M469 Spring 2020, Assignment 5, due Fri., Feb. 21

Suggested Reading 1. *Period Three Implies Chaos*, by T-Y. Li and J. A. Yorke, in *The American Mathematical Monthly* **82** (1975) 985-992. Although the main result of this paper was mostly established earlier (1964) by the Russian mathematician Sarkovskii (if you're curious, see *A theorem of Sarkovskii on the existence of periodic orbits of continuous endomorphisms of the real line*, by P. Stefan, in *Communications in Mathematical Physics* **54** (1977) 237-248), it doesn't appear to have become widely known in England and the U.S. until this paper appeared. I don't suggest you try to slug your way through this entire paper, but I do suggest the following: Read from the beginning until the *Proof of Theorem 1* on p. 987. (Skip the proof.) Then read all of Section 3, *Behavior near a periodic point*.

Suggested Reading 2. *The role of competition and clustering in population dynamics*, by A. Brannstrom and D. J. T. Sumpter, in *Proceedings of the Royal Society B* **272** (2005) 2065-2072. I said on the first day of class that mathematical models in the biological sciences are primarily phenomenological, so you might find the following quote from this paper curious: "The growing interest in the first principles derivations of population models is effecting a change in how population models are viewed. Traditionally, models have been treated phenomenologically, whereas first principles derivations demonstrate that these models are consequences of simple underlying mechanistic principles, most of which can be understood intuitively." By *first principles derivations* the authors mean derivations based on probabilistic assumptions. For example, they might start by assuming—as they do for certain models in this article—that the probability of an individual's being in a sub-region of area s of a larger region of area A is $\frac{s}{A}$. We won't discuss this approach in class (it would require too much development in probability theory), but I want to bring it to your attention. I don't suggest you read this article entirely unless you're particulary interested in probability. I suggest you read the introduction, including Table 1, which lists most of the commonly used single-species population models. Also, read the the first sentence of Section 3(a) (about scramble competition) and the first sentence of Section 3(b) (about contest competition). Finally, read Section 4 entitled *Discussion*.

1. [10 pts] A common method for finding roots of an equation of the form

$$h(y) = 0,$$

is Newton's method. For Newton's method, we start with an initial guess y_0 for the location of the root and compute the series of approximations

$$y_{t+1} = y_t - \frac{h(y_t)}{h'(y_t)}.$$

(The method is derived by noting that $h(y) \approx h(y_t) + h'(y_t)(y - y_t)$, so to get $h(y) = 0$, we solve $0 = h(y_t) + h'(y_t)(y - y_t)$ for y and take this result as our update y_{t+1}.) Show that if $\hat{y}$ denotes the root of the equation and $h'(\hat{y}) \neq 0$ then Newton's method is guaranteed to converge for initial values taken sufficiently close to $\hat{y}$.

2. [10 pts] Given that $\hat{y}_1 = \frac{7}{9}$ is a point of period 3 for the difference equation

$$y_{t+1} = 1 - 2|y_t|,$$

find a 3-cycle for this equation and determine whether or not it is stable.

Note. Of course $f(y) = 1 - 2|y|$ is not differentiable at $y = 0$, but you only require differentiability at the points in the 3-cycle, so this would only cause a problem if 0 was one of these points.

3. [10 pts] Solve the second order difference equation

$$y_{t+1} = -y_t + 6y_{t-1},$$

with initial conditions $y_0 = 1$ and $y_1 = 3$.

4. [10 pts] In this problem we'll consider the case in which a base obtained in solving a difference equation is repeated and the case in which a base is complex.

a. When the delay difference equation

$$y_{t+1} - y_t = ry_t(1 - y_{t-1})$$

is linearized about the fixed point $\hat{y} = 1$, the linearized equation is

$$z_{t+1} = z_t - rz_{t-1}.$$

We saw in class that in the case $r = \frac{1}{4}$ the base for this linear equation is $a = \frac{1}{2}$, repeated. This means that one solution is $(\frac{1}{2})^t$. Show that $t(\frac{1}{2})^t$ is a second solution in this case, and use this to show that $\hat{y} = 1$ is stable in this case. (By the way, if you have one solution y_t to an equation, it's often useful to look for a second solution of the form $y_t \cdot x_t$ for some function x_t. Clearly, in this case we find $x_t = t$.)

b. Solve the second order difference equation

$$y_{t+1} = 2y_t - 2y_{t-1},$$

with initial conditions $y_0 = 0$ and $y_1 = 1$. The imaginary number i should not appear in your final solution.

Hint on (b). Use polar form for the bases a_1 and a_2: $a_j = re^{i\theta_j}$. Then observe that we can write our general solution either as

$$y_t = C_1 r^t e^{i\theta_1 t} + C_2 r^t e^{i\theta_2 t},$$

or in the real-valued form

$$y_t = K_1 r^t \cos(\theta_1 t) + K_2 r^t \sin(\theta_1 t),$$

obtained by taking real and imaginary parts of $e^{i\theta_1 t} = \cos(\theta_1 t) + i \sin(\theta_1 t)$.