M-Cycles for Systems of Difference Equations, I

MATH 469, Texas A&M University

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Identifying M-Cycles

For a system of n difference equations

$$\vec{y}_{t+1} = \vec{f}(\vec{y}_t),$$

we say that a collection of m distinct vectors $\{\hat{y}_k\}_{k=1}^{m}$ ($\hat{y}_k \in \mathbb{R}^n$ for each $k = 1, 2, \ldots, m$) is an m-cycle if

$$\begin{aligned}
\hat{y}_2 &= \vec{f}(\hat{y}_1) \\
\hat{y}_3 &= \vec{f}(\hat{y}_2) \\
&\vdots \\
\hat{y}_m &= \vec{f}(\hat{y}_{m-1}) \\
\hat{y}_1 &= \vec{f}(\hat{y}_m).
\end{aligned}$$

Identifying M-Cycles

Recall our notation for the composition of functions,

$$\vec{f}^2(\vec{y}) = \vec{f}(\vec{f}(\vec{y}))$$
$$\vec{f}^3(\vec{y}) = \vec{f}(\vec{f}(\vec{f}(\vec{y})))$$
$$\vdots$$

For $m = 3$, we can write

$$\hat{y}_1 = \vec{f}(\hat{y}_3) = \vec{f}(\vec{f}(\hat{y}_2)) = \vec{f}(\vec{f}(\vec{f}(\hat{y}_1))) = \vec{f}^3(\hat{y}_1),$$

and more generally

$$\hat{y}_1 = \vec{f}(\hat{y}_m) = \vec{f}(\vec{f}(\hat{y}_{m-1})) = \cdots = \vec{f}^m(\hat{y}_1).$$

Identifying M-Cycles

Likewise, for each $k = 1, 2, \ldots, m$,

$$\hat{y}_k = \vec{f}^m(\hat{y}_k).$$

We see that each vector in the m-cycle is a fixed point for the difference equation

$$\vec{x}_{t+1} = \vec{f}^m(\vec{x}_t).$$

Identifying M-Cycles

Example. Find all 2-cycles for the system of difference equations

$$\vec{y}_{t+1} = \vec{f}(\vec{y}_t),$$

where

$$\vec{f}(y_1, y_2) = \begin{pmatrix} y_2^2 \\ y_1 \end{pmatrix}.$$

Recall that we found in a previous lecture that the fixed points for this system are $\binom{0}{0}$ and $\binom{1}{1}$. For the 2-cycles, we need to look for fixed points of the difference equation

$$\vec{x}_{t+1} = \vec{f}^2(\vec{x}_t) = \vec{f}(\vec{f}(\vec{x}_t)).$$

I.e., if $\hat{y} = \binom{y_1}{y_2}$ is one of the two vectors in a 2-cycle, it will satisfy

$$\hat{y} = \vec{f}(\vec{f}(\hat{y})).$$

Identifying M-Cycles

From the previous slide,

$$\hat{y} = \vec{f}(\vec{f}(\hat{y})). \tag{*}$$

We can compute

$$\vec{f}(\vec{f}) = \vec{f}(f_1, f_2) = \begin{pmatrix} f_2^2 \\ f_1 \end{pmatrix} = \begin{pmatrix} y_1^2 \\ y_2^2 \end{pmatrix}.$$

We see that (*) can be expressed as

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} y_1^2 \\ y_2^2 \end{pmatrix}.$$

This is the same equation in each component. For the first,

$$y_1 = y_1^2 \implies y_1(1 - y_1) = 0 \implies y_1 = 0, 1,$$

and likewise for y_2.

Identifying M-Cycles

In this case, any combination of the values for y_1 and y_2 will be a fixed point for $\vec{x}_{t+1} = \vec{f}^2(\vec{x}_t)$, so we have a total of four:

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Keep in mind that fixed points for $\vec{y}_{t+1} = \vec{f}(\vec{y}_t)$ are also fixed points for $\vec{x}_{t+1} = \vec{f}^2(\vec{x}_t)$. From this, we can eliminate $\binom{0}{0}$ and $\binom{1}{1}$ from consideration, and conclude that the 2-cycle is

$$\{\hat{y}_1, \hat{y}_2\} = \left\{ \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}.$$

Stability of M-Cycles

Before discussing the stability of m-cycles, we need one more result from multidimensional calculus.

First, recall that if f and g are both functions taking real values to real values, then the usual chain rule is

$$\frac{d}{dx}f(g(x)) = f'(g(x))g'(x).$$

We need to extend this to the case of vector-valued functions of vector variables.

Theorem (Multidimensional Chain Rule). Suppose $\vec{g} : \mathbb{R}^l \to \mathbb{R}^n$ is differentiable at a point $\vec{x} \in \mathbb{R}^l$, and $\vec{f} : \mathbb{R}^n \to \mathbb{R}^m$ is differentiable at $\vec{g}(\vec{x}) \in \mathbb{R}^n$. Then

$$\vec{f}(\vec{g}(\vec{x}))' = \vec{f}'(\vec{g}(\vec{x}))\vec{g}'(\vec{x}).$$

Stability of M-Cycles

From the previous slide,

$$\vec{f}(\vec{g}(\vec{x}))' = \vec{f}'(\vec{g}(\vec{x}))\vec{g}'(\vec{x}).$$

Here, $\vec{f}'(\vec{g}(\vec{x}))$ is an $m \times n$ matrix, and $\vec{g}'(\vec{x})$ is an $n \times l$ matrix, so $\vec{f}(\vec{g}(\vec{x}))'$ is an $m \times l$ matrix.

Now, suppose $\{\hat{y}_k\}_{k=1}^m$ is an m-cycle for the difference system

$$\vec{y}_{t+1} = \vec{f}(\vec{y}_t). \qquad (*)$$

By the same argument we used for single equations, the m-cycle $\{\hat{y}_k\}_{k=1}^m$ will be asymptotically stable as a solution of (*) if and only if each vector $\hat{y}_k$ in the m-cycle is asymptotically stable as a fixed point of

$$\vec{x}_{t+1} = \vec{f}^m(\vec{x}_t). \qquad (**)$$

Stability of M-Cycles

As a reminder of why this is the case, let's think about $m = 2$. If $\{\hat{y}_1, \hat{y}_2\}$ is an asyptotically stable 2-cycle for $\vec{y}_{t+1} = \vec{f}(\vec{y}_t)$, then as t gets large solutions to $\vec{y}_{t+1} = \vec{f}(\vec{y}_t)$ will approach the sequence

$$\hat{y}_1, \hat{y}_2, \hat{y}_1, \hat{y}_2, \ldots.$$

Solutions of $\vec{x}_{t+1} = \vec{f}^2(\vec{x}_t)$ will skip every other term, so will approach either the sequence

$$\hat{y}_1, \hat{y}_1, \ldots$$

or the sequence

$$\hat{y}_2, \hat{y}_2, \ldots,$$

depending on the initial vector $\vec{y}_0$. We see that if $\{\hat{y}_1, \hat{y}_2\}$ is an asyptotically stable 2-cycle for $\vec{y}_{t+1} = \vec{f}(\vec{y}_t)$, then both $\hat{y}_1$ and $\hat{y}_2$ are asymptotically stable fixed points for $\vec{x}_{t+1} = \vec{f}^2(\vec{x}_t)$, and we can turn the argument around to get the implication in the other direction.

Stability of M-Cycles

We also see from this that if one vector in the m-cycle is asymptotically stable as a fixed point of $\vec{x}_{t+1} = \vec{f}^m(\vec{x}_t)$, then the rest must be as well. Finally, the same argument can be made for stability and instability.

Stability of M-Cycles

This means that we have the following condition for stability of the m-cycle $\{\hat{y}_k\}_{k=1}^m$ as a solution of

$$\vec{y}_{t+1} = \vec{f}(\vec{y}_t). \qquad (*)$$

For any $\hat{y}_k$ in the m-cycle $\{\hat{y}_k\}_{k=1}^m$, let $\{\lambda_j\}_{j=1}^n$ denote the eigenvalues of the Jacobian matrix $\vec{f}^{m\prime}(\hat{y}_k)$.

(i) If $|\lambda_j| < 1$ for all $j = 1, 2, \ldots, n$, then the m-cycle $\{\hat{y}_k\}_{k=1}^m$ is asymptotically stable.

(ii) If $|\lambda_j| > 1$ for any $j \in \{1, 2, \ldots, n\}$ then the m-cycle $\{\hat{y}_k\}_{k=1}^m$ is unstable.

(iii) If $|\lambda_j| \leq 1$ for all $j = 1, 2, \ldots, n$ and $|\lambda_j| = 1$ for at least one $j \in \{1, 2, \ldots, n\}$, then this criterion for stability is inconclusive (stability is determined by the nonlinear terms).

Stability of M-Cycles

In order to check this stability condition, we need to compute $\vec{f}^{m\prime}(\hat{y}_k)$. Let's start with $m = 2$, in which case we have

$$\vec{f}^2(\vec{y}) = \vec{f}(\vec{f}(\vec{y})) \implies \vec{f}^{2\prime}(\vec{y}) = \vec{f}'(\vec{f}(\vec{y}))\vec{f}'(\vec{y}).$$

If we denote the 2-cycle by $\{\hat{y}_1, \hat{y}_2\}$, then for $\vec{y} = \hat{y}_1$ we see that

$$\vec{f}^{2\prime}(\hat{y}_1) = \vec{f}'(\vec{f}(\hat{y}_1))\vec{f}'(\hat{y}_1) = \vec{f}'(\hat{y}_2)\vec{f}'(\hat{y}_1).$$

In particular, $\vec{f}^{2\prime}(\hat{y}_1)$ is just the product of two Jacobian matrices, one evaluated at $\hat{y}_1$ and the other evaluated at $\hat{y}_2$.

Notice that if we use $\vec{y} = \hat{y}_2$, we get

$$\vec{f}^{2\prime}(\hat{y}_2) = \vec{f}'(\vec{f}(\hat{y}_2))\vec{f}'(\hat{y}_2) = \vec{f}'(\hat{y}_1)\vec{f}'(\hat{y}_2).$$

I.e., either order of matrix multiplication works, even though matrix multiplication isn't commutative.

Stability of M-Cycles

More generally,

$$\begin{aligned}\vec{f}^{m\,\prime}(\vec{y}) &= \vec{f}(\vec{f}^{m-1}(\vec{y}))' = \vec{f}'(\vec{f}^{m-1}(\vec{y}))\vec{f}^{m-1\,\prime}(\vec{y}) \\ &= \vec{f}'(\vec{f}^{m-1}(\vec{y}))\vec{f}(\vec{f}^{m-2}(\vec{y}))' = \vec{f}'(\vec{f}^{m-1}(\vec{y}))\vec{f}'(\vec{f}^{m-2}(\vec{y}))\vec{f}^{m-2\,\prime}(\vec{y}) \\ &= \cdots = \vec{f}'(\vec{f}^{m-1}(\vec{y}))\vec{f}'(\vec{f}^{m-2}(\vec{y}))\cdots\vec{f}'(\vec{y}).\end{aligned}$$

Using the relations $\hat{y}_2 = \vec{f}(\hat{y}_1)$, $\hat{y}_3 = \vec{f}(\vec{y}_2) = \vec{f}^2(\hat{y}_1)$, ..., $\vec{y}_m = \vec{f}^{m-1}(\hat{y}_1)$, we see that

$$\vec{f}^{m\,\prime}(\hat{y}_1) = \vec{f}'(\hat{y}_m)\vec{f}'(\hat{y}_{m-1})\cdots\vec{f}'(\hat{y}_1).$$

We would get a similar expression by evaluating at any other $\hat{y}_k$ in the cycle.

Stability of M-Cycles

Example 1. Let's check stability for the 2-cycle $\{\hat{y}_1, \hat{y}_2\} = \{\binom{0}{1}, \binom{1}{0}\}$ for

$$\vec{y}_{t+1} = \vec{f}(\vec{y}_t); \quad \vec{f}(y_1, y_2) = \begin{pmatrix} y_2^2 \\ y_1 \end{pmatrix}.$$

The Jacobian matrix is

$$\vec{f}'(\vec{y}) = \begin{pmatrix} 0 & 2y_2 \\ 1 & 0 \end{pmatrix},$$

so that

$$\vec{f}'(\hat{y}_1) = \vec{f}'(0,1) = \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}; \quad \vec{f}'(\hat{y}_2) = \vec{f}'(1,0) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

We need to compute

$$\vec{f}^{2\prime}(\hat{y}_1) = \vec{f}'(\hat{y}_2)\vec{f}'(\hat{y}_1) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}.$$

Stability of M-Cycles

The eigenvalues of this matrix are 0 and 2, so we can immediately conclude that this 2-cycle is unstable.

That finishes the example, but let's take a look at what would happen if we used $\hat{y}_2$ instead of $\hat{y}_1$. We would compute

$$\vec{f}^{2\prime}(\hat{y}_2) = \vec{f}'(\hat{y}_1)\vec{f}'(\hat{y}_2) = \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}.$$

We obtain a different matrix, but the conclusion about stability is the same.