M469 Spring 2020, Assignment 4, due Fri., Feb. 14

Suggested Reading. *Simple mathematical models with very complicated dynamics*, by Robert M. May, in *Nature* **261** (1976) 459-467. In this article, May focuses on the difference equation

$$X_{t+1} = aX_t(1 - X_t),$$

which (with different letters) is the model we derived in Problem 3 of the previous homework. In particular, he outlines precisely the dynamics we're emphasizing in class, the period-doubling cascade toward chaos. Here are a few highlights: (1) "The review ends with an evangelical plea for the introduction of these difference equations into elementary mathematics courses, so that students' intuition may be enriched by seeing the wild things that simple nonlinear equations can do." (2) "But, for smooth and 'sensible' functions $F(X)$ such as in equations (3) and (4), the underlying mathematical fact is that for any specified parameter value there is one unique cycle that is stable, and that attracts essentially all initial points." (3) "Putting all this together, we conclude that as the parameters in $F(X)$ are varied the fundamental, stable dynamical units are cycles of basic period k, which arise by tangent bifurcation, along with their associated cascade of harmonics of periods $k2^n$, which arise by pitchfork bifurcation."

1. [10 pts] If $\hat{y}$ denotes a fixed point for the difference equation

$$y_{t+1} = f(y_t)$$

and $f'(\hat{y}) = \pm 1$ then $\hat{y}$ can be stable, asymptotically stable, or unstable, depending on the higher order terms. In order to see this for the case $f'(\hat{y}) = 1$, categorize the fixed point $\hat{y} = 0$ as stable, asymptotically stable, or unstable for each of the following difference equations:

a. $y_{t+1} = y_t$

b. $y_{t+1} = y_t + y_t^3$

c. $y_{t+1} = y_t - y_t^3$.

2. [10 pts] Suppose that in the absence of fishermen the population of fish in a certain body of water is modeled by the discrete logistic model

$$y_{t+1} = y_t + ry_t(1 - \frac{y_t}{K}),$$

and that a constant harvest h is to be taken each period,

$$y_{t+1} = y_t + ry_t(1 - \frac{y_t}{K}) - h.$$

Find the maximum sustainable yield associated with this model, and observe that the prescribed yield gives a fish population with borderline stability. Determine whether or not you can obtain a stable equilibrium by taking slightly less than the optimal yield.

3. [10 pts] Suppose that in the absence of fishermen the population of fish in a certain body of water is modeled by the Beverton-Holt model,

$$y_{t+1} = \frac{(1+r)y_t}{1 + \frac{r}{K}y_t},$$

and that fishing effort is added as a percentage of population. Write down a difference equation that incorporates this fishing effort, and find the maximum sustainable yield associated with your model. Specify the yield obtained, the equilibrium fish population, and the parameter values for which your results are valid.

4. [10 pts] For the difference equation

$$y_{t+1} = a - y_t^2,$$

with $a > 0$, (1) find the non-negative fixed points and determine the values of a for which they are stable; and (2) find the 2-cycles and determine the values of a for which both populations in the 2-cycle are positive and for which the 2-cycle is stable.