Fixed Points and Stability, I

MATH 469, Texas A&M University

Spring 2020

Non-dimensionalization

As with single difference equations, it's convenient to non-dimensionalize a system of difference equations before analyzing it. Let's see how this works with our predator-prey model

$$y_{1_{t+1}} - y_{1_t} = ay_{1_t}(1 - \frac{y_{1_t}}{K}) - by_{1_t}y_{2_t}$$
$$y_{2_{t+1}} - y_{2_t} = -ry_{2_t} + cy_{1_t}y_{2_t}.$$

We set

$$Y_{1_t} = \frac{y_{1_t}}{A}$$
$$Y_{2_t} = \frac{y_{2_t}}{B},$$

where A and B are constants with the dimension of biomass that will be chosen to put the system in a convenient form.

Non-dimensionalization

We substitute $y_{1_t} = AY_{1_t}$ and $y_{2_t} = BY_{2_t}$ into the system to obtain

$$
\begin{aligned}
AY_{1_{t+1}} - AY_{1_t} &= aAY_{1_t}(1 - \frac{AY_{1_t}}{K}) - bABY_{1_t}Y_{2_t} \\
BY_{2_{t+1}} - BY_{2_t} &= -rBY_{2_t} + cABY_{1_t}Y_{2_t}.
\end{aligned}
$$

We notice that A can be divided into the first equation, and B can be divided into the second, giving

$$
\begin{aligned}
Y_{1_{t+1}} - Y_{1_t} &= aY_{1_t}(1 - \frac{AY_{1_t}}{K}) - bBY_{1_t}Y_{2_t} \\
Y_{2_{t+1}} - Y_{2_t} &= -rY_{2_t} + cAY_{1_t}Y_{2_t}.
\end{aligned}
$$

It's natural to choose $A = K$ and $B = \frac{1}{b}$, and this leads to the non-dimensionalized system

$$
\begin{aligned}
Y_{1_{t+1}} - Y_{1_t} &= aY_{1_t}(1 - Y_{1_t}) - Y_{1_t}Y_{2_t} \\
Y_{2_{t+1}} - Y_{2_t} &= -rY_{2_t} + \delta Y_{1_t}Y_{2_t}, \quad \delta = cK.
\end{aligned}
$$

Fixed Points

As with single difference equations, we say that $\hat{y}$ is a fixed point for the system of difference equations

$$\vec{y}_{t+1} = \vec{f}(\vec{y}_t)$$

if

$$\hat{y} = \vec{f}(\hat{y}).$$

Notice that in this case $\hat{y}$ is a vector with the same number of components as $\vec{y}_t$.

Example. Find all fixed points for the system

$$\begin{aligned} y_{1_{t+1}} &= y_{2_t}^2 \\ y_{2_{t+1}} &= y_{1_t}. \end{aligned}$$

Fixed Points

For this example, we have

$$\vec{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \quad \text{and} \quad \vec{f}(\vec{y}) = \begin{pmatrix} y_2^2 \\ y_1 \end{pmatrix}.$$

The fixed point equation $\hat{y} = \vec{f}(\hat{y})$ is

$$\begin{aligned} \hat{y}_1 &= \hat{y}_2^2 \\ \hat{y}_2 &= \hat{y}_1. \end{aligned}$$

Upon substitution of the second into the first, we see that

$$\hat{y}_1 = \hat{y}_1^2 \implies \hat{y}_1(\hat{y}_1 - 1) = 0 \implies \hat{y}_1 = 0, 1.$$

We conclude that the fixed points are $\binom{0}{0}$ and $\binom{1}{1}$.

Stability of Fixed Points

In order to discuss the stability of fixed points for systems, we need one more review of multivariate differentiation: the case of vector functions of a vector variable.

First, for a vector function of a vector variable $\vec{f}(\vec{y})$, with $\vec{y} \in \mathbb{R}^n$ and $\vec{f}(\vec{y}) \in \mathbb{R}^m$ (we typically write $\vec{f} : \mathbb{R}^n \to \mathbb{R}^m$), the Jacobian matrix is

$$\vec{f}'(\vec{y}) = \begin{pmatrix} \frac{\partial f_1}{\partial y_1} & \frac{\partial f_1}{\partial y_2} & \cdots & \frac{\partial f_1}{\partial y_n} \\ \frac{\partial f_2}{\partial y_1} & \frac{\partial f_2}{\partial y_2} & \cdots & \frac{\partial f_2}{\partial y_n} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial f_m}{\partial y_1} & \frac{\partial f_m}{\partial y_2} & \cdots & \frac{\partial f_m}{\partial y_n} \end{pmatrix}.$$

Notice that $\vec{f}'(\vec{y})$ is an $m \times n$ matrix.

Some Technical Stuff

We say that a function $\vec{f} : \mathbb{R}^n \to \mathbb{R}^m$ is differentiable at a point $\vec{y} \in \mathbb{R}^n$ if the partial derivatives in $\vec{f}'(\vec{y})$ all exist at $\vec{y}$, and

$$\lim_{|\vec{h}| \to 0} \frac{|\vec{f}(\vec{y} + \vec{h}) - \vec{f}(\vec{y}) - \vec{f}'(\vec{y})\vec{h}|}{|\vec{h}|} = 0.$$

Equivalently: if there exists a function $\vec{\epsilon}(\vec{h}; \vec{y})$ so that

$$\vec{f}(\vec{y} + \vec{h}) = \vec{f}(\vec{y}) + \vec{f}'(\vec{y})\vec{h} + \vec{\epsilon}(\vec{h}; \vec{y}),$$

where

$$\lim_{|\vec{h}| \to 0} \frac{|\vec{\epsilon}(\vec{h}; \vec{y})|}{|\vec{h}|} = 0.$$

I.e., $|\vec{\epsilon}(\vec{h}; \vec{y})| = \mathbf{o}(|\vec{h}|)$.

Back to Stability

Let $\hat{y}$ denote a fixed point for

$$\vec{y}_{t+1} = \vec{f}(\vec{y}_t), \qquad (*)$$

and set

$$\vec{y}_t = \hat{y} + \vec{z}_t. \qquad (**)$$

We want to determine whether $\vec{y}_t$ approaches $\hat{y}$ as $t \to \infty$ (asymptotic stability), and this means we want to determine whether $\vec{z}_t$ approaches 0 as $t \to \infty$. If we substitute (**) into (*), we get:

$$\begin{aligned} \hat{y} + \vec{z}_{t+1} &= \vec{f}(\hat{y} + \vec{z}_t) \\ &= \vec{f}(\hat{y}) + \vec{f}'(\hat{y})\vec{z}_t + \vec{\epsilon}(\vec{z}_t; \hat{y}). \end{aligned}$$

Back to Stability

Since $\hat{y} = \vec{f}(\hat{y})$ and $\vec{\epsilon}(\vec{z}_t; \hat{y})$ is smaller than $\vec{z}_t$, we have the approximate equation

$$\vec{z}_{t+1} \cong \vec{f}'(\hat{y})\vec{z}_t. \qquad (***)$$

This is a linear system of difference equations, and we know how to solve such equations.

Let $\{\lambda_j\}_{j=1}^n$ denote the eigenvalues of $\vec{f}'(\hat{y})$, and for simplicity assume these eigenvalues are distinct. In this case, we can associate them with a linearly independent collection of eigenvectors $\{\vec{v}_j\}_{j=1}^n$. We've seen that we can solve (***) with

$$\vec{z}_t = \sum_{j=1}^{n} c_j \lambda_j^t \vec{v}_j,$$

for some collection of constants $\{c_j\}_{j=1}^n$.

Back to Stability

From the previous page,

$$\vec{z}_t = \sum_{j=1}^{n} c_j \lambda_j^t \vec{v}_j.$$

If the eigenvalues $\{\lambda_j\}_{j=1}^{n}$ all satisfy $|\lambda_j| < 1$ (possibly complex modulus), then $\hat{y}$ must be asymptotically stable. If $|\lambda_j| > 1$ for any j, then $\hat{y}$ must be unstable. If $|\lambda_j| = 1$, stability will be determined by the nonlinear terms. In particular, stability is not determined by this criterion.

This criterion is valid even if the eigenvalues of $\vec{f}'(\hat{y})$ are not distinct.

Easy Example

Let's return to our example

$$\vec{y}_{t+1} = \vec{f}(\vec{y}_t),$$

where

$$\vec{f}(\vec{y}) = \begin{pmatrix} y_2^2 \\ y_1 \end{pmatrix}.$$

Recall that we've already found the fixed points to be $\binom{0}{0}$ and $\binom{1}{1}$. In order to write down the Jacobian matrix, we need to compute

$$\frac{\partial f_1}{\partial y_1} = 0; \quad \frac{\partial f_1}{\partial y_2} = 2y_2$$
$$\frac{\partial f_2}{\partial y_1} = 1; \quad \frac{\partial f_2}{\partial y_2} = 0.$$

This gives

$$\vec{f}'(\vec{y}) = \begin{pmatrix} 0 & 2y_2 \\ 1 & 0 \end{pmatrix}.$$

Easy Example

For $\hat{y} = \binom{0}{0}$, we have

$$\vec{f}'(0,0) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

The eigenvalues of this matrix are $\lambda_1 = \lambda_2 = 0$ (repeated), so this fixed point is asymptotically stable.

For $\hat{y} = \binom{1}{1}$, we have

$$\vec{f}'(1,1) = \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}.$$

In this case, we compute

$$\det \begin{pmatrix} -\lambda & 2 \\ 1 & -\lambda \end{pmatrix} = \lambda^2 - 2 = 0 \implies \lambda = \pm\sqrt{2}.$$

Since $\sqrt{2} > 1$, we can conclude that this fixed point is unstable.