Iterated Monodromy Groups
Lecture 4

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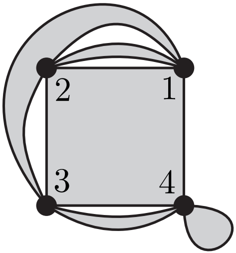
August, 2009
Bath

Dendroid sets of permutations

Let $\alpha_1, \alpha_2, \dots, \alpha_n \in S_d$. A *cycle diagram* of the sequence is the CW complex with the set of vertices $\{1, 2, \dots, d\}$ with 2-cells corresponding to the cycles of the permutations α_i.



The sequence is called *dendroid* if its cycle diagram is contractible.

$d = 2$

$d = 3$

$d = 4$

$d = 5$

Properties of dendroid sets

If $\alpha_1, \alpha_2, \ldots, \alpha_n$ is dendroid, then $\alpha_1\alpha_2\cdots\alpha_n$ is a transitive cycle.

If $\alpha_1, \alpha_2, \ldots, \alpha_n$ is dendroid, then

$$\alpha_1\cdots\alpha_{k_1}, \quad \alpha_{k_1+1}\cdots\alpha_{k_2}, \quad \ldots, \quad \alpha_{k_m+1}\cdots\alpha_n$$

is a dendroid sequence.

IMGs of polynomial iterations

A sequence

$$\mathbb{C} \xleftarrow{f_1} \mathbb{C} \xleftarrow{f_2} \mathbb{C} \xleftarrow{f_3} \cdots$$

of polynomials is *post-critically finite* if there is a finite set $P \subset \mathbb{C}$ such that for every n the set of critical values of $f_1 \circ f_2 \circ \cdots \circ f_n$ is contained in P. Examples are constant sequences of p.c.f. polynomials or any sequence of z^2 and $1 - z^2$.

Theorem

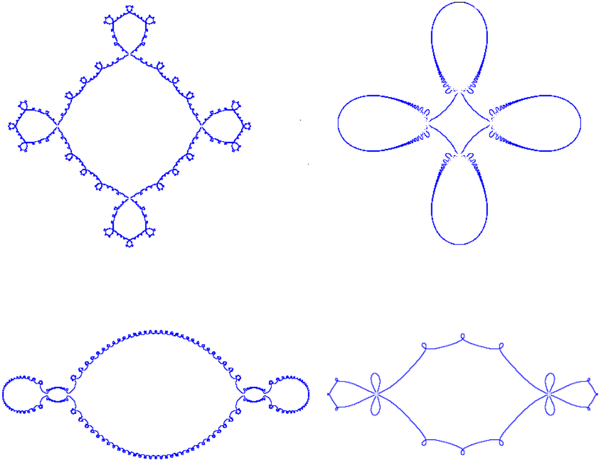
A group acting on a rooted tree T is the iterated monodromy group of a post-critically finite sequence of polynomials iff it has a generating set $g_1, \ldots, g_n$ such that for every n the sequence of restrictions of g_i onto the nth level of T is dendroid.

Let $f_1, f_2, \dots,$ be a sequence of z^2 and $1 - z^2$ in some order. Then its IMG is generated by a_1, b_1, which are given by

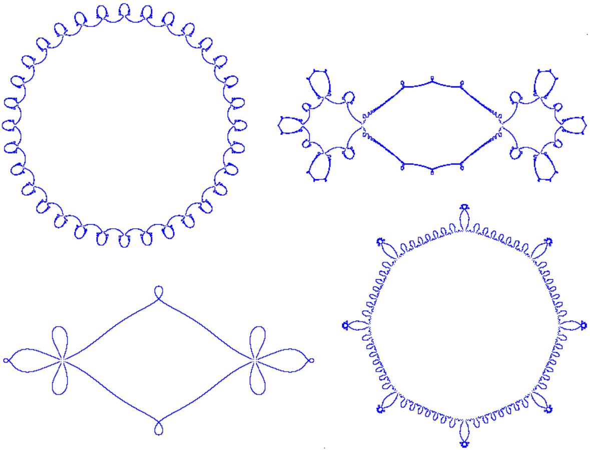
$$a_n = \begin{cases} \sigma(1, a_{n+1}) & \text{if } f_n(z) = z^2 \\ \sigma(1, b_{n+1}) & \text{if } f_n(z) = 1 - z^2 \end{cases}$$

$$b_n = \begin{cases} (1, b_{n+1}) & \text{if } f_n(z) = z^2 \\ (1, a_{n+1}) & \text{if } f_n(z) = 1 - z^2 \end{cases}$$

Julia sets of forward iterations of z^2 and $1 - z^2$



Julia sets of forward iterations of z^2 and $1 - z^2$



Quadratic polynomials

Post-critically finite quadratic polynomials $z^2 + c$ are parametrized by rational angles $\theta \in \mathbb{R}/\mathbb{Z}$ in the following way.

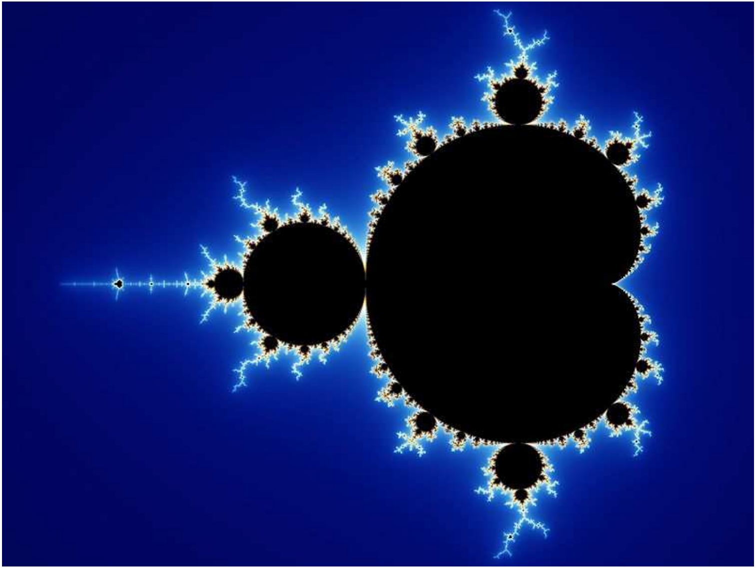
Mandelbrot set is the set M of numbers $c \in \mathbb{C}$ such that the sequence

$$0, f(0), f^{\circ 2}(0), \ldots, f^{\circ n}(0), \ldots$$

is bounded, where $f(z) = z^2 + c$.

There exists a unique bi-holomorphic isomorphism $\Phi : \mathbb{C} \setminus \overline{\mathbb{D}} \longrightarrow \mathbb{C} \setminus M$ tangent to identity at infinity. Here $\overline{\mathbb{D}} = \{z \in \mathbb{C} \ : \ |z| \leq 1\}$.

Mandelbrot set



The image R_θ of the ray $\{r \cdot e^{\theta \cdot 2\pi i} \ : \ r \in (1, +\infty)\}$ under Φ is called the *parameter ray at the angle* θ.

We say that R_θ *lands* on a point $c \in M$ if $c = \lim_{r \searrow 1} \Phi\left(r \cdot e^{\theta \cdot 2\pi i}\right)$. It is known that rays with $\theta \in \mathbb{Q}/\mathbb{Z}$ land.

If the orbit $\{f^{\circ n}(c)\}_{n\geq 1}$ of c is *pre-periodic*, then c belongs to the boundary of M and it is a landing point of a finite number of parameter rays R_θ. Each such θ is a rational number with even denominator.

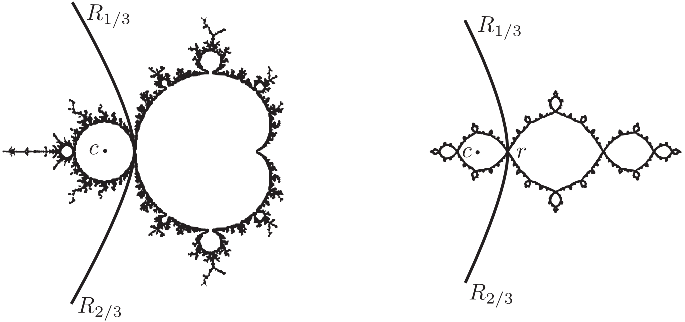
If $\theta \in \mathbb{Q}/\mathbb{Z}$ has even denominator, then the ray R_θ lands on a point $c \in M$ such that the orbit of c under action of $f(z) = z^2 + c$ is pre-periodic.

For example, the landing point of $R_{1/6}$ is i. The orbit of i under $z^2 + i$ is $i \mapsto -1 + i \mapsto -i \mapsto -1 + i$. The orbit of $1/6$ under angle doubling is $1/6 \mapsto 1/3 \mapsto 2/3 \mapsto 4/3 = 1/3$.

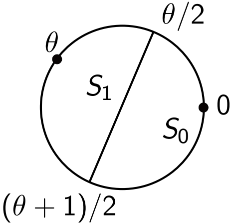
If c is periodic, then c is an internal point of M. There are two rays $R_{\theta_1}, R_{\theta_2}$ landing on the *root* of the component of $\overset{\circ}{M}$ to which c belongs.

Both angles θ_i have odd denominators and their periods under angle doubling are equal to the period of c under the action of $z^2 + c$.

For example, the orbit of -1 under $z^2 - 1$ is $-1 \mapsto 0 \mapsto -1$. The corresponding angles are $1/3$ and $2/3$. The action of angle doubling is $1/3 \mapsto 2/3 \mapsto 4/3 = 1/3$.



Fix $\theta \in \mathbb{Q}/\mathbb{Z}$. The points $\theta/2$ and $(\theta+1)/2$ divide the circle $\mathbb{R}/\mathbb{Z}$ into two open semicircles S_0, S_1. Here S_0 is the semicircle containing 0.



Kneading sequence $\widehat{\theta}$ is $x_1x_2\ldots$, where

$$x_k = \begin{cases} 0 & \text{if } 2^k\theta \in S_0 \\ 1 & \text{if } 2^k\theta \in S_1 \\ * & \text{if } 2^k\theta \in \{\theta/2, (\theta+1)/2\} \end{cases}$$

Denote for $v = x_1 \dots x_{n-1}$ by $\mathfrak{K}(v)$ the group generated by

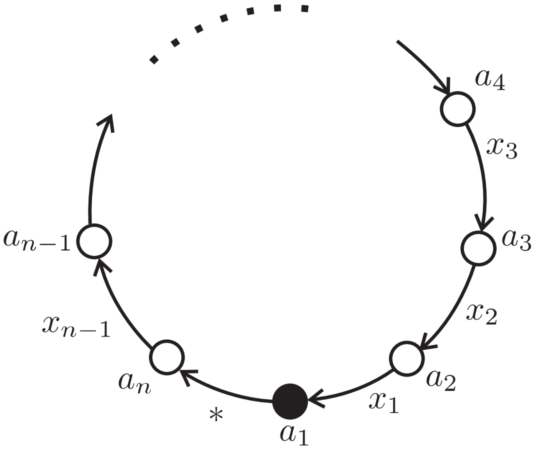
$$a_1 = \sigma(1, a_n), \qquad a_{i+1} = \begin{cases} (a_i, 1) & \text{if } x_i = 0, \\ (1, a_i) & \text{if } x_i = 1, \end{cases}$$

Denote for $w = y_1 \dots y_k \in \mathsf{X}^*$ and $v = x_1 \dots x_n \in \mathsf{X}^*$ such that $y_k \neq x_n$ by $\mathfrak{K}(w, v)$ the group generated by

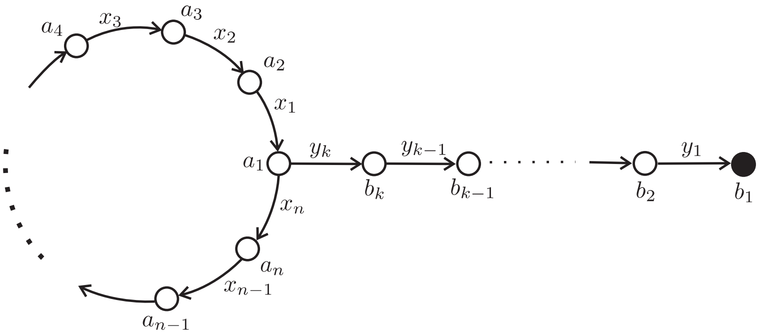
$$b_1 = \sigma, \qquad b_{j+1} = \begin{cases} (b_j, 1) & \text{if } y_j = 0 \\ (1, b_j) & \text{if } y_j = 1 \end{cases}$$

$$a_1 = \begin{cases} (b_k, a_n) & \text{if } y_k = 0 \text{ and } x_n = 1, \\ (a_n, b_k) & \text{if } y_k = 1 \text{ and } x_n = 0, \end{cases} a_{i+1} = \begin{cases} (a_i, 1) & \text{if } x_i = 0 \\ (1, a_i) & \text{if } x_i = 1 \end{cases}$$

The automaton generating $\mathfrak{K}(x_1 x_2 \ldots x_{n-1})$



The automaton generating $\mathfrak{K}(y_1 \ldots y_k, x_1 \ldots x_n)$



Theorem (L. Bartholdi, V. N.)

Denote by $z^2 + c_\theta$ *the polynomial corresponding to the angle* $\theta \in \mathbb{Q}/\mathbb{Z}$*.*
If $\widehat{\theta} = (x_1 x_2 \ldots x_{n-1} *)^\infty$*, then*

$$\mathrm{IMG}\left(z^2 + c_\theta\right) = \mathfrak{K}\left(x_1 x_2 \ldots x_{n-1}\right).$$

If $\widehat{\theta} = y_1 y_2 \ldots y_k (x_1 x_2 \ldots x_n)^\infty$*, then*

$$\mathrm{IMG}\left(z^2 + c_\theta\right) = \mathfrak{K}\left(y_1 y_2 \ldots y_k, x_1 x_2 \ldots x_n\right).$$

"Smooth" examples: for $\theta = 0$: $\mathrm{IMG}\left(z^2\right) = \mathfrak{K}\left(\varnothing\right) = \mathbb{Z}$, for $\theta = 1/2$: $\mathrm{IMG}\left(z^2 - 2\right) = \mathfrak{K}\left(1, 0\right) = \mathbb{D}_\infty$.

If we take $\theta = 1/3$, then $\widehat{1/3} = (1*)^\infty$ and hence $\mathrm{IMG}\left(z^2 - 1\right)$ is generated by

$$a_1 = \sigma(1, a_2), \qquad a_2 = (1, a_1).$$

L-presentation

Fix $v = x_1 \dots x_{n-1}$. Define the following endomorphism of the free group:

$$\varphi(a_n) = a_1^2, \qquad \varphi(a_i) = \begin{cases} a_{i+1} & \text{if } x_i = 0 \\ a_{i+1}^{a_1} & \text{if } x_i = 1 \end{cases}$$

Let R be the set of commutators

$$\left[a_i, a_j^{a_1^k}\right],$$

where $2 \le i, j \le n$, and $k = 0, 2$ if $x_{i-1} \neq x_{j-1}$ and $k = 1$ if $x_{i-1} = x_{j-1}$.

Theorem (L. Bartholdi, V. N.)

$$\mathfrak{K}(v) = \left\langle a_1, \dots, a_n \,\middle|\, \varphi^\ell(R) \text{ for all } \ell \ge 0 \right\rangle.$$

Corollary

Write $p(t) = x_{n-1}t + x_{n-2}t^2 + \cdots + x_1 t^{n-1} \in \mathbb{Z}[t]$. Then the group $\mathfrak{K}(v)$ is isomorphic to the subgroup $\langle a, a^t, a^{t^2}, \ldots, a^{t^{n-1}} \rangle$ of the finitely presented group

$$\left\langle a, t \,\middle|\, a^{t^n - 2a^{p(t)}}, \left[a^{t^i}, a^{t^j a}\right], \left[a^{t^i}, a^{t^j a^3}\right] \text{ for all } 1 \leq i, k < n \right\rangle.$$

Open problem: Find similar embeddings for other IMGs and their relation with the topology of the respective maps.

Theorem (D. Schleicher, V. N.)

Let f_1 and f_2 be post-critically finite quadratic polynomials. The following conditions are equivalent.

1. $\mathrm{IMG}(f_1)$ *and* $\mathrm{IMG}(f_2)$ *are isomorphic as abstract groups.*
2. *There is a homeomorphism between the Julia sets of f_1 and f_2, conjugating the corresponding dynamical systems.*
3. *The corresponding kneading sequences coincide.*

In particular, if $\mathrm{IMG}(f_1)$ and $\mathrm{IMG}(f_2)$ are isomorphic, then the Julia sets of f_1 and f_2 are homeomorphic.

Example: rabbit and airplane

Consider the groups:

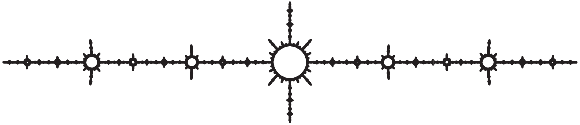
$$G_1 = \langle a_1 = \sigma(1, c_1), \quad b_1 = (1, a_1), \quad c_1 = (1, b_1)\rangle,$$

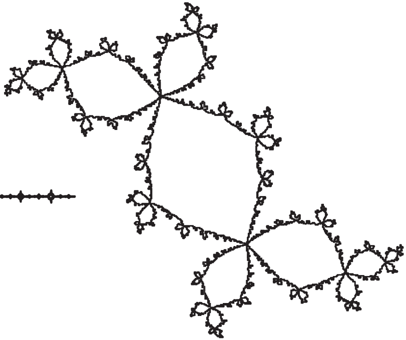
$$G_2 = \langle a_2 = \sigma(1, c_2), \quad b_2 = (1, a_2), \quad c_2 = (b_2, 1)\rangle.$$

They are IMGs of two polynomials with critical point of period 3:

$$z^2 - 0.1226\ldots + 0.7449\ldots i, \quad z^2 - 1.7549\ldots.$$

They are not isomorphic, since the Julia sets of these polynomials (known as "Douady Rabbit" and "Airplane") are not homeomorphic.

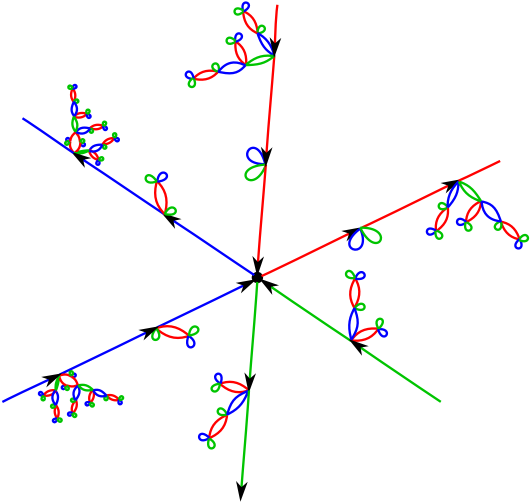




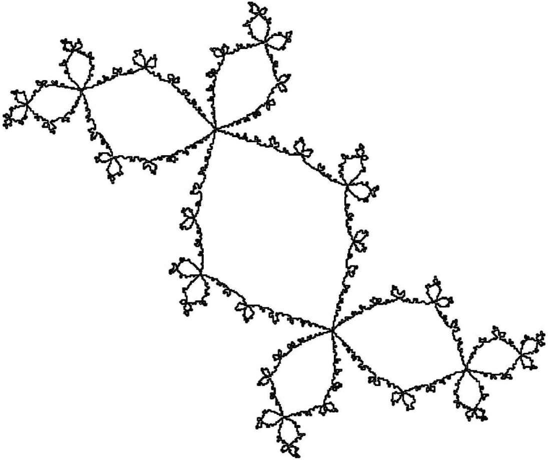
Theorem

The closures of the groups G_1 and G_2 in the automorphism group of the binary tree coincide.
For every finite sets of relations and inequalities between the generators a_1, b_1, c_1 of G_1 there exists a generating set a_1', b_1', c_1' of G_2 satisfying the same relations and inequalities.

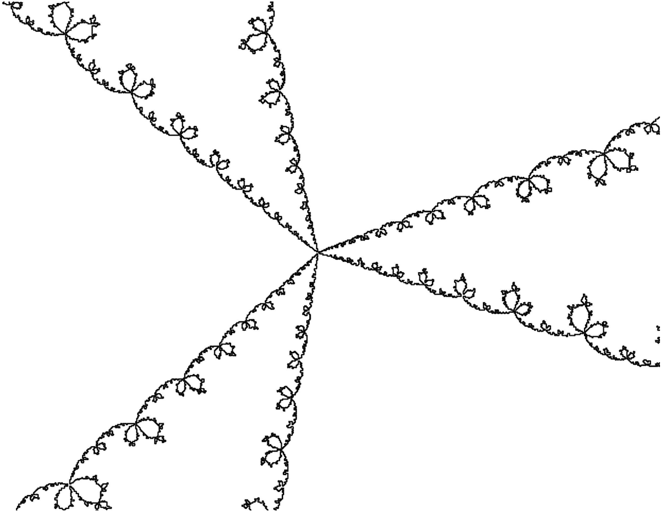
a_1, b_1, c_1 generate a free monoid



A zoom of the Douady Rabbit



A zoom of the Douady Rabbit



Theorem

Let f be a post-critically finite polynomial. If there exist two finite Fatou components of f with intersecting closures, then $\mathrm{IMG}(f)$ *contains a free subsemigroup.*

There are more examples of $\mathrm{IMG}(f)$ of exponential growth, since every semi-conjugacy of dynamical systems induces an embedding of the IMGs.

The following is a result of K.-U. Bux and R. Perez.

Theorem

$\mathrm{IMG}\left(z^2 + i\right)$ *has intermediate growth.*

An earlier example is the Gupta-Fabrikowski group, which is $\mathrm{IMG}\left(z^3(-3/2 + i\sqrt{3}/2) + 1\right)$.

Which polynomials have IMG of intermediate growth?