M469 Spring 2020, Assignment 9, due Fri., Apr. 17

1. [10 pts] The first two equations of the discrete SIR model are

$$\begin{aligned} y_{1_{t+1}} - y_{1_t} &= -\, a y_{1_t} y_{2_t} \\ y_{2_{t+1}} - y_{2_t} &= a y_{1_t} y_{2_t} - b y_{2_t}, \end{aligned}$$

where a and b are positive parameters. Non-dimensionalize this equation and then find the fixed points for your non-dimensional form. Find the range of parameters for which each fixed point is asymptotically stable and the range for which each is unstable. State your findings with respect to your original variables, and try to interpret your results on physical grounds.

2. [10 pts] According to our stability criterion for nonlinear systems of difference equations, if $|\lambda| = 1$ for at least one eigenvalue and there are no eigenvalues for which $|\lambda| > 1$, then the associated fixed point can be any of the following: asymptotically stable, stable (and not asymptotically stable), or unstable. Working with the system

$$\begin{aligned} y_{1_{t+1}} - y_{1_t} &= -\, \frac{1}{2} y_{1_t} \\ y_{2_{t+1}} - y_{2_t} &= r y_{2_t}^3, \end{aligned}$$

show that the only fixed point is $(0, 0)$, with eigenvalues $\frac{1}{2}$ and 1, and that the choice of r determines its stability.

3. [10 pts] In this problem we'll consider a simple difference equation

$$\begin{aligned} y_{1_{t+1}} &= y_{2_t}^2 \\ y_{2_{t+1}} &= y_{1_t}^2. \end{aligned}$$

a. Find the fixed points for this system and classify each as asymptotically stable or unstable.

b. Find the 2-cycles for this system and classify each as asymptotically stable or unstable.

4. [10 pts] We showed in Problem 1 that the non-dimensionalized form of the discrete SIR model is (in lower case variables for notational convenience)

$$\begin{aligned} y_{1_{t+1}} - y_{1_t} &= -\, y_{1_t} y_{2_t} \\ y_{2_{t+1}} - y_{2_t} &= y_{1_t} y_{2_t} - b y_{2_t}. \end{aligned}$$

Explain why we would not expect to find 2-cycles for this model, and then verify that 2-cycles do not generally exist. What is the exception? In the exceptional case, what can you say about stability?

5. [10 pts] If we assume that infected individuals become susceptible again as soon as they recover, the SIR model becomes

$$\begin{aligned} y_{1_{t+1}} - y_{1_t} &= -\, a y_{1_t} y_{2_t} + b y_{2_t} \\ y_{2_{t+1}} - y_{2_t} &= a y_{1_t} y_{2_t} - b y_{2_t}. \end{aligned}$$

a. Non-dimensionalize this equation, find the fixed points associated with your non-dimensionalized equation, and determine the parameter values for which each is asymptotically stable and the parameter values for which each is unstable.

b. Explain why you would expect this system to have 2-cycle solutions, and find a general expression for one of the points in each 2-cycle. Set $b = 1$ and find the full 2-cycle associated with $y_1 = 2$. Determine whether or not it is asymptotically stable.

6. [10 pts] Suppose M grams of a certain heart medication are injected into a patient at time 0, and that whenever the drug is present in the body (excluding the heart) its rate of absorption out of the bloodstream is proportional to the concentration in the body (excluding the heart) with proportionality constant $r_B\ s^{-1}L$, while whenever the drug is in the heart its rate of absorption out of the bloodstream is proportional to the concentration in the heart with proportionality constant $r_H\ s^{-1}L$. If blood flows into the patient's heart with variable rate $r_I(t)\ L/s$ and out with variable rate $r_O(t)\ L/s$, and if the initial volume of blood in the heart is $V_H\ L$ while the initial volume of blood in the body (excluding the heart) is $V_B\ L$, develop a model for the amount of drug absorbed into heart tissue by t.