that they contain).

Contract the components of D''_m in D_m to points. We will obtain a CW-complex homotopically equivalent to D_m, i.e., contractible. Every preimage of $e \in E$ in D_m belongs to boundary of a disc, which after contraction becomes a disc corresponding to a cycle of b_e. Hence the obtained CW-complex is isomorphic to the cycle diagram of $\{b_e\}_{e \in E}$ and is contractible. □

Note that some of the generators b_e from the proof may be trivial. In this case we can remove them from the generating set.

Let us call the generating set $\{b_e\}_{e \in E}$ (with trivial elements removed) the *induced generating set* of $G|_v$.

The induced generating set of $G|_v$ depends on the choice of marked edges, however, it is not hard to see that the conjugacy classes in $\mathrm{Aut}\,(T_v)$ of its elements depend only on A.

In particular, the cardinality of the induced generated set depends only on A and the level number of v. It can be also found in the following way.

Definition 4.2. A *support* of a group $H < \mathrm{Aut}\,(T)$ is the set of vertices $v \in T$ such that the stabilizer H_v acts non-trivially on the sub-tree T_v.

It is easy to see that the support of a group H is a sub-tree containing the root of the tree.

Definition 4.3. We call a cycle of the action of an automorphism $a \in \mathrm{Aut}\,(T)$ on vertices *active* if it is contained in the support of $\langle a \rangle$.

Proposition 4.2. *Let $A \subset \mathrm{Aut}(T)$ be a dendroid set and let $G = \langle A \rangle$ be the group it generates. The cardinality of the induced generating set of $G|_v$ is equal to the sum over $a \in A$ of the number of active cycles of a on the level of v.*

Proof. Let $x_1, x_2, \ldots, x_k$ be a cycle of the action of $a \in A$ on the level of v. If a^k acts trivially on the subtrees T_{x_i}, then the vertices x_i do not belong to the support of $\langle a \rangle$ and also the corresponding generator b_e of $G|_v$ will be trivial, since it is conjugate to the restriction of a^k onto T_v. Otherwise, if b_e is not trivial, then the action of a^k on the subtrees T_{x_i} will be conjugate to b_e, and thus non-trivial. In this case the points x_i will belong to the support of $\langle a \rangle$. □

4.2. **Dendroid automata.** Let $A = \{a_1, a_2, \ldots, a_n\}$ be a dendroid set of automorphisms of a rooted tree T. Denote by D_n the cycle diagram of the action of A on the nth level L_n of T. Recall, that a *marking* of D_n is a set E of (*marked*) edges containing exactly one edge from the boundary of every 2-cell.

Choose a vertex $v \in L_n$ of D_n. Our aim is to describe the bimodule $\mathfrak{M}_{\varnothing,v}$.

Denote, for $u \in L_n$, by $\gamma_{v,u}$ a path in D_n from v to u not containing marked edges. Let $h_u = \pi(\gamma_{v,u})$. The automorphisms h_u does not depend on the choice of $\gamma_{v,u}$, since the complement D'_n of the set of marked edges is a spanning tree of the 1-skeleton of D_n.

Denote by $x_u : T_v \longrightarrow T_u$ the element of $\mathfrak{M}_{\varnothing,v}$ equal to the restriction of h_u onto the sub-tree T_v. The set $\{x_u\}_{u \in L_n}$ is a basis of the bimodule $\mathfrak{M}_{\varnothing,v}$.

Let us describe now the bimodule $\mathfrak{M}_{\varnothing,v}$ with respect to the basis $\mathsf{X} = \{x_u\}_{u \in L_n}$, the generating set $A = \{a_1, \ldots, a_m\}$ of G and the induced generating set $\{b_e\}_{e \in E}$ of $G|_v$, constructed in the proof of Proposition 4.1.

Recall that b_e is the restriction of $h_{r(e)}^{-1}\pi(e)h_{s(e)}$ onto T_v, where $s(e)$ is the beginning and $r(e)$ is the end of the edge e.

Let $a_i \in A$ and $x_u \in \mathsf{X}$ be arbitrary. Denote by e_{a_i,x_u} the edge of D_n starting in u and corresponding to a_i. If e_{a_i,x_u} is not marked, then $a_i \cdot h_u = h_{a_i(u)}$, hence

$$a_i \cdot x_u = x_{a_i(u)}. \tag{4.1}$$

If e_{a_i,x_u} is marked, then $b_{e_{a_i,x_u}} = h_{a_i(u)}^{-1} a_i h_u$, hence

$$a_i \cdot x_u = x_{a_i(u)} \cdot b_{e_{a_i,x_u}}. \tag{4.2}$$

Let us generalize the properties of the obtained automaton in the following definition.

Definition 4.4. A group automaton with the set of states X, input set A and output set B is said to be a *dendroid* automaton if the following conditions are satisfied:

1. The set of permutations defined by A on X is dendroid.
2. For every $b \in B$ there exists a unique pair $a \in A, x \in \mathsf{X}$ such that $a \cdot x = y \cdot b$ for some $y \in \mathsf{X}$.
3. For any cycle $(x_1, x_2, \ldots, x_k)$ of the action of $a \in A$ on X we have $a \cdot x_i = x_{i+1} \cdot 1$ for all but possible one index i. (Here indices are taken modulo k.)

We will draw the dual Moore diagram of dendroid automata as the cycle diagram of A in which a cell is labeled by the corresponding element of A. If $a \cdot x = y \cdot b$ for $a \in A$ and $b \in B$, then we label the edge from x to y corresponding to a by $b \in B$. If $a \cdot x = y \cdot 1$, then we do not label the corresponding edge (since we typically

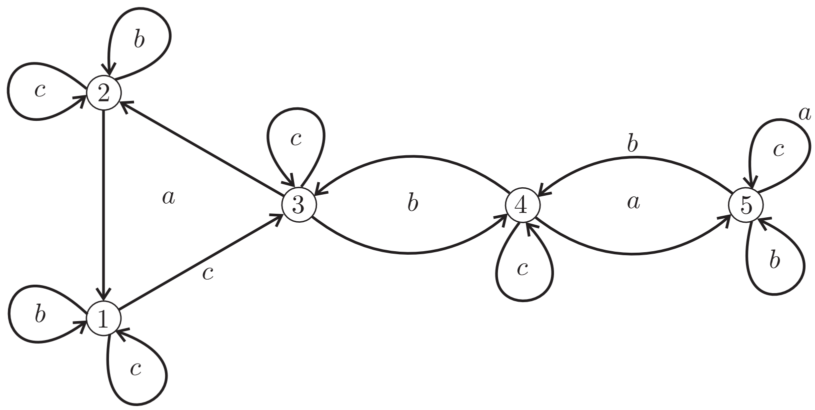

FIGURE 4. A dendroid automaton

ignore trivial states). This labeled diagram completely describes the automaton. It coincides (up to a different labeling convention) with its dual Moore diagram.

Condition (1) of Definition 4.4 is equivalent to contractibility of the diagram; condition (2) means that every element $b \in B$ appear exactly once as a label of an edge; and condition (3) means that every 2-cell of the diagram has at most one label of its boundary edge. See an example of a diagram of a dendroid automaton on Figure 4.

By equations (4.1) and (4.2), the automaton describing the bimodule $\mathfrak{M}_{\varnothing,v}$ with respect to the basis $\{x_u\}_{u\in L_n}$, the (input) generating set A and the (output) induced generating set $\{b_e\}$ is a dendroid automaton. The third condition follows from the fact that every cycle of a has exactly one marked edge. (The element b_e might be trivial, though).

Our aim now is to show that dendroid sets of automorphisms of a rooted tree are exactly the sets defined by sequences of dendroid automata.

Proposition 4.3. *Product of two dendroid automata is a dendroid automaton.*

The proof of this proposition also gives a description of a procedure of constructing the diagram of the product of dendroid automata.

Proof. The proof essentially coincides with the proof of Proposition 6.7.5 of [Nek05]. We rewrite it here, making the necessary changes.

Let $\mathcal{A}_1$ be a dendroid automaton over the alphabet X_1, input set A_1 and output set A_2. Let $\mathcal{A}_2$ be a dendroid automaton over X_2 with input and output sets A_2 and A_3, respectively.

If we have $a_1 \cdot x_1 = y_1 \cdot 1$ in $\mathcal{A}_1$, then $a_1 \cdot x_1x_2 = y_1x_2 \cdot 1$ in $\mathcal{A}_1 \otimes \mathcal{A}_2$. If $a_1 \cdot x_1 = y_1 \cdot a_2$ in $\mathcal{A}_2$, then $a_1 \cdot x_1x_2 = y_1 \cdot a_2 \cdot x_2$.

Consequently, the diagram of the automaton $\mathcal{A}_1 \otimes \mathcal{A}_2$ can be described in the following way.

Take $|\mathsf{X}_2|$ copies of the diagram D_1 of $\mathcal{A}_1$. Each copy will correspond to a letter $x_2 \in \mathsf{X}_2$ and the respective copy will be denoted D_1x_2. If $x_1 \in \mathsf{X}_1$ is a vertex of D_1, then the corresponding vertex of the copy D_1x_2 will become the vertex x_1x_2 of the diagram of $\mathcal{A}_1 \otimes \mathcal{A}_2$.

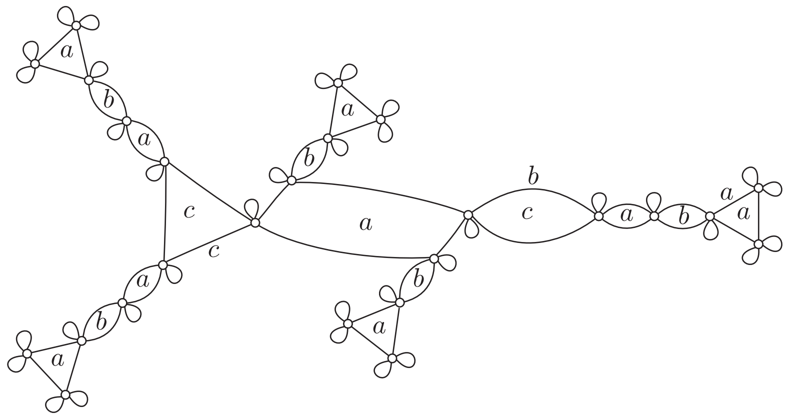

FIGURE 5. Diagram of $\mathcal{A} \otimes \mathcal{A}$

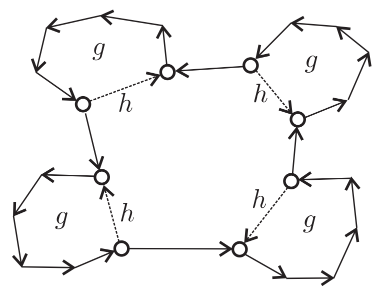

FIGURE 6. A cell of the diagram of $\mathcal{A}_1 \otimes \mathcal{A}_2$

If we have an arrow labeled by a_2 in the copy D_1x_2, (i.e., if we have $a_1 \cdot x_1 = y_1 \cdot a_2$) then we detach it from its end $y_1x_2 \in D_1x_2$ and attach it to the vertex $y_1a_2(x_2) \in D_1a_2(x_2)$. If $a_2|_{x_2} \neq 1$, then we label the obtained arrow by $a_2|_{x_2}$. The rest of the arrows of $\bigsqcup_{x_2 \in \mathsf{X}_2} D_1x_2$ are not changed.

It is easy to see that in this way we get the diagram of $\mathcal{A}_1 \otimes \mathcal{A}_2$. We see that the copies of D_1 are connected in the same way as the vertices of D_2 are. See, for example, in Figure 5 the diagram of $\mathcal{A} \otimes \mathcal{A}$, where $\mathcal{A}$ is the automaton from Figure 4.

It follows immediately that every element $a_3 \in \mathsf{X}_3$ is a label of exactly one arrow of the diagram of $\mathcal{A}_1 \otimes \mathcal{A}_2$.

Let us reformulate the procedure of construction of the diagram of $\mathcal{A}_1 \otimes \mathcal{A}_2$ in a more geometric way. The diagram is obtained by gluing discs, corresponding to the cells of D_2, to the copies of D_1 along their labeled edges. Namely, if the edge (a_1, x_1) is labeled in D_1 by $a_2 = a_1|_{x_1}$ and $x_2 \in \mathsf{X}_2$ belongs to a cycle $\big(x_2, a_2(x_2), \ldots, a_2^{k-1}(x_2)\big)$ of length k under the action of a_2, then we have to take a $2k$-sided polygon and glue its every other side to the copies of the edge (a_1, x_1) in the diagrams D_1x_2, $D_1a_2(x_2)$, $\ldots$, $D_1a_2^{k-1}(x_2)$ in the given cyclic order. We will glue in this way the k copies of a cell of D_1 together and get a cell of the diagram of $\mathcal{A}_1 \otimes \mathcal{A}_2$. See, for example, Figure 6, where the case $k = 4$ is shown. It is easy to see that in this procedure is equivalent to the one described before.

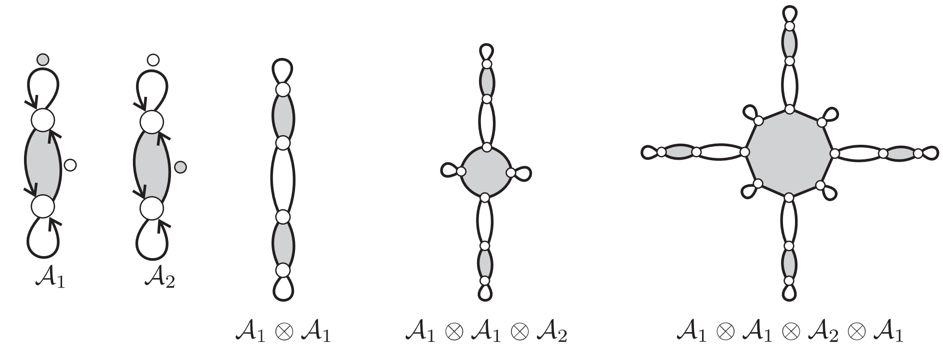

FIGURE 7. Products of dendroid automata

It follows that we can contract the copies of D_1 in the diagram of $\mathcal{A}_1 \otimes \mathcal{A}_2$ to single points, and get a cellular complex homeomorphic to D_2, which is contractible. This proves that the diagram of $\mathcal{A}_1 \otimes \mathcal{A}_2$ is contractible.

We also see that every cell of this diagram has at most 1 labeled side, since the labels come only from the attached $2k$-sided polygons, whose sides are labeled in the same way as the corresponding cell of D_2. □

Theorem 4.4. *A set $A = \{a_1, \ldots, a_m\}$ is a dendroid set of automorphisms of T if and only if there exists a sequence of automata $\mathcal{A}_n$, $n \geq 1$, and sequences of finite sets B_n and X_n such that $B_0 = A$ and*

- *$\mathcal{A}_n$ is a dendroid automaton over alphabet X_n, input set B_{n-1} and output set B_n;*
- *the action of the elements a_i on the rooted tree $\mathsf{X}^* = \bigcup_{n\geq 0} X_1 \times \cdots \times X_n$ defined by the sequence of automata $\mathcal{A}_n$ is conjugate to the action of a_i on T.*

In other words, all dendroid sets of automorphisms of a rooted tree are defined taking products of kneading automata.

The actions defined by sequences of automata is described in Definition 3.9.

Proof. Sequence of dendroid automata define dendroid sets of automorphisms of a rooted tree by Proposition 4.3.

On the other hand, we have seen in Proposition 4.2 that the automaton describing the bimodules $\mathfrak{M}_{\varnothing,v}$ with respect to the basis x_v and the generating set A is a dendroid automaton with a dendroid output set B, which is the generating set of $\langle A\rangle|_v$. It follows now from Corollary 3.2 that the action of A on the tree T is conjugate to the action defined by a sequence of dendroid automata. □

See on Figure 7 an example of products of a sequence of dendroid automata. It shows two dendroid automata $\mathcal{A}_1$ and $\mathcal{A}_2$ over the binary alphabet and their products $\mathcal{A}_1 \otimes \mathcal{A}_1$, $\mathcal{A}_1 \otimes \mathcal{A}_1 \otimes \mathcal{A}_2$ and $\mathcal{A}_1 \otimes \mathcal{A}_1 \otimes \mathcal{A}_2 \otimes \mathcal{A}_1$. We use colors “grey” and “white” instead of letters to mark the cells and arrows of the dual Moore diagrams (arrows are marked by dots near them). The diagrams of the products are shown without marking of the edges and orientation.

5. Backward polynomial iteration

5.1. **Iterated monodromy groups.** Let f_n be a sequence of complex polynomials and consider them as an inverse sequence of maps between complex planes:

$$\mathbb{C} \xleftarrow{f_1} \mathbb{C} \xleftarrow{f_2} \mathbb{C} \xleftarrow{f_3} \cdots .$$

We call such sequences *backward iterations*. More generally, we may consider backward iterations of *topological polynomials*. Here a topological polynomial is a continuous map $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ which is an orientation preserving local homeomorphism at all but a finite number of (*critical*) points of $\mathbb{R}^2$. For more on topological polynomials, see [BFH92] and [BN06b].

A backward iteration $(f_n)_{n\geq 1}$ is said to be *post-critically finite* if there exists a finite set $P \subset \mathbb{C}$ such that the set of critical values of the composition $f_1 \circ f_2 \circ \cdots \circ f_n$ is contained in P for all n. The smallest such set P is called the *post-critical set* of the iteration.

Example 1. Every sequence $(f_1, f_2, \ldots)$ such that f_i is either z^2 or $1 - z^2$ is post-critically finite with post-critical set a subset of $\{0, 1\}$.

If the backward iteration $f_1, f_2, \ldots$ is post-critically finite, then the shifted iteration $f_2, f_3, \ldots$ is also post-critically finite. Let us denote by P_n the post-critical set of the iteration $f_n, f_{n+1}, \ldots$.

Choose a basepoint $t \in \mathbb{C} \setminus P_1$ and consider the rooted *tree of preimages*

$$T_t = \{t\} \sqcup \bigsqcup_{n\geq 1} (f_1 \circ f_2 \circ \cdots \circ f_n)^{-1}(t),$$

where a vertex $z \in (f_1 \circ f_2 \circ \cdots \circ f_n)^{-1}(t)$ is connected by an edge with the vertex $f_n(z) \in (f_1 \circ f_2 \circ \cdots \circ f_{n-1})^{-1}(t)$ and the vertex t is considered to be the root.

The fundamental group $\pi_1(\mathbb{C} \setminus P_1, t)$ of the punctured plane acts on the tree T_t by the monodromy action. The image of a point $z \in (f_1 \circ f_2 \circ \cdots \circ f_n)^{-1}(t)$ under the action of a loop $\gamma \in \pi_1(\mathbb{C} \setminus P_1, t)$ is the end of the unique lift of γ by $f_1 \circ f_2 \circ \cdots \circ f_n$ starting at z. This is clearly an action by automorphisms of the tree T_t. This action is called the *iterated monodromy action*.

The *iterated monodromy group* of the iteration $f_1, f_2, \ldots$ is the quotient of the fundamental group by the kernel of the iterated monodromy action, i.e., the group of automorphisms of T_t defined by the loops $\gamma \in \pi_1(\mathbb{C} \setminus P_1, t)$.

Example 2. This is a slight generalization of a construction due to Richard Pink (see also [AHM05] and [BJ07]). Consider the field of rational functions $\mathbb{C}(t)$ and the polynomial over $\mathbb{C}(t)$

$$F_n(z) = f_1 \circ f_2 \circ \cdots \circ f_n(z) - t.$$

Let Ω_n be the splitting field of F_n in an algebraic closure of $\mathbb{C}(t)$. It is not hard to see that $\Omega_{n+1} \supset \Omega_n$. Denote by Ω the field $\bigcup_{n\geq 1} \Omega_n$. Then the Galois group of the extension $\Omega/\mathbb{C}(t)$ is isomorphic to the closure of the iterated monodromy group of the iteration $f_1, f_2, \ldots$ in the automorphism group of the tree of preimages T_t (see [Nek05] Proposition 6.4.2).

Definition 5.1. Let $P = \{z_1, z_2, \ldots, z_n\}$ be a finite subset of the plane $\mathbb{R}^2$. A *planar* generating set of the fundamental group $\pi_1(\mathbb{R}^2 \setminus P, t)$ is a collection of simple (i.e., without self-intersections) loops γ_i such that the region Γ_i bounded

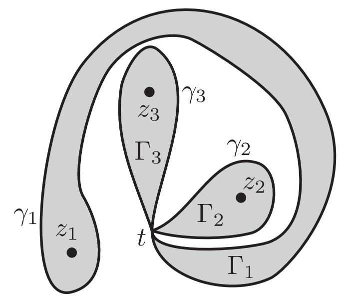

FIGURE 8. A planar generating set

by γ_i contains only one point z_i of P, γ_i goes around the region in the positive direction and Γ_i are disjoint for different i.

See Figure 8 for an example of a planar generating set of the fundamental group of a punctured plane.

There exists a unique cyclic order $\gamma_1, \gamma_2, \ldots, \gamma_n$ on a planar generating set such that the loop $\gamma_1\gamma_2\cdots\gamma_n$ has no transversal self-intersections. We call this order the *natural cyclic order* of the planar generating set.

Proposition 5.1. *Let $f_1, f_2, \ldots$ be a post-critically finite backward iteration of topological polynomials. Let P be its post-critical set and let $t \in \mathbb{R}^2 \setminus P$ be a basepoint. Let $\{\gamma_z\}_{z\in P}$ be a planar generating set of $\pi_1(\mathbb{R}^2 \setminus P, t)$, where γ_z goes around z. Let a_z be the corresponding elements of the iterated monodromy group of the backward iteration. Then the set $(a_z)_{z\in P}$ of automorphisms of the tree of preimages T_t is dendroid and generates the iterated monodromy group of the backward iteration.*

Proof. The loops γ_z generate the fundamental group, hence the elements a_z generate the iterated monodromy group. Denote $F = f_1 \circ f_2 \circ \cdots \circ f_n$. Let us prove that the generators a_z define a dendroid set of permutations of the nth level $F^{-1}(t)$ of the tree T_t.

Let $\Delta = \overline{\bigcup_{z\in P} \Gamma_z}$ be the closed part of the plane bounded by the curves γ_z. Then the set $\mathbb{C} \setminus \Delta$ contains no critical values of F and is a homeomorphic to the cylinder. Consequently, the map $F : F^{-1}(\mathbb{C} \setminus \Delta) \longrightarrow \mathbb{C} \setminus \Delta$ is a $\deg F$-covering. In particular, $F^{-1}(\mathbb{C} \setminus \Delta)$ is also an annulus, hence the set $F^{-1}(\Delta)$ is contractible. But it is easy to see that $F^{-1}(\Delta)$ is homeomorphic to the cycle diagram of the action of the set $(a_z)_{z\in P}$ on the nth level $F^{-1}(t)$ of the tree T_t. Hence the set of permutations of $F^{-1}(t)$ defined by $(a_z)_{z\in P}$ is dendroid. □

Consequently, the iterated monodromy group of any post-critically finite backward iteration is defined by a sequence of dendroid automata, due to Theorem 4.4.

Example 3. Consider, as in Example 1, a backward iteration $(f_1, f_2, \ldots)$, where $f_i(z) = z^2$ or $f_i(z) = 1 - z^2$ for every i. Then the iterated monodromy group is defined by the sequence of automata $(\mathcal{A}_{i_1}, \mathcal{A}_{i_2}, \ldots)$, where $\mathcal{A}_{i_j}$ is the automaton $\mathcal{A}_1$ from Figure 7, if $f_i(z) = 1 - z^2$ and $\mathcal{A}_2$ if $f_i(z) = z^2$.

Let us show that converse to Proposition 5.1 is also true.

Proposition 5.2. *Let $(a_1, \ldots, a_m)$ be a cyclically ordered dendroid set of automorphisms of a rooted tree T. Then there exists a post-critically finite backward iteration* *of complex polynomials $f_1, f_2, \dots$ with post-critical set $P = \{z_1, z_2, \dots, z_m\}$, a planar generating set $\{\gamma_i\}_{i=1,\dots,m}$ of $\pi_1(\mathbb{C} \setminus P, t)$ with the natural order $(\gamma_1, \dots, \gamma_m)$ (where γ_i goes around the point z_i), and an isomorphism of T with the preimage tree T_t conjugating the iterated monodromy action of the loops γ_i with the automorphisms a_i.*

Proof. Denote by D_n the cycle diagram of the sequence of permutations induced by $(a_1, \dots, a_m)$ on the nth level L_n of the tree T. The product $a_m \cdots a_1$ acts on L_n as a transitive cycle by Proposition 2.4. Choose a vertex $x \in L_n$ of D_n and consider the path γ_n starting at x and passing along the arrows

$$e_1, e_2, \dots, e_m, e_{m+1}, e_{m+2}, \dots, e_{m\cdot|L_n|},$$

where e_i is the edge corresponding to the action of a_i, where the indices of a_i are taken modulo m. The path γ_n is closed and contains every arrow of D_n exactly once.

Embed D_n into the plane so that γ_n is embedded as the path going around the image of D_n in the positive direction (without transverse self-intersections). Choose a point in each of the (images of the) 2-cells of D_n, which we will call *centers*.

The 1-skeleton of D_{n+1} covers of the 1-skeleton of D_n by the map carrying a vertex $v \in L_{n+1}$ to the adjacent vertex $u \in L_n$. This covering induces an $|L_{n+1}|/|L_n|$-fold covering of the curve γ_n by the curve γ_{n+1}. Let us extend this covering to a covering of the complement of the image of D_n in the plane by the complement of the image of D_{n+1}, so that we get a covering of the union of the complements with the images of the 1-skeletons of diagrams. We can then extend this covering inside the image of the diagrams so that we get an orientation-preserving branched covering of the planes, mapping D_{n+1} onto D_n so that the 1-skeletons are mapped as before, while the branching points are the centers of the 2-cells of D_{n+1} (not all centers must be branching points). We also require that images of centers are centers. See a more explicit construction in Theorem 6.10.4 of [Nek05], which can be easily adapted to our more general situation.

We get in this way a sequence of branched coverings $\mathbb{R}^2 \xleftarrow{f_1} \mathbb{R}^2 \xleftarrow{f_2} \cdots$ with embedded diagrams $D_0, D_1, \dots$. Note that D_0 is a bouquet of m circles going around m centers. The backward iteration $f_1, f_2, \dots$ is post-critically finite with the post-critical set equal to the set of centers of D_0. The set of preimages of the basepoint of D_0 under $f_1 \circ \cdots \circ f_n$ is equal to the image of $L_n \subset D_n$. It follows from the construction of the maps f_i that the action of the loops of D_n on the tree of preimages of the basepoint of D_0 coincides with the action of the corresponding elements a_i.

It remains now to introduce a complex structure on the first plane and pull it back by the branched coverings $f_1 \circ \cdots \circ f_n$ so that the maps f_i become complex polynomials. $\square$

5.2. **Bimodules associated with coverings.** We multiply paths (in particular in the fundamental groups) as functions: in a product $\gamma_1\gamma_2$ the path γ_2 is passed first.

Definition 5.2. Let $P_1, P_2 \subset \mathbb{R}^2$ be finite sets of points and let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be a continuous map such that $f : \mathbb{R}^2 \setminus f^{-1}(P_2) \longrightarrow \mathbb{R}^2 \setminus P_2$ is a d-fold covering and $f^{-1}(P_2) \supseteq P_1$. Denote $\mathcal{M}_i = \mathbb{R}^2 \setminus P_i$. We say that f is a *partial covering* of $\mathcal{M}_2$

by (a subset of) $\mathcal{M}_1$. Choose a basepoint $t_2 \in \mathcal{M}_2$ and let $t_1 \in f^{-1}(t_2)$ be some of its preimages.

The *bimodule* $\mathfrak{M}_f$ *associated with the covering* f is the $(\pi_1(\mathcal{M}_2, t_2)-\pi_1(\mathcal{M}_1, t_1))$-bimodule consisting of homotopy classes of paths in $\mathcal{M}_2$ starting in t_1 and ending in any point of $f^{-1}(t_2)$. The right action $\pi_1(\mathcal{M}_1, t_1)$ is appending the loop $\gamma \in \pi_1(\mathcal{M}_1, t_1)$ at the beginning of the path $\ell \in \mathfrak{M}_f$. The left action of $\pi_1(\mathcal{M}_2, t_2)$ is appending the f-lift of the loop $\gamma \in \pi_1(\mathcal{M}_2, t_2)$ to the end of the path $\ell \in \mathfrak{M}_f$.

It is not hard to prove that the bimodule $\mathfrak{M}_f$ does not depend (up to an isomorphism) on the choice of the basepoints t_1 and t_2, if we identify the respective fundamental groups in the usual way using paths.

Moreover, if $P_1 = P_2$, then we may identify the fundamental groups $\pi_1(\mathcal{M}_1, t_1)$ and $\pi_1(\mathcal{M}_2, t_2)$ by a path from t_1 to t_2, and assume that $\mathfrak{M}_f$ is a $\pi_1(\mathcal{M})$-bimodule, where $\mathcal{M} = \mathcal{M}_1 = \mathcal{M}_2$. The isomorphism class of the $\pi_1(\mathcal{M})$-bimodule $\mathfrak{M}_f$ will not depend on the choice of the connecting path.

Note also that the bimodule $\mathfrak{M}_f$ has free right action and that two elements belong to the same orbit of the right action if and only if the ends of the corresponding paths coincide. Hence the number of the right orbits is equal to the degree of the covering f. The induced left action of the group $\pi_1(\mathcal{M}_2, t_2)$ on the right orbits coincides with the monodromy action on the fiber $f^{-1}(t_2)$, if we identify orbits with the corresponding endpoints.

Proposition 5.3. *Let $P_1, P_2, P_3 \subset \mathbb{R}^2$ be finite sets and let $f_i : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ for $i = 1, 2$ be branched coverings such that the set of critical values of f_i is contained in P_i and $f_i(P_{i+1}) \subseteq P_i$ for $i = 1, 2$. Let $\mathfrak{M}_{f_i}$ be the bimodules associated with the respective partial coverings f_i of $\mathbb{R}^2 \setminus P_i$ by a subset of $\mathbb{R}^2 \setminus P_{i+1}$.*

Then the bimodule $\mathfrak{M}_{f_1} \otimes \mathfrak{M}_{f_2}$ is isomorphic to the bimodule $\mathfrak{M}_{f_1 \circ f_2}$ associated with the partial covering of $\mathbb{R}^2 \setminus P_1$ by a subset of $\mathbb{R}^2 \setminus P_3$.

Proof. Denote $\mathcal{M}_i = \mathbb{R}^2 \setminus P_i$ and let the basepoints $t_i \in \mathcal{M}_i$ be chosen so that $f_i(t_{i+1}) = t_i$. Then $\mathfrak{M}_{f_i}$ consists of the homotopy classes of paths in $\mathcal{M}_{i+1}$ starting in t_{i+1} and ending in a point from the set $f_i^{-1}(t_i)$.

For $\ell_i \in \mathfrak{M}_{f_i}$ for $i = 1, 2$ be arbitrary elements. Then it is an easy exercise to prove that the map

$$\ell_1 \otimes \ell_2 \mapsto f_2^{-1}(\ell_1)_{\ell_2} \ell_2$$

is an isomorphism of bimodules, where $f_2^{-1}(\ell_1)_{\ell_2}$ is the lift of ℓ_1 by f_2 to a path starting at the end of ℓ_2. □

Proposition 5.4. *Let f_n be a post-critically finite backward iteration of topological polynomials. Let P_n be the post-critical set of the iteration $f_n, f_{n+1}, \ldots$. Then the iterated monodromy action of $\pi_1(\mathbb{R}^2 \setminus P_1)$ is conjugate to the action associated with the infinite tensor product $\mathfrak{M}_{f_1} \otimes \mathfrak{M}_{f_2} \otimes \cdots$.*

Proof. The bimodule $\mathfrak{M}_{f_1} \otimes \cdots \otimes \mathfrak{M}_{f_n}$ is isomorphic to the bimodule $\mathfrak{M}_{f_1 \circ \cdots \circ f_n}$, hence the corresponding left action on the right orbits coincides with the monodromy action associated with the covering $f_1 \circ \cdots \circ f_n$. □

We have seen that the bimodules associated with dendroid sets of automorphisms of a rooted tree can be put in a simple form of dendroid automata. Let us describe how it is done for the bimodule $\mathfrak{M}_f$ associated with a partial covering directly and interpret the dendroid automata geometrically. The construction that we are going to describe is nothing more than just a translation of the proof of Proposition 4.1 in terms of paths and monodromy actions.

Let, as above P_1, P_2 be finite subsets of the plane and let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be an orientation preserving branched covering such that P_2 contains all critical values of f and $f(P_1) \subseteq P_2$. Denote $\mathcal{M}_i = \mathbb{R}^2 \setminus P_i$ and choose a basepoint $t_2 \in \mathcal{M}_2$ and $f_1 \in f^{-1}(t_2)$. Let $\mathfrak{M}_f$ be the associated $(\pi_1(\mathcal{M}_2, t_2) - \pi_1(\mathcal{M}_1, t_1))$-bimodule.

Let $\gamma_1, \ldots, \gamma_n$ be a planar generating set of $\pi_1(\mathcal{M}_2, t_2)$, where γ_i goes around a point $z_i \in P_2$.

The union of the preimages of γ_i under f is a oriented graph isomorphic to the 1-skeleton of the cycle diagram of the monodromy action of the sequence $\gamma_1, \gamma_2, \ldots, \gamma_n$ on the set $f^{-1}(t_2)$. Every connected component of the total inverse image $f^{-1}(\gamma_i)$ corresponds to a cycle of the monodromy action of γ_i and is a closed simple path containing a unique point of $f^{-1}(P_2)$. If this point belongs to P_1, then we call the component *active*. The topological disc bounded by a component of $f^{-1}(\gamma_i)$ is called a *cell*. Every cell corresponds to a 2-cell of the cycle diagram of the monodromy action.

Choose one lift e_i of γ_i in each active component of $f^{-1}(\gamma_i)$ and call the chosen arcs *marked*. The choice of the marked paths (together with the choice of the generators γ_i) determine a basis of $\mathfrak{M}_f$ and a generating set of $\pi_1(\mathcal{M}_1, t_1)$ with respect to which it is given by a dendroid automaton. Namely, for $z \in P_1$ define γ'_z as the loop going along unmarked paths of $f^{-1}(\gamma_1\gamma_2\cdots\gamma_n)$ to the beginning of the marked path e of the component containing z, then going back to t_1 along unmarked paths. Since the union of unmarked paths is contractible in $\mathcal{M}_1$, this description defines $\gamma'_z \in \pi_1(\mathcal{M}_1, t_1)$ uniquely. It is also easy to see that the loops γ'_z generate freely the fundamental group $\pi_1(\mathcal{M}_1, t_1)$.

For every $t \in f^{-1}(t_2)$ the corresponding element x_t of the basis of $\mathfrak{M}_f$ is the path connecting t_1 to t along the unmarked paths of $f^{-1}(\gamma_1\gamma_2\cdots\gamma_n)$.

The following proposition is checked directly.

Proposition 5.5. *The recursion defined by the generating sets γ_i, γ'_z and the basis x_t is given by a dendroid automaton. The generating set $\{\gamma'_z\}$ of $\pi_1(\mathcal{M}_1, t_1)$ is planar.*

5.3. **Cyclically ordered dendroid automata.** Let $\mathcal{A}$ be a dendroid automaton over the alphabet X with the input set A and output set B. Let D be the dual Moore diagram of $\mathcal{A}$. Let $(a_1, a_2, \ldots, a_m)$ be a cyclic order on the input set A. Dendroid automata with a cyclic order of the input set are called *cyclically ordered*.

It follows from Proposition 2.4 that the product $a = a_m \cdots a_2 a_1$ of the permutations of X defined by the states a_i acts transitively on the alphabet X.

Let us denote by (a_i, y) for $y \in \mathsf{X}$ the edge of D corresponding to a_i and starting in y. Pick a letter $x \in \mathsf{X}$. Since the product $a = a_m \cdots a_2 a_1$ is transitive on X, the oriented edge path

$$\begin{gathered}
(a_1, x), (a_2, a_1(x)), \ldots, (a_m, a_{m-1}\cdots a_2 a_1(x)) \\
(a_1, a(x)), (a_2, a_1 a(x)), \ldots, (a_m, a_{m-1}\cdots a_1 a(x)) \\
\vdots \\
(a_1, a^{d-1}(x)), (a_2, a_1 a^{d-1}(x)), \ldots, (a_m, a_{m-1}\cdots a_1 a^{d-1}(x)),
\end{gathered}$$

where $d = |\mathsf{X}|$, is a closed cycle containing every edge of D exactly once. Consequently, the sequence of their labels

$$\begin{gathered} a_1|_x, a_2|_{a_1(x)}, \ldots, a_m|_{a_{m-1}\cdots a_2 a_1(x)} \\ a_1|_{a(x)}, a_2|_{a_1 a(x)}, \ldots, a_m|_{a_{m-1}\cdots a_1 a(x)} \\ \vdots \\ a_1|_{a^{d-1}(x)}, a_2|_{a_1 a^{d-1}(x)}, \ldots, a_m|_{a_{m-1}\cdots a_1 a^{d-1}(x)}, \end{gathered}$$

contains every element of the output set B of the dendroid automaton $\mathcal{A}$ once (while the rest of its elements are trivial). Let $(b_1, b_2, \ldots, b_k)$ be the cyclic order on B of appearance in the above sequence. We call it the *induced cyclic ordering of the output set*. The cyclic ordering of the induced generating set depends only on the cyclic ordering of the original set $(a_1, a_2, \ldots, a_m)$. Note that different choice of the initial letter x changes only the initial point of the sequence $(b_1, b_2, \ldots, b_k)$, but does not change the cyclic ordering.

Suppose that we have a sequence $\mathcal{A}_1, \mathcal{A}_2, \ldots$ of dendroid automata over a sequence of alphabets $(X_1, X_2, \ldots)$, such that $\mathcal{A}_i$ has an input set A_{i-1} and output set A_i. Then every cyclic order on A_0 induced a cyclic order on the output set A_1 of $\mathcal{A}_1$, which in turn induces a cyclic order on the output set A_2 of $\mathcal{A}_2$ (since A_1 is the input set of $\mathcal{A}_1$), and so on. Thus, every cyclic order on A_0 induces cyclic orders on each of the sets A_i.

We leave the proof of the following proposition to the reader.

Proposition 5.6. *Let $\mathfrak{M}_f$ be the $(\pi_1(\mathcal{M}_2), \pi_1(\mathcal{M}_1))$-bimodule of a partial covering f of punctured planes. Let $A = (\gamma_1, \gamma_2, \ldots, \gamma_n)$ be a planar generating set of $\pi_1(\mathcal{M}_2)$ in its natural order. Choose a marking of the inverse image of this generating set and let $\mathcal{A}$ be the corresponding dendroid automaton. Then the cyclic ordering on the output set of $\mathcal{A}$ induced by the natural ordering of A coincides with its natural cyclic ordering as a planar generating set of $\pi_1(\mathcal{M}_1)$.*

5.4. **Action of the braid groups.** Let us denote the set of all dendroid sequences of length m of elements of $\mathrm{Aut}\,(T)$ by $\Delta_m(T)$. If $(a_1, a_2, \ldots, a_m) \in \Delta_m(T)$ and g belongs to the group generated by $\{a_1, \ldots, a_m\}$, then the sequence $(a_1^g, a_2^g, \ldots, a_m^g)$ belongs to $\Delta_m(T)$ and generates the same group.

We also know that the map

$$\beta_i : (a_1, a_2, \ldots, a_i, a_{i+1}, \ldots, a_m) \mapsto (a_1, a_2, \ldots, a_{i+1}, a_i^{a_{i+1}}, \ldots, a_m)$$

is an invertible transformation of $\Delta_m(T)$ (by Corollary 2.3), which does not change the group generated by the sequence.

The transformations β_i satisfy the usual defining relations for the generators of the braid group B_m on m strands, hence we get an action of B_m on $\Delta_m(T)$.

We will need the action of the braid group on dendroid sequence due to the following fact.

Proposition 5.7. *If $(\gamma_1, \gamma_2, \ldots, \gamma_n)$ and $(\delta_1, \delta_2, \ldots, \delta_n)$ are planar generating sets of the fundamental group of a punctured plane in their natural orders, then there exists an element of the braid group $\alpha \in B_n$ and an element γ of the fundamental group such that*

$$(\gamma_1^\gamma, \gamma_2^\gamma, \ldots, \gamma_n^\gamma)^\alpha = (\delta_1, \delta_2, \ldots, \delta_n).$$

Proof. By a classical result, the braid group is the mapping class group of the n-punctured disc. Conjugating by γ, we may achieve that $(\gamma_1\gamma_2\cdots\gamma_n)^\gamma = \delta_1\delta_2\cdots\delta_n$. After that α is the braid representing the isotopy class of the homeomorphisms mapping γ_i^γ to δ_i. $\square$

6. Iteration of a single polynomial

6.1. **Planar generating sets.** Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be a post-critically finite topological, so that the backward iteration $(f, f, \ldots)$ is post-critically finite. Let P be the post-critical set of this iteration. Denote $\mathcal{M} = \mathbb{R}^2 \setminus P$ and let $t \in \mathcal{M}$ be an arbitrary basepoint.

Consider a planar generating set $\{\gamma_z\}_{z\in P}$ of $\pi_1(\mathcal{M}, t)$. Then construction of Proposition 5.5 gives a basis $\{x_p\}_{p\in f^{-1}(t)}$ of $\mathfrak{M}_f$ and the induced planar generating set $\{\gamma'_z\}_{z\in P}$ of the fundamental group $\pi_1(\mathcal{M}, t_1)$, where $t_1 \in f^{-1}(t)$.

Let us identify the fundamental groups $\pi_1(\mathcal{M}, t)$ and $\pi_1(\mathcal{M}, t_1)$ by a path, i.e., by the isomorphism

$$L : \pi_1(\mathcal{M}, t) \longrightarrow \pi_1(\mathcal{M}, t_1) : \gamma \mapsto \ell^{-1}\gamma\ell,$$

where ℓ is a path starting at t_1 and ending in t. This identification is well defined, up to inner automorphisms of the fundamental groups.

After the identification by the isomorphism L the bimodule $\mathfrak{M}_f$ becomes a bimodule over one group. More formally, denote by F_n the free group with the free generating set $g_1, g_2, \ldots, g_n$. Let us identify F_n with the fundamental group $\pi_1(\mathcal{M}, t)$ by the isomorphism $\phi_0 : g_i \mapsto \gamma_{z_i}$ and with $\pi_1(\mathcal{M}, t_1)$ by the isomorphism $L\circ\phi_0$. Then, by Proposition 5.7, $(L\circ\phi_0)^{-1}(\gamma'_{z_{i_k}}) = g_k^{g\alpha}$ for some $g \in F_n$ and braid $\alpha \in B_n < \mathrm{Aut}\,(F_n)$. Composing the wreath recursion with an inner automorphism of the wreath product $\mathfrak{S}\,(d) \wr F_n$, we may assume that $g = 1$.

Then the bimodule $\mathfrak{M}_f$ is described by an ordered automaton with the input set $(g_1, g_2, \ldots, g_n)$ and the output set $(g_1, g_2, \ldots, g_n)^\alpha$. Here the ordering of the output set is induced by the ordering of the input set and corresponds, by Proposition 5.6, to the natural ordering of the respective planar generating set.

Definition 6.1. A *twisted kneading automaton* is a dendroid automaton with a cyclically ordered input set $(g_1, \ldots, g_n)$ and output set $(g_1, \ldots, g_n)^\alpha$ for some $\alpha \in B_n$.

Question 1. Is every bimodule given by a twisted kneading automaton sub-hyperbolic.

The twisted kneading automaton comes with a cyclic orderings of the input and output sets agreeing with the natural order on the corresponding planar generating sets of the fundamental groups. The cyclic order on the output set coincides with the order induced by the cyclic order on the input set (see Proposition 5.6).

For the definition of Thurston combinatorial equivalence of post-critically finite branched coverings of the sphere, see [DH93].

Proposition 6.1. *The twisted kneading automaton associated to the topological polynomial f together with the cyclic order $g_1, g_2, \ldots, g_n$ of the generators of F_n uniquely determines the Thurston combinatorial class of the polynomial f.*

Proof. The Proposition is a direct corollary of the following fact, which is Theorem 6.5.2 of [Nek05].

Theorem 6.2. *Let f_1, f_2 be post-critically finite orientation preserving self-coverings of the sphere S^2 with post-critical sets P_{f_1}, P_{f_2} and let $\mathfrak{M}_{f_i}$, $i = 1, 2$, be the respective $\pi_1\left(S^2 \setminus P_{f_i}\right)$-bimodules.*

Then the maps f_1 and f_2 are combinatorially equivalent if and only if there exists an isomorphism $h_* : \pi_1\left(S^2 \setminus P_{f_1}\right) \longrightarrow \pi_1\left(S^2 \setminus P_{f_2}\right)$ conjugating the bimodules $\mathfrak{M}(f_1)$ and $\mathfrak{M}(f_2)$ and induced by an orientation preserving homeomorphism $h : S^2 \longrightarrow S^2$ such that $h\left(P_{f_1}\right) = P_{f_2}$.

Here we say that an isomorphism conjugates the bimodules if the bimodules become isomorphic if we identify the groups by the isomorphism. □

Of course, it would be nice to be able to find the simplest possible twisted kneading automaton describing $\mathfrak{M}_f$. In particular, we would like to know if α can be made trivial. We will call the generating set $\{\gamma_{z_i}\}$ *invariant* (for a given marking), if $\alpha = 1$, i.e., if $L(\gamma_{z_i}) = \gamma'_{z_i}$. In this case the twisted kneading automaton describing the bimodule $\mathfrak{M}_f$ is called a *kneading automaton* (see Section 6.7 of [Nek05]).

The following theorem is proved in [Nek05] (Theorems 6.8.3 and 6.9.1), where more details can be found.

Theorem 6.3. *If a post-critically finite polynomial f is* hyperbolic *(i.e., if every post-critical cycle contains a critical point), then $\pi_1(\mathcal{M}, t)$ has an invariant generating set for some choice of marking.*

In general, if f is post-critically finite, then there exists n such that there exists an invariant generating set of $\pi_1(\mathcal{M}, t)$ for the nth iteration of f.

Sketch of the proof. The idea of the proof is to find an *invariant spider* of the polynomial using external angles, i.e., a collection of disjoint curves p_{z_i} connecting the post-critical points to infinity such that $\bigcup_{z_i \in P} f^{-1}(p_{z_i})$ contains $\bigcup_{z_i \in P} p_{z_i}$ (up to homotopies). If we can find such a spider, then the generators γ_{z_i} are uniquely defined by the condition that γ_{z_i} is a simple loop going around z_i in the positive direction and intersects only p_{z_i} and only once.

Every active component of $f^{-1}(\gamma_{z_i})$ will go around one post-critical point, and hence will intersect only one path p_{z_j}. The arc intersecting p_{z_j} will be marked. It is easy to show that the chosen generating set will be invariant with respect to the given marking, if we identify $\pi_1(\mathcal{M}, t)$ with $\pi_1(\mathcal{M}, t_1)$ by a path disjoint with the legs (i.e., paths) of the spider.

If f is hyperbolic, then there exists an invariant spider constructed using "external-internal" rays. In the general case there will be no way to choose an invariant collection of external rays, but any such collection will be periodic, hence after passing to some iteration of f, we may find an invariant spider. □

6.2. **Example: quadratic polynomials.** If f is a post-critically finite hyperbolic quadratic polynomial, then $\mathfrak{M}_f$ can be described by a planar kneading automaton (see Theorem 6.3). The following description of such kneading automata is proved in [BN06a] (see also [Nek05] Section 6.11).

If $(x_1 x_2 \ldots x_n *)^\omega$ is the kneading sequence of the polynomial f (for a definition of the kneading sequence see the above references and [BS02]), then the bimodule

$\mathfrak{M}_f$ is described by the following wreath recursion

$$a_1 = \sigma(1, a_n)$$
$$a_{i+1} = \begin{cases} (a_i, 1) & \text{if } x_i = 0 \\ (1, a_i) & \text{if } x_i = 1 \end{cases}, \quad i = 1, \ldots, n-1,$$

where $\sigma \in \mathfrak{S}(2)$ is the transposition.

In general, if one deletes the trivial state and all arrows coming into it in the Moore diagram of a kneading automaton and inverts the direction of all arrows, then one will get a graph isomorphic to the graph of the action of f on its post-critical set.

Not for every sub-hyperbolic polynomial f the bimodule $\mathfrak{M}_f$ can be represented by a kneading automaton. However, it follows from the results of [BN06a] that for every sub-hyperbolic iteration $(f, f, \ldots)$ of the same post-critically finite quadratic polynomial there exists a constant sequence of kneading automata $(\mathcal{A}, \mathcal{A}, \ldots)$, such that the iterated monodromy action is conjugate to the action defined by this sequence of automata. However, the input-output set of the automaton $\mathcal{A}$ will correspond to different generating sets of the fundamental group for different instances of the bimodule in the sequence (the sequence of generating sets is though periodic).

6.3. **Combinatorial spider algorithm.** The proof of Theorem 6.3 is analytic, which seems to be not satisfactory. At least it would be interesting to understand invariance of generating sets (or, which is equivalent, of spiders) in purely algebraic terms. For instance, it would be nice to be able to find algorithmically the simplest automaton describing the bimodule $\mathfrak{M}_f$ starting from a given twisted kneading automaton (which can be easily found for a given topological polynomial). This will give a way to decide when two given topological polynomials are combinatorially equivalent (due to Proposition 6.1 below) and possibly will give a better understanding of Thurston obstructions of topological polynomials.

We propose here an algorithm, inspired by the "spider algorithm" by J. Hubbard and D. Schleicher from [HS94] and by the solution of the "Hubbard's twisted rabbit problem" in [BN06b]. Note that this algorithm provides only a combinatorial information about the polynomial and it lacks an important part of the original spider algorithm of J. Hubbard and D. Schleicher: numerical values of coefficients of the polynomial.

Suppose that we have a post-critically finite topological polynomial f and suppose that we have chosen an arbitrary planar generating set $\{\gamma_{z_i}\}_{z_i \in P}$ of $\pi_1(\mathcal{M}, t)$ and a marking, so that we have a bimodule over the free group F_n given by a twisted kneading automaton with the input generating set $(g_1, \ldots, g_n)$ and the output generating set $(g_1, \ldots, g_n)^\alpha$ for some braid $\alpha \in B_n$. Let $\psi : F_n \longrightarrow \mathfrak{S}(\mathsf{X}) \wr F_n$ be the associated wreath recursion. We write it as a list

$$\psi(g_i) = \sigma_i(h_{1,i}, h_{2,i}, \ldots, h_{d,i}).$$

The elements $h_{k,i} \in F_n$ are either trivial or of the form g_j^α.

Our aim is to simplify α by changing the generating set of F_n. Namely, we replace the ordered generating set (g_i) of F_n by the generating set $(g_{i,1}) = (g_i)^\alpha$, computing the images of g_i^α under ψ and rewriting the coordinates of the wreath product as words in the generating set $\{g_{i,1}\}$.

The new wreath recursion on the ordered generating set $(g_{i,1})$ will not correspond to a dendroid automaton, but since the corresponding generating set of the fundamental group is planar we can post-conjugate the recursion (i.e., change the basis of the bimodule), using Proposition 5.5, so that the new recursion will correspond to a twisted kneading automaton, i.e., to a dendroid automaton with the output set (i.e., the induced generating set) of the form $(g_{i,1})^{\alpha_1}$ for some new element $\alpha_1 \in B_n$. Note that the element α_1 is defined only up to an inner automorphism of F_n. This means that we actually work with the quotient $\overline{B}_n$ of the braid group B_n by its center.

The idea is that α_1 will be shorter than α and therefore iterations of this procedure will give us simple wreath recursions. In many cases an invariant generating set can be found in this way. We will formalize the algorithm and the question of its convergence in more algebraic terms in the next subsection. Here we present examples of work of this algorithm.

Example 4. A detailed analysis of this example (in a more general setting) is described in [BN06b]. Consider the "rabbit polynomial", which is a quadratic polynomial f for which the bimodule $\mathfrak{M}_f$ is described by the wreath recursion

$$\begin{aligned} a &= \sigma(1, c), \\ b &= (1, a), \\ c &= (1, b), \end{aligned}$$

with the cyclic order (a, b, c), where $\sigma \in \mathfrak{S}(2)$ is the transposition. Note that $abc = \sigma(1, cab)$, hence this cyclic order agrees with the structure of the kneading automaton.

Let us pre-compose now this polynomial with the Dehn twist around the curve bc. The obtained post-critically finite topological polynomial will correspond to the following wreath recursion

$$\begin{aligned} a &= \sigma(1, c^{bc}), \\ b &= (1, a), \\ c &= (1, b^{bc}), \end{aligned}$$

which is now a twisted kneading automaton with the same cyclic order of the generators. Let us run the combinatorial spider algorithm on this example. With respect to the new generating set $a_1 = a, b_1 = b^{bc} = b^c, c_1 = c^{bc}$ the wreath recursion is

$$\begin{aligned} a_1 &= \sigma(1, c_1), \\ b_1 &= (1, a)^{(1, b^c)} = (1, a_1^{b_1}), \\ c_1 &= (1, b^c)^{(1,a)(1,b^c)} = (1, b_1^{a_1 b_1}). \end{aligned}$$

This is still not a kneading automaton, but a dendroid automaton with the output set $a_2 = a_1^{b_1}, b_2 = b_1^{a_1 b_1}, c_2 = c_1$. We have to rewrite now the last recursion in terms of the new generating set a_2, b_2, c_2 (with the cyclic order (a_2, b_2, c_2)).

$$\begin{aligned} a_2 &= (1, a_1^{b_1})^{-1}\sigma(1, c_1)(1, a_1^{b_1}) = \sigma((a_1^{b_1})^{-1}, c_1 a_1^{b_1}) = \sigma(a_2^{-1}, c_2 a_2), \\ b_2 &= (1, a_1^{b_1})^{\sigma(1,c_1)(1,a_1^{b_1})} = (a_1^{b_1}, 1) = (a_2, 1), \\ c_2 &= (1, b_2). \end{aligned}$$

This is not a planar automaton, but composing the wreath recursion with conjugation by $(a_2, 1)$, we get

$$\begin{aligned} a_2 &= \sigma(1, c_2^{a_2}) \\ b_2 &= (a_2, 1) \\ c_2 &= (1, b_2). \end{aligned}$$

Again, we have to change the generating set to $a_3 = a_2, b_3 = b_2, c_3 = c_2^{a_2}$. Note that this generating set is ordered (a_3, c_3, b_3), since we have applied one generator of the braid group and $c_2 a_2 b_2 = a_2 c_2^{a_2} b_2 = a_3 c_3 b_3$. Then

$$\begin{aligned} a_3 &= \sigma(1, c_3), \\ b_3 &= (a_3, 1), \\ c_3 &= (1, b_2)^{\sigma(1,c_3)} = (b_2, 1) = (b_3, 1). \end{aligned}$$

Hence this topological polynomial is described by the kneading automaton

$$a_3 = \sigma(1, c_3), \quad b_3 = (a_3, 1), \quad c_3 = (b_3, 1),$$

with the cyclic order (a_3, c_3, b_3) of the generators. Conjugating the wreath recursion by σ, we get

$$a_3 = \sigma(c_3, 1), \quad b_3 = (1, a_3), \quad c_3 = (1, b_3),$$

which is exactly the recursion for a^{-1}, b^{-1}, c^{-1}. This implies that this recursion corresponds to the polynomial, which is complex conjugate to the original "rabbit polynomial".

6.4. **The bimodule of twisted kneading automata.** We will need some more technical notions related to permutational bimodules.

Definition 6.2. If α is an automorphism of a group G, then the associated bimodule $[\alpha]$ is the set of expressions of the form $\alpha \cdot g$ for $g \in G$, where the actions are given by

$$(\alpha \cdot g) \cdot h = \alpha \cdot gh$$

and

$$h \cdot (\alpha \cdot g) = \alpha \cdot h^{\alpha} g.$$

We denote the element $(\alpha \cdot 1)$ just by α.

Proposition 6.4. *If α is an inner automorphism, then the bimodule $[\alpha]$ is isomorphic to the* trivial bimodule G *with the natural left and right actions of G on itself. In particular, if $\mathfrak{M}$ is a G-bimodule, then $\mathfrak{M} \otimes [\alpha]$ and $[\alpha] \otimes \mathfrak{M}$ are isomorphic to $\mathfrak{M}$.*

Proof. Suppose that α is conjugation by g, i.e., that $h^{\alpha} = h^{g}$ for all $h \in G$. Then the map $x \mapsto \alpha \cdot g^{-1}x$ is an isomorphism of the bimodules G and $[\alpha]$, since

$$h \cdot (\alpha \cdot g^{-1}x) = \alpha \cdot g^{-1}hg \cdot g^{-1}x = \alpha \cdot g^{-1} \cdot hx$$

and

$$(\alpha \cdot g^{-1}x) \cdot h = \alpha \cdot g^{-1} \cdot (xh)$$

for all $h, x \in G$.

The rest of the proposition follows from the fact that $\mathfrak{M} \otimes G$ and $G \otimes \mathfrak{M}$ are isomorphic to $\mathfrak{M}$ by the isomorphisms $x \otimes g \mapsto x \cdot g$ and $g \otimes x \mapsto g \cdot x$. $\square$

Let, as above, f be a post-critically finite topological polynomial with the post-critical set P. Fix a basepoint $t \in \mathcal{M} = \mathbb{R}^2 \setminus P$ and $t_1 \in f^{-1}(t)$. Let $\mathfrak{M}_f$ be the associated $(\pi_1(\mathcal{M}, t) - \pi_1(\mathcal{M}, t_1))$-bimodule.

Denote by F_n the free group of rank $n = |P|$ with the basis $g_1, g_2, \ldots, g_n$. Let us choose, as in the previous subsections, a planar generating set $\{\gamma_i\}$ of $\pi_1(\mathcal{M}, t)$. Choose marked lifts of γ_i and let $\gamma_1', \gamma_2', \ldots, \gamma_m'$ be the induced generating set of $\pi_1(\mathcal{M}, t_1)$. Connect t and t_1 by a path and let $L : \pi_1(\mathcal{M}, t) \longrightarrow \pi_1(\mathcal{M}, t_1)$ be the corresponding isomorphism of fundamental groups.

We get then a pair of isomorphisms $\phi : F_n \longrightarrow \pi_1(\mathcal{M}, t)$ and $\phi_1 : F_n \longrightarrow \pi_1(\mathcal{M}, t_1)$ given by

$$\phi(g_i) = \gamma_i, \qquad \phi_1(g_i) = L(\gamma_i).$$

Let $\mathfrak{M}_0$ be the F_n-bimodule obtained from $\mathfrak{M}_f$ by identification of the group F_n with $\pi_1(\mathcal{M}, t)$ and $\pi_1(\mathcal{M}, t_1)$ by the isomorphisms ϕ and ϕ_1. More formally, it is the bimodule equal to $\mathfrak{M}_f$ as a set, with the actions given by

$$g \cdot x \cdot h = \phi(g) \cdot x \cdot \phi_1(h),$$

where on the right hand side the original action of $\pi_1(\mathcal{M}, t)$ and $\pi_1(\mathcal{M}, t_1)$ are used.

Note that it follows from Proposition 6.4 that the isomorphism class of $\mathfrak{M}_0$ does not depend on a particular choice of the connecting path defining the isomorphism L.

Recall, that the *outer automorphism group* of the free group F_n is

$$\mathrm{Out}(F_n) = \mathrm{Aut}\,(F_n) \,/\, \mathrm{Inn}(F_n),$$

where $\mathrm{Inn}(F_n)$ is the group of inner automorphisms of F_n. Let $\mathcal{G}_n$ be the image in $\mathrm{Out}(F_n)$ of the group generated by the automorphisms of F_n of the form

$$g_k^{a_{ij}} = \begin{cases} g_k & \text{if } k \neq i, \\ g_i^{g_j} & \text{if } k = i. \end{cases}$$

Note that it follows from Corollary 2.3 that image of a dendroid sequence of automorphisms of a tree under the action of the elements of $\mathcal{G}_n$ is a dendroid sequence.

Denote by $\mathfrak{G}_f$ the set of isomorphism classes of F_n-bimodules $[\alpha] \otimes \mathfrak{M}_0 \otimes [\beta]$ for all pairs $\alpha, \beta \in \mathcal{G}_n$.

Recall that for every $\alpha \in \mathrm{Out}(F_n)$ and every F_n-bimodule $\mathfrak{M}$ the bimodules $[\alpha] \otimes \mathfrak{M}$ and $\mathfrak{M} \otimes [\alpha]$ are well defined, up to an isomorphism of bimodules, by Proposition 6.4.

Note also that the automorphisms of F_n coming from the braid group B_n become also elements of $\mathcal{G}_n$, if we permute the images of the generators, so that the image of every generators g_i is conjugate to g_i.

Proposition 6.5. *Every element of $\mathfrak{G}_f$ is equal to $\mathfrak{M} \otimes [\alpha]$ for some $\alpha \in \mathcal{G}_n$ and an F_n-bimodule $\mathfrak{M}$ given with respect to the input set $\{g_i\}$ by a kneading automaton (in some basis of $\mathfrak{M}$).*

Recall that a kneading automaton is a dendroid automaton with the same input and output sets.

Proof. It is sufficient to show that for any kneading automaton $\mathcal{A}$ defining a bimodule $\mathfrak{M}$ and every generator a_{ij} of $\mathcal{G}_n$ the bimodule $[a_{ij}] \otimes \mathfrak{M}$ is of the form $\mathfrak{M}' \otimes [\alpha]$ for some $\alpha \in \mathcal{G}_n$ and a bimodule $\mathfrak{M}'$ given by a kneading automaton.

We can find an ordering of the generating set of F_n such that a_{ij} is a generator of the braid group (one just has to put the generators g_i and g_j next to each other).

We can assume then that $[a_{ij}] \otimes \mathfrak{M}$ is a bimodule associated with a post-critically finite topological polynomial. But any such a bimodule can be represented as a twisted kneading automaton, i.e., is of the form $\mathfrak{M}' \otimes [\alpha]$ for some bimodule $\mathfrak{M}'$ given by a kneading automaton and an element $\alpha \in \mathcal{G}_n$. □

The combinatorial spider algorithm can be formalized now in the following way. The cyclically ordered generating set $(\phi_1^{-1}(\gamma_1'), \ldots, \phi_1^{-1}(\gamma_n'))$ of F_n corresponding to the induced generating set of $\pi_1(\mathcal{M}, t_1)$ is the image of the standard generating set $(g_i = \phi^{-1}(\gamma_i))$ of F_n under an element of the braid group, which modulo permutation of the images of the generators is equal to some $\alpha \in \mathcal{G}_n$. Consequently, $\mathfrak{M}_0 = \mathfrak{M}_0' \otimes [\alpha]$, where $\mathfrak{M}_0'$ is an F_n-bimodule given by a kneading automaton. Our aim is to find a planar generating set of $\pi_1(\mathcal{M}, t)$ for which α is trivial, or as short as possible. Since all planar generating sets are obtained from any planar generating set by application of elements of the braid group, we will change the generating set of F_n applying elements of $\mathcal{G}_n$.

Changing the generating set of F_n from (g_i) to $(g_i)^\alpha$ corresponds to conjugation of the bimodule $\mathfrak{M}_0$ by α^{-1}, i.e., to passing from $\mathfrak{M}_0' \otimes [\alpha]$ to $\mathfrak{M}_1 = [\alpha] \otimes \mathfrak{M}_0'$. The bimodule $[\alpha] \otimes \mathfrak{M}_0'$ can be also written in the form $\mathfrak{M}_1' \otimes [\alpha_1]$, where $\mathfrak{M}_1'$ is a kneading bimodule and $\alpha_1 \in \mathcal{G}_n$. Our hope is that α_1 will be shorter than α and iterating this procedure we will find a simple representation of the bimodule $\mathfrak{M}_f$. Keeping track of the cyclic order of the generators of F_n (and passing each time to the induced order) will hopefully provide an ordered kneading automaton.

Question 2. Is the bimodule $\mathfrak{G}_f$ always sub-hyperbolic?

If the bimodule $\mathfrak{G}_f$ is sub-hyperbolic, then the combinatorial spider algorithm will always converge to a finite cycle of kneading automata.

If the bimodule $\mathfrak{G}_f$ is always sub-hyperbolic, then the answer on Question 1 is positive, since then for every bimodule $\mathfrak{M}$ given by a twisted kneading automaton there will exist (by Proposition 2.11.5 of [Nek05]) a finite subset $N \subset \mathcal{G}_n$ and a number $m \in \mathbb{N}$, such that $\mathfrak{M}^{\otimes km}$ is isomorphic to $\mathfrak{M}_1^k \otimes \alpha_k$ for some $\alpha_k \in N$ and a bimodule $\mathfrak{M}_1$ not depending on k and given by a kneading automaton. Since bimodules given by kneading automata are sub-hyperbolic (as kneading automata are *bounded*, see [BN03] and Section 3.9 of [Nek05]), this implies sub-hyperbolicity of $\mathfrak{M}$.

6.5. **An example of the bimodule $\mathfrak{G}_f$.** The following example is considered in [Nek07a]. Let F_3 be the free group on three free generators a, b, c. Denote by $\mathfrak{M}_0$ and $\mathfrak{M}_1$ the bimodules given by the recursions

$$a = \sigma(1, c), \quad b = (1, a), \quad c = (1, b)$$

and

$$a = \sigma(1, c), \quad b = (a, 1), \quad c = (1, b),$$

respectively.

Consider the following elements of the group $\mathcal{G}_3$:

$$a^\alpha = a, \quad b^\alpha = b^a, \quad c^\alpha = c,$$
$$a^\beta = a, \quad b^\beta = b, \quad c^\beta = c^b,$$
$$a^\gamma = a^c, \quad b^\gamma = b, \quad c^\gamma = c.$$

Note that these three automorphisms of F_3 generate $\mathcal{G}_3$ (for instance, the automorphism $a \mapsto a, b \mapsto b, c \mapsto c^a$ is equal to α, modulo conjugation by a, i.e., modulo an inner automorphism).

The bimodule $[\alpha] \otimes \mathfrak{M}_0$ is given by the recursion

$$\begin{aligned} a &= \sigma(1, c), \\ b &= (1, a)^{\sigma(1,c)} = (a, 1), \\ c &= (1, b), \end{aligned}$$

hence $[\alpha] \otimes \mathfrak{M}_0 = \mathfrak{M}_1$.

The bimodule $[\alpha] \otimes \mathfrak{M}_1$ is given by

$$\begin{aligned} a &= \sigma(1, c), \\ b &= (a, 1)^{\sigma(1,c)} = (1, a^c), \\ c &= (1, b), \end{aligned}$$

hence $[\alpha] \otimes \mathfrak{M}_1 = \mathfrak{M}_0 \otimes [\gamma]$. Similarly, $[\beta] \otimes \mathfrak{M}_0 = \mathfrak{M}_0 \otimes [\alpha]$ and $[\beta] \otimes \mathfrak{M}_1 = \mathfrak{M}_1$.

The bimodule $[\gamma] \otimes \mathfrak{M}_0$ is given by

$$\begin{aligned} a &= (1, b^{-1})\sigma(1, c)(1, b) = \sigma(b^{-1}, cb), \\ b &= (1, a), \\ c &= (1, b), \end{aligned}$$

composing with conjugation by $(b, 1)$, we get

$$\begin{aligned} a &= \sigma(1, c^b), \\ b &= (1, a), \\ c &= (1, b), \end{aligned}$$

hence $[\gamma] \otimes \mathfrak{M}_0$ is isomorphic $\mathfrak{M}_0 \otimes [\beta]$. Similarly, $[\gamma] \otimes \mathfrak{M}_1$ is isomorphic to $\mathfrak{M}_1$.

We see that the bimodule $\mathfrak{G}_f$ is given by the recursion

$$\begin{aligned} \alpha &= \sigma(1, \gamma), \\ \beta &= (\alpha, 1), \\ \gamma &= (\beta, 1). \end{aligned}$$

Note that in this case $\mathcal{G}_3$ is isomorphic to the free group on 3 generators, and the bimodule $\mathfrak{G}_f$ is conjugate with the bimodule $\mathfrak{M}_0$, i.e., with the bimodule associated with the "rabbit polynomial".

The computations in the example of Subsection 6.3 can be written now as a the following sequence of equalities in $\mathfrak{G}_f$. We have started with the twisted kneading automaton $\mathfrak{M}_0 \otimes [\beta\gamma^{-1}]$, since

$$a^{\beta\gamma^{-1}c} = a, \quad b^{\beta\gamma^{-1}c} = b^c, \quad c^{\beta\gamma^{-1}c} = c^{bc}.$$

Then we have run through the following sequence of automata

$$\begin{aligned} [\beta\gamma^{-1}] \otimes \mathfrak{M}_0 &= \mathfrak{M}_0 \otimes [\alpha\beta^{-1}], \\ [\alpha\beta^{-1}] \otimes \mathfrak{M}_0 &= \mathfrak{M}_1 \otimes [\alpha^{-1}], \\ [\alpha^{-1}] \otimes \mathfrak{M}_1 &= \mathfrak{M}_0. \end{aligned}$$

Looking at the parity of the corresponding braids, we see that the cyclic order of the input set has been changed.

Now it is easy to run the combinatorial spider algorithm for any composition of the rabbit polynomial with a homeomorphism of the plane fixing the post-critical set pointwise. The corresponding computations is the essence of a solution of J. Hubbard's "twisted rabbit problem", given in [BN06b].

6.6. **The bimodule over the pure braid group.** The quotient $\overline{\mathcal{P}}_n$ of the pure braid group P_n by the center (i.e., its image in $\mathrm{Out}(F_n)$) is a subgroup of $\mathcal{G}_n$. Consequently, instead of looking at the bimodule $\mathfrak{G}_f$ of the isomorphism classes of the bimodules $[\alpha]\otimes\mathfrak{M}_0\otimes[\beta]$ for $\alpha,\beta\in\mathcal{G}_n$, one can consider the $\overline{\mathcal{P}}_n$-bimodule of the isomorphism classes of the bimodules $[\alpha]\otimes\mathfrak{M}_0\otimes[\beta]$ for $\alpha,\beta\in\overline{\mathcal{P}}_n$. This bimodule was considered in [Nek05] (see Proposition 6.6.1 about the bimodule $\mathfrak{F}$) and in [BN06b]. It is isomorphic to the bimodule associated with a correspondence on the moduli space of the puncture sphere coming from the pull-back map of complex structures by the topological polynomial f.

In particular, in the above example of the rabbit polynomial, the pure braid group $\overline{\mathcal{P}}_3<\mathcal{G}_3$ is the sub-group generated by the automorphisms $T=\beta^{-1}\alpha$ and $S=\gamma^{-1}\beta$. It follows from the recursion defining α,β,γ, that

$$T=(\alpha^{-1},1)\sigma(1,\gamma)=\sigma(1,\alpha^{-1}\gamma)=\sigma(1,T^{-1}S^{-1})$$

and

$$S=(\beta^{-1},1)(\alpha,1)=(\beta^{-1}\alpha,1)=(T,1).$$

This recursion was used in [BN06b] and it is the recursion associated with the post-critical rational function $1-1/z^2$, which is the map on the moduli space induced by the rabbit polynomial. For more details see [BN06b, Nek07a] and [Nek05] Section 6.6.

Question 3. Does there exist an analytic interpretation of the bimodule $\mathfrak{G}_f$ similar to the description of the bimodule over the pure braid group? In particular, are they always associated with some post-critically finite multidimensional rational maps (correspondences)?

6.7. **Limit space and symbolic presentation of the Julia set.** Let $\mathfrak{M}$ be a hyperbolic G-bimodule. Fix a basis X of $\mathfrak{M}$. By $\mathsf{X}^{-\omega}=\{\ldots x_2x_1\ :\ x_i\in\mathsf{X}\}$ we denote the space of the left-infinite sequences with the direct product topology.

Definition 6.3. We say that two sequences $\ldots x_2x_1$ and $\ldots y_2y_1$ are *equivalent* if there exists a finite set $N\subset G$ and a sequence $g_k\in N$, $k=1,2,\ldots$, such that

$$g_k(x_k\ldots x_1)=y_k\ldots y_1$$

for all k.

The quotient of $\mathsf{X}^{-\omega}$ by the equivalence relation is called the *limit space* of the bimodule $\mathfrak{M}$ and is denoted $\mathcal{J}_{\mathfrak{M}}$.

Note that the equivalence relation on $\mathsf{X}^{-\omega}$ is invariant under the shift $\ldots x_2x_1\mapsto\ldots x_3x_2$, hence the shift induces a continuous self-map $\mathsf{s}:\mathcal{J}_{\mathfrak{M}}\longrightarrow\mathcal{J}_{\mathfrak{M}}$ of the limit space. The obtained dynamical system $(\mathcal{J}_{\mathfrak{M}},\mathsf{s})$ is called the *limit dynamical system* of the bimodule.

In some cases the following description of the equivalence relation on $\mathsf{X}^{-\omega}$ may be more convenient. For its proof see [Nek05] Proposition 3.2.7.

Proposition 6.6. *Let S be a* state-closed generating set *of G, i.e., such a generating set that for every $g \in S$ and $x \in \mathsf{X}$ we have $g|_x \in S$.*

Let $\mathcal{S} \subset \mathsf{X}^{-\omega} \times \mathsf{X}^{-\omega}$ be the set of pairs of sequences read on the labels of the left-infinite paths in the Moore diagram of S, i.e.,

$$\mathcal{S} = \{(\ldots x_2 x_1, \ldots y_2 y_1) \;:\; \textit{there exist } g_k \in S \textit{ such that } g_k \cdot x_k = y_k \cdot g_{k-1}\}.$$

Then the asymptotic equivalence relation on $\mathsf{X}^{-\omega}$ is the equivalence relation generated by $\mathcal{S}$.

If $\mathfrak{M}$ is a sub-hyperbolic bimodule, then its limit dynamical system $(\mathcal{J}_{\mathfrak{M}}, \mathsf{s})$ is defined to be the limit dynamical system of its faithful quotient.

More about the limit spaces of hyperbolic bimodules, see Chapter 3 of [Nek05].

A corollary of Theorem 5.5.3 of [Nek05] is the following description of a topological model of the Julia set of a post-critically finite rational map.

Theorem 6.7. *Let f be a post-critically finite rational function with post-critical set P. Then the $\pi_1(\mathbb{PC}^2 \setminus P)$-bimodule $\mathfrak{M}_f$ is sub-hyperbolic and the limit dynamical system $(\mathcal{J}_{\mathfrak{M}}, \mathsf{s})$ is topologically conjugate with the restriction of f on its Julia set.*

Question 4. Find an interpretation of the limit space of the bimodule $\mathfrak{G}_f$, defined in Subsection 6.4. An interpretation of the limit space of the bimodule over the pure braid group can be found in [BN06b, Nek07a].

6.8. **Realizability and obstructions.**

Proposition 6.8. *Every twisted kneading automaton over F_n is associated with some post-critically finite topological polynomial.*

Proof. It is proved in Theorem 6.10.4 of [Nek05] that every cyclically ordered kneading automaton can be realized by a topological polynomial (see also the proof of Proposition 5.2 in this paper). It remains to compose it with the homeomorphism realizing the respective element of the braid group. □

A natural question now is which twisted kneading automata can be realized by *complex* polynomials. This question for *kneading automata* was answered in [Nek05] in Theorem 6.10.7. We reformulate this theorem here perhaps in slightly more accessible terms.

Let $\mathcal{A}$ be an ordered kneading automaton over the alphabet X and input-output set A. Denote by D the set of sequences $x_1 x_2 \ldots \in \mathsf{X}^\omega$ such that there exists a non-trivial element $g \in A \cup A^{-1}$ such that $g|_{x_1 \ldots x_m}$ is non-trivial for every $k \geq 1$.

Note that then $g(x_1 x_2 \ldots)$ also belongs to D. Connect $x_1 x_2 \ldots$ to $g(x_1 x_2 \ldots)$, if $g \in A$, by an arrow, thus transforming D into a graph (multiple edges and loops are allowed, since elements of A may have fixed points or act in the same way on some sequences). We call D the *boundary graph* of the kneading automaton $\mathcal{A}$.

It also follows directly from the definition of a kneading automaton that every element of D is periodic and there is no more than one sequence $x_1 x_2 \ldots \in D$ for every $g \in A \cup A^{-1} \setminus \{1\}$. More explicitly, if m is sufficiently big (e.g. bigger than the lengths of cycles in the Moore diagram of $\mathcal{A}$), then D is isomorphic to the subgraph of the *dual* Moore diagram of $\mathcal{A}^{\otimes m}$ consisting of the edges marked by non-finitary elements of $\mathcal{A}$. (Here a state g of $\mathcal{A}$ is called finitary if there exists k such that $g|_v = 1$ in $\mathcal{A}^{\otimes k}$ for all $v \in \mathsf{X}^k$.) In particular, every component of the boundary graph is a tree with loops.

Theorem 6.9. *The kneading automaton $\mathcal{A}$ can be realized by a complex polynomial if and only if every connected component of its boundary graph has at most one loop.*

We leave to the reader to check that the condition of this theorem is equivalent to the condition of Theorem 6.10.7 of [Nek05].

Question 5. Find a simple criterion of absence of obstruction for a topological polynomial given by an arbitrary twisted kneading automaton.

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