Nonlinear Systems of ODE: Nullcline Diagrams and Integral Curves

MATH 469, Texas A&M University

Spring 2020

The Non-dimensionalized Lotka-Volterra System

This lecture will focus on the Lotka-Volterra System

$$\begin{aligned}\frac{dy_1}{dt} &= ay_1 - by_1y_2; \quad y_1(0) = y_{1_0}\\ \frac{dy_2}{dt} &= -ry_2 + cy_1y_2; \quad y_2(0) = y_{2_0},\end{aligned}$$

and we'll begin by non-dimensionalizing it. For this, we introduce three dimensionless variables

$$\tau = \frac{t}{A}; \quad Y_1(\tau) = \frac{y_1(t)}{B}; \quad Y_2(\tau) = \frac{y_2(t)}{C}.$$

The constant A will be chosen with the dimension time, and the constants B and C will both be chosen with dimension biomass.

The Non-dimensionalized Lotka-Volterra System

First, using the chain rule, we compute

$$\frac{dy_1}{dt} = B\frac{d}{dt}Y_1(\tau) = B\frac{d}{d\tau}Y_1(\tau)\frac{d\tau}{dt} = \frac{B}{A}\frac{dY_1}{d\tau}$$
$$\frac{dy_2}{dt} = C\frac{d}{dt}Y_2(\tau) = C\frac{d}{d\tau}Y_2(\tau)\frac{d\tau}{dt} = \frac{C}{A}\frac{dY_2}{d\tau}.$$

If we now substitute these dimensionless variables into the Lotka-Volterra system, we get

$$\frac{B}{A}\frac{dY_1}{dt} = aBY_1 - bBCY_1Y_2; \quad Y_1(0) = \frac{y_{1_0}}{B}$$
$$\frac{C}{A}\frac{dY_2}{dt} = -rCY_2 + cBCY_1Y_2; \quad Y_2(0) = \frac{y_{2_0}}{C}.$$

The Non-dimensionalized Lotka-Volterra System

We multiply by A and divide by B (respectively C) to arrive at

$$\frac{dY_1}{dt} = aAY_1 - bACY_1Y_2; \quad Y_1(0) = \frac{y_{1_0}}{B}$$
$$\frac{dY_2}{dt} = -rAY_2 + cABY_1Y_2; \quad Y_2(0) = \frac{y_{2_0}}{C}.$$

As always, our goal is to choose the constants A, B, and C in a way that simplifies the system, while also ensuring that they have the correct dimensions. We'll take

$$A = \frac{1}{a}; \quad B = \frac{a}{c}; \quad C = \frac{a}{b}.$$

The system becomes

$$\frac{dY_1}{dt} = Y_1 - Y_1Y_2; \quad Y_1(0) = \frac{c}{a}y_{1_0}$$
$$\frac{dY_2}{dt} = -\frac{r}{a}Y_2 + Y_1Y_2; \quad Y_2(0) = \frac{b}{a}y_{2_0}.$$

The Non-dimensionalized Lotka-Volterra System

Recall that one of the things that we accomplish with non-dimensionalization is that we identify useful combinations of parameters. In this case, we set

$$s = \frac{r}{a}.$$

This allows us to write our system in the form we'll use for analysis,

$$\begin{aligned}
\frac{dY_1}{dt} &= Y_1 - Y_1 Y_2; \quad Y_1(0) = \frac{c}{a} y_{1_0} \\
\frac{dY_2}{dt} &= -sY_2 + Y_1 Y_2; \quad Y_2(0) = \frac{b}{a} y_{2_0}.
\end{aligned}$$

Equilibrium Points and Stability

For the rest of the lecture, we'll express the non-dimensionalized Lotka-Volterra system as

$$\frac{dy_1}{dt} = y_1 - y_1 y_2; \quad y_1(0) = y_{1_0}$$
$$\frac{dy_2}{dt} = -sy_2 + y_1 y_2; \quad y_2(0) = y_{2_0}.$$

We identify the equilibrium points $\hat{y} = \binom{\hat{y}_1}{\hat{y}_2}$ by solving the system

$$0 = \hat{y}_1 - \hat{y}_1 \hat{y}_2$$
$$0 = -s\hat{y}_2 + \hat{y}_1 \hat{y}_2.$$

For the first equation, we have

$$\hat{y}_1(1 - \hat{y}_2) = 0 \implies \hat{y}_1 = 0 \text{ or } \hat{y}_2 = 1.$$

We now substitute each of these into the second equation.

Equilibrium Points and Stability

For $\hat{y}_1 = 0$, the second equation becomes $-s\hat{y}_2 = 0$, and this implies $\hat{y}_2 = 0$. I.e., our first equilibrium point is $\hat{y} = \binom{0}{0}$.

For $\hat{y}_2 = 1$, the second equation becomes

$$-s + \hat{y}_1 = 0 \implies \hat{y}_1 = s,$$

our second equilibrium point is $\hat{y} = \binom{s}{1}$. In total, we have two equilibrium points,

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} s \\ 1 \end{pmatrix}.$$

We'll analyze the stability of each of these.

Equilibrium Points and Stability

We need to compute $\vec{f}'(\vec{y})$, and for this, we start by observing that

$$\vec{f}(\vec{y}) = \begin{pmatrix} y_1 - y_1 y_2 \\ -sy_2 + y_1 y_2 \end{pmatrix}.$$

This gives

$$\vec{f}'(\vec{y}) = \begin{pmatrix} 1 - y_2 & -y_1 \\ y_2 & -s + y_1 \end{pmatrix}.$$

For $\hat{y} = \binom{0}{0}$, we compute

$$\vec{f}'(0,0) = \begin{pmatrix} 1 & 0 \\ 0 & -s \end{pmatrix}.$$

The eigenvalues of this matrix are $-s$ and 1, and since 1 is positive, we can conclude that $\binom{0}{0}$ is unstable.

Equilibrium Points and Stability

For $\hat{y} = \binom{s}{1}$, we compute

$$\vec{f}'(s,1) = \begin{pmatrix} 0 & -s \\ 1 & 0 \end{pmatrix}.$$

In this case, we find the eigenvalues by computing

$$\det \begin{pmatrix} -\lambda & -s \\ 1 & -\lambda \end{pmatrix} = \lambda^2 + s = 0 \implies \lambda_\pm = \pm i\sqrt{s}.$$

We see that $\mathrm{Re}\lambda_\pm = 0$, and so the derivative test is inconclusive.

In cases like this, we need some other way of understanding the system dynamics. We'll discuss two methods for obtaning additional information, but we note at the outset that while our stability criterion works for systems with any number of equations, these two methods are generally only useful for systems of two equations.

Nullcline Diagrams

We're still considering the non-dimensionalized Lotka-Volterra system

$$\frac{dy_1}{dt} = y_1 - y_1 y_2; \quad y_1(0) = y_{1_0}$$
$$\frac{dy_2}{dt} = -sy_2 + y_1 y_2; \quad y_2(0) = y_{2_0},$$

and in this case we'll work with *nullclines* for the system. These are:

- y_1-nullclines: curves along which $\frac{dy_1}{dt} = 0$;
- y_2-nullclines: curves along which $\frac{dy_2}{dt} = 0$.

For the Lotka-Volterra system, these are easy to identify.

Nullcline Diagrams

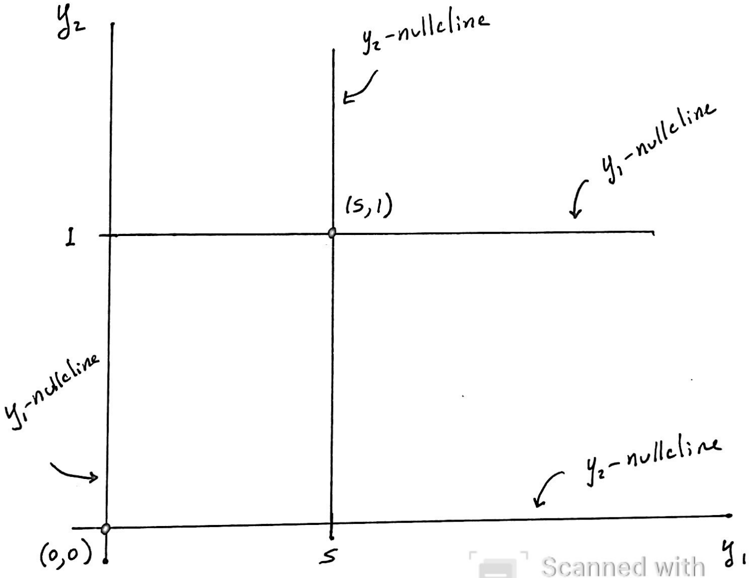
For the y_1-nullclines, we require

$$y_1 - y_1 y_2 = 0 \implies y_1(1 - y_2) = 0 \implies y_1 = 0 \text{ or } y_2 = 1.$$

Likewise, for the y_2-nullclines, we require

$$-sy_2 + y_1 y_2 = 0 \implies y_2(-s + y_1) = 0 \implies y_2 = 0 \text{ or } y_1 = s.$$

The associated nullclines are depicted on the next slide.



y_2
y_2-nullcline
y_1-nullcline
(S, I)
I
y_1-nullcline
y_2-nullcline
$(0,0)$
S
y_1

Nullcline Diagrams

Let's observe the following:

- If a y_1-nullcline intersects a y_2-nullcline, the point of intersection is an equilibrium point;
- The direction of change can only be vertical along a y_1-nullcline (because $\frac{dy_1}{dt} = 0$);
- The direction of change can only be horizontal along a y_2-nullcline (because $\frac{dy_2}{dt} = 0$).

In order to determine directions along these nullclines, we use the equations in our system. For y_1-nullclines, we use

$$\frac{dy_2}{dt} = -sy_2 + y_1 y_2.$$

For $y_1 = 0$, we have $\frac{dy_2}{dt} = -sy_2 < 0$, so the motion is downward. For $y_2 = 1$, $\frac{dy_2}{dt} = -s + y_1$. In this case, the direction is downward for $y_1 < s$ and upward for $y_1 > s$.

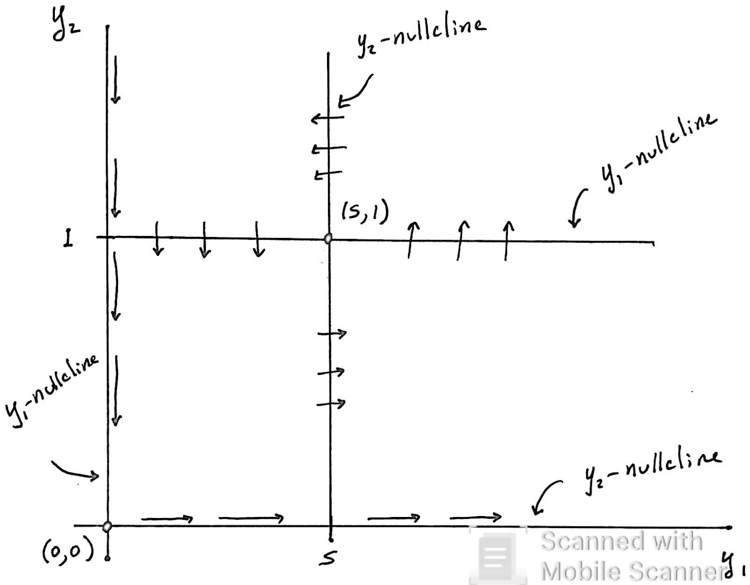
Nullcline Diagrams

For y_2-nullclines, we use

$$\frac{dy_1}{dt} = y_1 - y_1 y_2.$$

For $y_2 = 0$, we have $\frac{dy_1}{dt} = y_1$, and so the motion is to the right. For $y_1 = s$, we have $\frac{dy_1}{dt} = s(1 - y_2)$, so the motion is to the right for $y_2 < 1$ and to the left for $y_2 > 1$.

We typically add this information to our diagram with arrows, as depicted on the next slide.

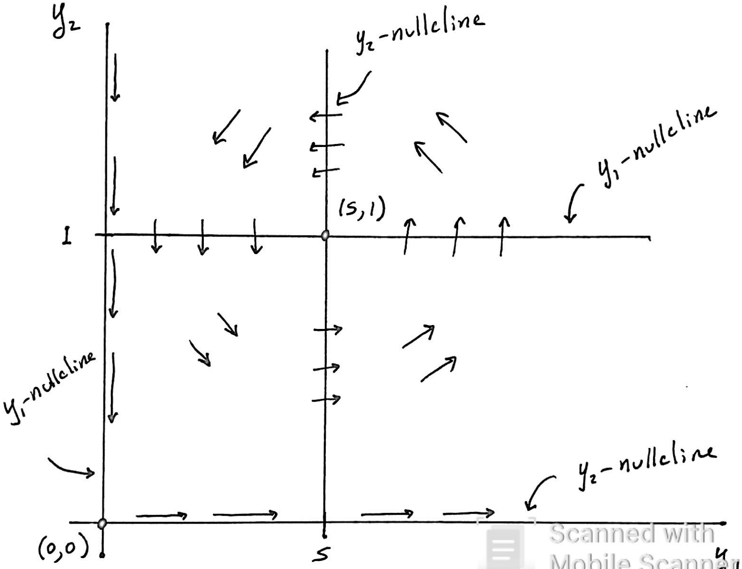

y_2
y_2-nullcline
y_1-nullcline
(S,I)
I
y_1-nullcline
y_2-nullcline
$(0,0)$
S
y_1

Nullcline Diagrams

Solutions have to follow the arrows, so we're starting to get a better of idea of how they behave. For example, solutions near $(0,0)$ tend to move toward it from above, but then away from it to the right, corresponding with instability. Solutions near $(s,1)$ seem to be cycling around it.

We can say more as well, because up-down direction and right-left direction can't change except at nullclines. E.g., if $\frac{dy_1}{dt} > 0$, then it cannot become negative without first satisfying $\frac{dy_1}{dt} = 0$, which can only occur along a y_1-nullcline.

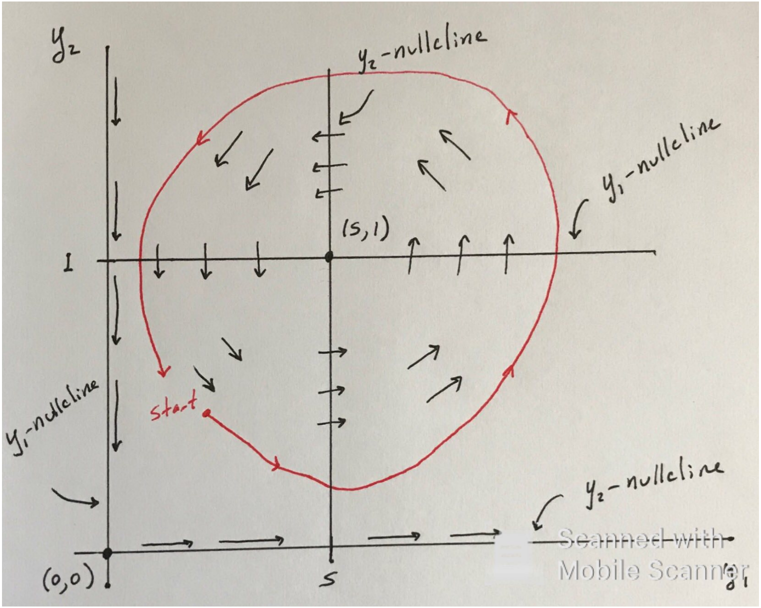
From this, we see that all arrows in the bottom left quadrant must be downward and to the right, all arrows in the bottom right quadrant upward and to the right etc. This is depicted on the next slide.


y_2
y_2-nullcline
y_1-nullcline
$(S,1)$
I
y_1-nullcline
y_2-nullcline
$(0,0)$
S
y_1

Nullcline Diagrams

Suppose we now want to trace out the trajectory of a solution starting at some point (y_{1_0}, y_{2_0}), and to be precise, let's suppose (y_{1_0}, y_{2_0}) lies in the lower left quadrant.

Then we can trace out its trajectory as depicted on the next slide by following the arrows indicating the direction it can follow.


y_2
y_2-nullcline
y_1-nullcline
(S, I)
I
start
y_1-nullcline
y_2-nullcline
$(0,0)$
S
y_1

Nullcline Diagrams

Let's notice two things about this trajectory. First, the trajectory can never hit either the y_1-axis or the y_2-axis. This is because if a solution is ever on one of these axes it will remain there for all time (because the motion is either entirely horizontal or entirely vertical), and this includes both forward in time and backward in time. But if a trajectory were to hit one of these axes, then in backward time it would be leaving it.

Second, while a trajectory can approach an equilibrium point, and in that case will move slower and slower, it can never stop. (It can't ever actually hit the equilibrium point for exactly the same reason as above: if it is ever at the equilbrium point, then it must remain there for all time, forward and backward.)

Notice that we're still left with the following question: does the trajectory actually complete a full loop and start over? To answer this, we'll need to look at integral curves.

Integral Curves

Recall from our discussion of the phase line that the *phase variables* for an equation are those that determine all future behavior. For the Loktka-Volterra system, or any other first order system of two equations, these are y_1 and y_2. That is, if we ever know the values of y_1 and y_2 at some time, then we will know their values for all future times. In fact, we saw an example of how this works with the trajectory we drew on our nullcline diagram.

Let's see how we can be more precise about such diagrams than we were with our nullcline diagrams.

Integral Curves

As a start, we can we can think about how we might plot y_2 as a function of y_1, cutting out the middle-man t.

Suppose we have a general first-order autonomous system of two equations

$$\begin{aligned} \frac{dy_1}{dt} &= f_1(y_1, y_2) \\ \frac{dy_2}{dt} &= f_2(y_1, y_2). \end{aligned}$$

Formally, we can obtain an equation for y_2 as a function of y_1 by writing

$$\frac{dy_2}{dy_1} = \frac{\frac{dy_2}{dt}}{\frac{dy_1}{dt}} = \frac{f_2(y_1, y_2)}{f_1(y_1, y_2)}.$$

In some cases, we can solve this single ODE to get an explicit relationship between y_1 and y_2.

Integral Curves

Let's see how this works for the non-dimensionalized Lotka-Volterra system

$$\frac{dy_1}{dt} = y_1 - y_1 y_2; \quad y_1(0) = y_{1_0}$$
$$\frac{dy_2}{dt} = -sy_2 + y_1 y_2; \quad y_2(0) = y_{2_0}.$$

In this case, we have

$$\frac{dy_2}{dy_1} = \frac{\frac{dy_2}{dt}}{\frac{dy_1}{dt}} = \frac{-sy_2 + y_1 y_2}{y_1 - y_1 y_2}.$$

The question is, can we solve this ODE

$$\frac{dy_2}{dy_1} = \frac{-sy_2 + y_1 y_2}{y_1 - y_1 y_2}?$$

Integral Curves

We can solve this equation by separating variables. If we write

$$\frac{dy_2}{dy_1} = \frac{y_2(-s + y_1)}{y_1(1 - y_2)},$$

then we see that

$$\frac{1 - y_2}{y_2} dy_2 = \frac{-s + y_1}{y_1} dy_1,$$

which is

$$(\frac{1}{y_2} - 1) dy_2 = (-\frac{s}{y_1} + 1) dy_1.$$

We now integrate both sides to see that

$$\ln|y_2| - y_2 = -s \ln|y_1| + y_1 + C,$$

where C is a constant of integration.

Integral Curves

Since we're interested in positive populations, we can drop the absolute values and rearrange terms to see that

$$\ln y_2 + \ln y_1^s = y_1 + y_2 + C \implies \ln(y_2 y_1^s) = y_1 + y_2 + C.$$

Exponentiating both sides, we find

$$y_2 y_1^s = e^{y_1+y_2+C} = e^{y_1} e^{y_2} e^C.$$

Let's write $K = e^C$ (not a carrying capacity!), so that we have the relation

$$y_2 y_1^s = K e^{y_1} e^{y_2}.$$

This is an algebraic relationship between y_1 and y_2, and for each positive constant K it defines a curve in the (y_1, y_2)-plane (or has no solutions). These curves are called *integral curves*.

Integral Curves

In order to get an idea of what these integral curves look like, we'll plot a few with MATLAB. We'll fix $s = \frac{1}{2}$, and consider a range of values for K.

The one calculation we'll do by hand is to find a starting value for K. For this, we're particularly interested in the dynamics near the equilibrium point $(s, 1)$, and at this point, our relationship between y_1 and y_2 becomes

$$s^s = Ke^{(1+s)}.$$

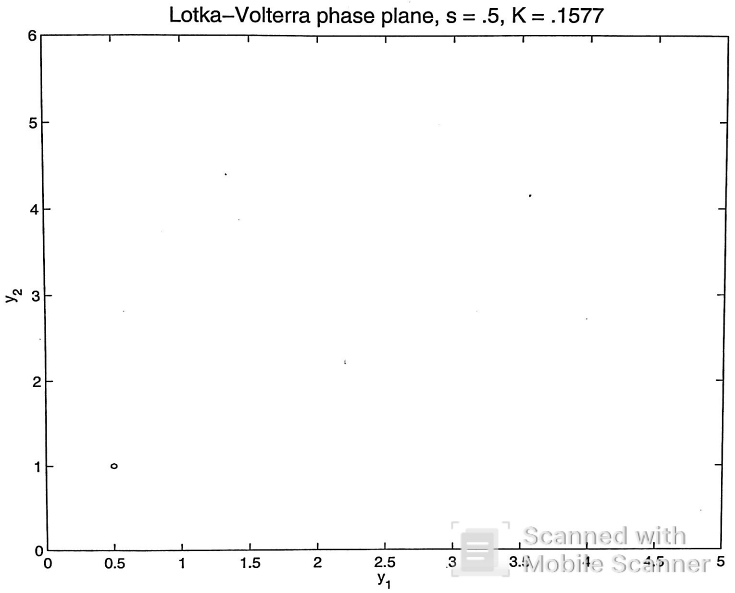
For $s = \frac{1}{2}$, this is

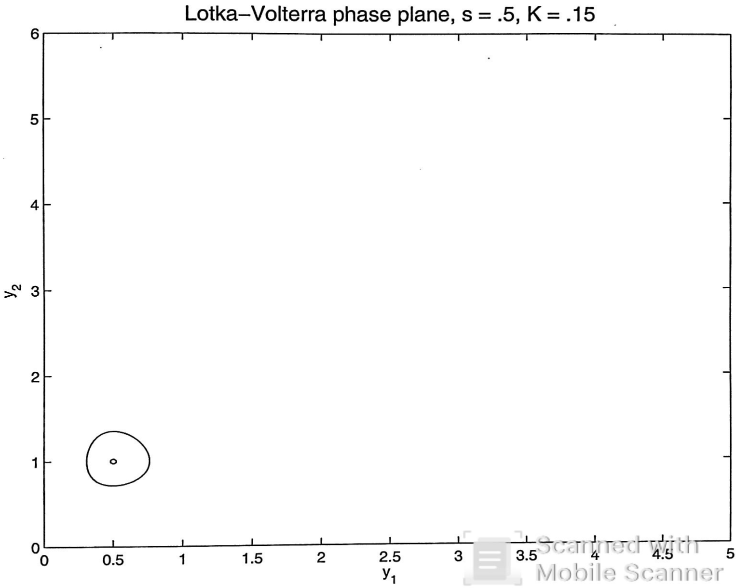
$$K = \frac{(\frac{1}{2})^{1/2}}{e^{3/2}} = .1578.$$

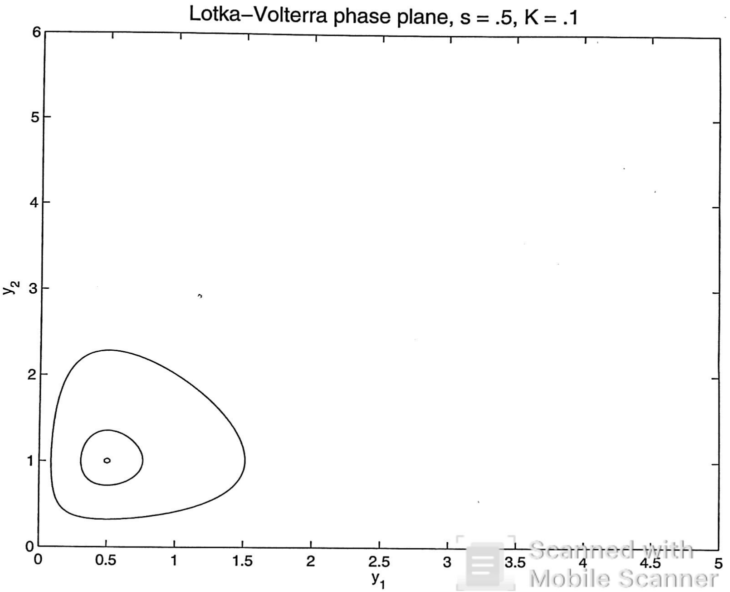
In order to avoid plotting a dot (which is difficult to see), we'll start with $K = .1577$. Additional values will be given on the figures. In each case, the previous integral curves have been kept on the figure for comparison.

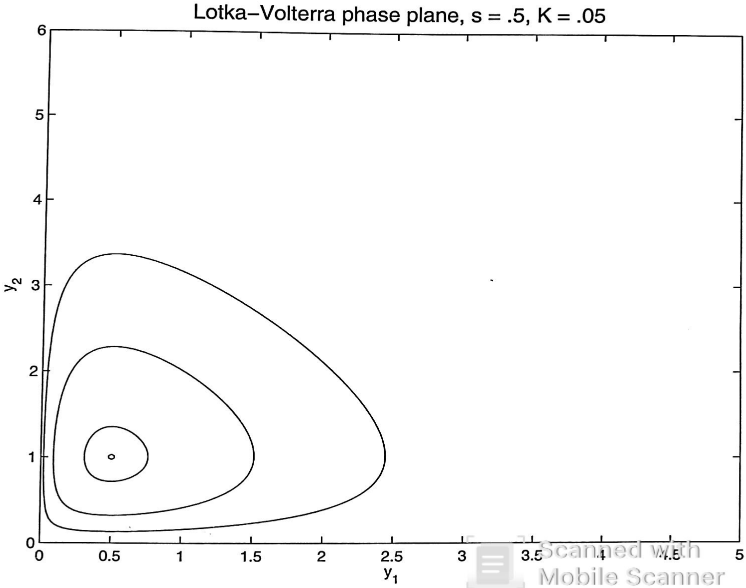
Integral Curves

The direction of flow can be obtained as with nullclines by using derivative values from the system of equations.

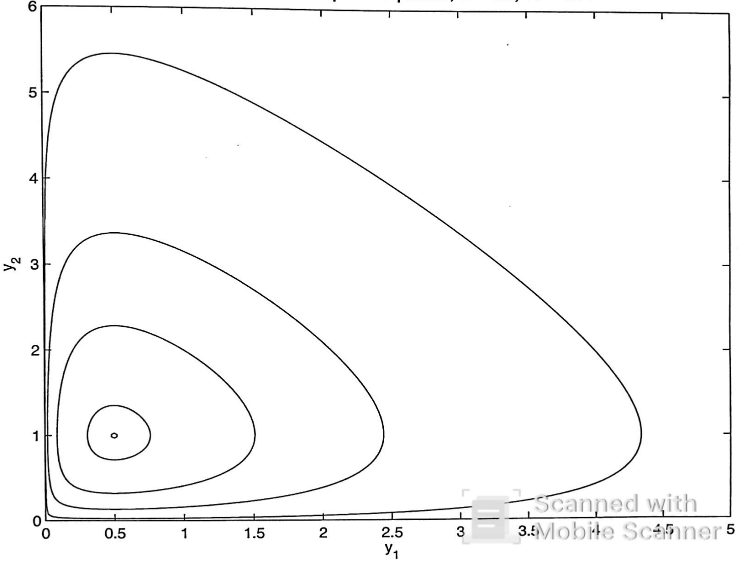

Lotka–Volterra phase plane, s = .5, K = .1577
y2
y1
0
1
2
3
4
5
6
0
0.5
1
1.5
2
2.5
3
3.5
4
4.5
5


Lotka–Volterra phase plane, s = .5, K = .15
y_2
y_1
0
0.5
1
1.5
2
2.5
3
3.5
4
4.5
5
6


Lotka–Volterra phase plane, s = .5, K = .1
0
0.5
1
1.5
2
2.5
3
3.5
4
4.5
5
y1
1
2
3
4
5
6
y2


Lotka–Volterra phase plane, s = .5, K = .05
y2
y1
0
1
2
3
4
5
6
0
0.5
1
1.5
2
2.5
3
3.5
4
4.5
5

Lotka–Volterra phase plane, s = .5, K = .01



Integral Curves

Final note: We see from this that the equilibrium point $\binom{1/2}{1}$ is stable, but not asymptotically stable, and in fact this is true in general for the equilibrium point $\binom{s}{1}$.