Linear Systems of ODE

MATH 469, Texas A&M University

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Overview

Recall that in our lecture on compartment models, we wrote down the ODE system

$$\frac{dA}{dt} = r_A \frac{y(t)}{V_H(t)}; \quad A(0) = 0$$
$$\frac{dy}{dt} = r_I(t) \frac{M - y(t) - A(t)}{V_B(t)} - (r_O(t) + r_A) \frac{y(t)}{V_H(t)}; \quad y(0) = 0.$$

We can express this system in vector form by setting $\vec{y} = \binom{y_1}{y_2} = \binom{A}{y}$, and writing

$$\frac{d\vec{y}}{dt} = M(t)\vec{y} + \vec{g}(t); \quad \vec{y}(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

where

$$M(t) = \begin{pmatrix} 0 & \frac{r_A}{V_H(t)} \\ -\frac{r_I(t)}{V_B(t)} & -\frac{r_I(t)}{V_B(t)} - \frac{r_O(t)+r_A}{V_H(t)} \end{pmatrix}; \quad \vec{g}(t) = \begin{pmatrix} 0 \\ \frac{r_I(t)M}{V_B(t)} \end{pmatrix}.$$

Overview

Generally, a system of the form

$$\frac{d\vec{y}}{dt} = M(t)\vec{y} + \vec{g}(t); \quad \vec{y}(0) = \vec{y}_0,$$

for some matrix function $M(t)$ and some vector function $\vec{g}(t)$ is referred to as a linear system of ODE. If $\vec{g}(t)$ is identically 0, we refer to the system as homogeneous, and otherwise we refer to it as inhomogeneous.

Example: Better Protection of the Ozone Layer

This example is taken from the article, "Better protection of the ozone layer," by M. K. W. Ko, N-D. Sze, and M. J. Prather, in *Nature* **367** (1994) 505-508.

One element known to contribute to the depletion of ozone (O_3) in the stratosphere (10 km - 50 km above the earth's surface) is chlorine (Cl). (Other big culprits, which won't have a role here, include bromine (Br), hydroxyl radicals (OH^-), and nitric oxide (NO).)

Chlorine often gets into the atmosphere via halocarbons, which are compounds in which one or more carbon atoms are linked with one or more halogen atoms (i.e., fluorine (F), chlorine, bromine (Br), iodine (I), and astatine (At); Group VIIA in the periodic table). Halocarbons are widely used as solvents, pesticides, refigerants, adhesives, sealants, electrically insulated coatings, and so on.

Example: Better Protection of the Ozone Layer

Once a halocarbon gets into the stratosphere, it can be catalyzed by sunlight to break down and release a halogen. For example, the authors are interested in chlorofluorocarbons (CFC's), and one such molecule is trichlorofluoromethane (CCl_3F). In the presence of sunlight, CCl_3F breaks down via the reaction

$$CCl_3F \longrightarrow CCl_2F + Cl.$$

Once free chlorine is in the stratosphere, it interacts with ozone via the reaction

$$\begin{aligned} Cl + O_3 &\longrightarrow ClO + O_2 \\ ClO + O_3 &\longrightarrow Cl + 2O_2. \end{aligned}$$

Example: Better Protection of the Ozone Layer

The authors focus on tracking the amount of free chlorine in the stratosphere as a function of time.

The basic dynamics are as follows:

- *CFC* gets into the troposphere (0 km - 10 km above the earth's surface) and transfers back and forth into the stratosphere.
- *CFC* in the stratosphere breaks down, releasing free chlorine into the stratosphere.

The authors use the following variables, measured in kilograms:

$$\begin{aligned} C_T &= \text{amount of } Cl \text{ in } CFC \text{ in the troposphere} \\ C_S &= \text{amount of } Cl \text{ in } CFC \text{ in the stratosphere} \\ C &= \text{amount of free chlorine in the stratosphere.} \end{aligned}$$

Example: Better Protection of the Ozone Layer

The system is as follows:

$$\begin{aligned}
\frac{dC_T}{dt} &= -\frac{1}{L_T}C_T + \frac{1}{\tau}C_S - \frac{f}{\tau}C_T \\
\frac{dC_S}{dt} &= -\frac{1}{L_S}C_S - \frac{1}{\tau}C_S + \frac{f}{\tau}C_T \\
\frac{dC}{dt} &= +\frac{1}{L_S}C_S - \frac{1}{\tau}C.
\end{aligned}$$

The term $-\frac{1}{L_T}C_T$ corresponds with the breakdown of *CFC* in the troposphere. Similarly as with our discussion of death rates and life expectancies for difference equations, L_T denotes the average lifetime of a kilogram of C_T.

The term $\frac{1}{\tau}C_S$ corresponds with transfer of *CFC* from the stratosphere to the troposphere.

Example: Better Protection of the Ozone Layer

The system is as follows:

$$\frac{dC_T}{dt} = -\frac{1}{L_T}C_T + \frac{1}{\tau}C_S - \frac{f}{\tau}C_T$$
$$\frac{dC_S}{dt} = -\frac{1}{L_S}C_S - \frac{1}{\tau}C_S + \frac{f}{\tau}C_T$$
$$\frac{dC}{dt} = +\frac{1}{L_S}C_S - \frac{1}{\tau}C.$$

The term $-\frac{f}{\tau}C_T$ corresponds with transfer of *CFC* from the troposphere to the stratosphere. Here, $f < 1$, reflecting that *CFC* is more likely to diffuse downward.

The term $-\frac{1}{L_S}C_S$ corresponds with breakdown of *CFC* in the stratosphere, and L_S denotes the average lifetime of a kilogram of C_S. In general, L_S is much smaller than L_T. I.e., breakdown is much faster in the stratosphere than in the troposphere.

Example: Better Protection of the Ozone Layer

The system is as follows:

$$\begin{aligned} \frac{dC_T}{dt} &= -\frac{1}{L_T}C_T + \frac{1}{\tau}C_S - \frac{f}{\tau}C_T \\ \frac{dC_S}{dt} &= -\frac{1}{L_S}C_S - \frac{1}{\tau}C_S + \frac{f}{\tau}C_T \\ \frac{dC}{dt} &= +\frac{1}{L_S}C_S - \frac{1}{\tau}C. \end{aligned}$$

The term $-\frac{1}{\tau}C$ corresponds with the transport of chlorine from the stratosphere to the troposphere.

We can write this system in vector form with

$$\vec{y} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} C_T \\ C_S \\ C \end{pmatrix}.$$

Example: Better Protection of the Ozone Layer

We get

$$\frac{d\vec{y}}{dt} = A\vec{y}; \quad A = \begin{pmatrix} -\frac{1}{L_T} - \frac{f}{\tau} & \frac{1}{\tau} & 0 \\ \frac{f}{\tau} & -\frac{1}{L_S} - \frac{1}{\tau} & 0 \\ 0 & \frac{1}{L_S} & -\frac{1}{\tau} \end{pmatrix}.$$

Some parameter values given in the article are as follows:

$$\begin{aligned} L_T &= 1000 \text{ years} \\ L_S &= 5 \text{ years} \\ \tau &= 3 \text{ years} \\ f &= .1765. \end{aligned}$$

Solving Linear Systems with Constant Coefficients

For a fixed $n \times n$ matrix A, we consider the linear system of ODE

$$\frac{d\vec{y}}{dt} = A\vec{y}; \quad \vec{y}(0) = \vec{y}_0.$$

We begin by looking for solutions of the form

$$\vec{y}(t) = e^{\lambda t}\vec{v},$$

where λ is a constant value and $\vec{v}$ is a contant vector. If we substitute $\vec{y}(t)$ into the equation, we find

$$\lambda e^{\lambda t}\vec{v} = e^{\lambda t}A\vec{v}.$$

Dividing both sides by $e^{\lambda t}$, we obtain the eigenvalue problem

$$A\vec{v} = \lambda\vec{v}.$$

Solving Linear Systems with Constant Coefficients

Let $\{\lambda_j\}_{j=1}^n$ denote the eigenvalues of A, and suppose for simplicity that these values are all distinct. Then there is a corresponding collection of n linearly independent eigenvectors $\{\vec{v}_j\}_{j=1}^n$.

The general solution for our equation is

$$\vec{y}(t) = \sum_{j=1}^{n} c_j e^{\lambda_j t} \vec{v}_j,$$

where the constants $\{c_j\}_{j=1}^n$ can be determined from $\vec{y}_0$. I.e., we have the relation

$$\vec{y}_0 = \vec{y}(0) = \sum_{j=1}^{n} c_j \vec{v}_j,$$

and this is a system of n equations for the n constants.

Solving Linear Systems with Constant Coefficients

For the constants, we obtained precisely the same system while solving linear systems of difference equations, and we noticed that if we write

$$\vec{c} = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}, \quad V = \begin{pmatrix} \vec{v}_1 & \vec{v}_2 & \dots & \vec{v}_n \end{pmatrix},$$

then we can express our equation for the constants $\{c_j\}_{j=1}^n$ as

$$V\vec{c} = \vec{y}_0 \implies \vec{c} = V^{-1}\vec{y}_0.$$

Here, we mean that $\vec{v}_1$ is the first column of V, $\vec{v}_2$ is the second column of V, and so on.

Solving Linear Systems with Constant Coefficients

Example. Let's solve the system

$$\frac{dy_1}{dt} = y_1 + y_2; \quad y_1(0) = 1$$
$$\frac{dy_2}{dt} = 4y_1 + y_2; \quad y_2(0) = -1.$$

First, if we express this system in the matrix form

$$\frac{d\vec{y}}{dt} = A\vec{y}; \quad \vec{y}(0) = \begin{pmatrix} 1 \\ -1 \end{pmatrix},$$

we see that

$$A = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix}.$$

We begin by computing the eigenvalues of A.

Solving Linear Systems with Constant Coefficients

We compute

$$\det \begin{pmatrix} 1-\lambda & 1 \\ 4 & 1-\lambda \end{pmatrix} = (1-\lambda)^2 - 4 = 1 - 2\lambda + \lambda^2 - 4$$
$$= \lambda^2 - 2\lambda - 3 = (\lambda - 3)(\lambda + 1) = 0.$$

We see that the eigenvalues are $\lambda_1 = -1$, $\lambda_2 = 3$.

For the eigenvectors, we'll start with $\lambda_1 = -1$, and we'll write the corresponding eigenvector as $\vec{v}_1 = \binom{v_{11}}{v_{21}}$. We must have the relation

$$\begin{pmatrix} 2 & 1 \\ 4 & 2 \end{pmatrix} \begin{pmatrix} v_{11} \\ v_{21} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

from which we see that

$$2v_{11} + v_{21} = 0.$$

(After dividing by 2, the second equation would be the same thing.)

Solving Linear Systems with Constant Coefficients

We have the freedom to choose one component of $\vec{v}_1$, and we take $v_{11} = 1$, which implies $v_{21} = -2$. I.e.,

$$\vec{v}_1 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}.$$

Likewise, for $\lambda_2 = 3$, we'll write the corresponding eigenvector as $\vec{v}_2 = \binom{v_{12}}{v_{22}}$. In this case, we must have the relation

$$\begin{pmatrix} -2 & 1 \\ 4 & -2 \end{pmatrix} \begin{pmatrix} v_{12} \\ v_{22} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

from which we see that

$$-2v_{12} + v_{22} = 0.$$

We choose $v_{12} = 1$, and this implies $v_{22} = 2$, so that

$$\vec{v}_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

Solving Linear Systems with Constant Coefficients

We've now identified the eigenvalues and eigenvectors, so we can write the general solution to this equation as

$$\vec{y}(t) = c_1 e^{-t} \begin{pmatrix} 1 \\ -2 \end{pmatrix} + c_2 e^{3t} \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

Last, to find the constants c_1 and c_2, we set $t = 0$ to get

$$\begin{pmatrix} 1 \\ -1 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ -2 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix},$$

which we can write out as

$$\begin{aligned} 1 &= c_1 + c_2 \\ -1 &= -2c_1 + 2c_2. \end{aligned}$$

If we multiply the first equation by 2 and subtract the second equation from the result, we see that

$$3 = 4c_1 \implies c_1 = \frac{3}{4} \implies c_2 = \frac{1}{4}.$$

Solving Linear Systems with Constant Coefficients

We conclude that

$$\vec{y}(t) = \frac{3}{4}e^{-t}\begin{pmatrix} 1 \\ -2 \end{pmatrix} + \frac{1}{4}e^{3t}\begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

In component form, the solution is

$$\begin{aligned} y_1(t) &= \frac{3}{4}e^{-t} + \frac{1}{4}e^{3t} \\ y_2(t) &= -\frac{3}{2}e^{-t} + \frac{1}{2}e^{3t}. \end{aligned}$$