M-Cycles for Systems of Difference Equations, II

MATH 469, Texas A&M University

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Stability of M-Cycles

Example 2. Let's return to our non-dimensionalized predator-prey model

$$
\begin{aligned}
y_{1_{t+1}} - y_{1_t} &= a y_{1_t}(1 - y_{1_t}) - y_{1_t} y_{2_t} \\
y_{2_{t+1}} - y_{2_t} &= -r y_{2_t} + \delta y_{1_t} y_{2_t}, \quad \delta = cK.
\end{aligned}
$$

I.e., this is $\vec{y}_{t+1} = \vec{f}(\vec{y}_t)$, with

$$
\vec{f}(\vec{y}) = \begin{pmatrix} (1+a)y_1 - a y_1^2 - y_1 y_2 \\ y_2 - r y_2 + \delta y_1 y_2 \end{pmatrix},
$$

and we will use the parameter values we obtained from the hare-lynx data, $a = 1.4974$, $r = .5820$, and $\delta = 1.9675$. (Also $c = .0239$, $K = 82.3206$, and $b = .0425$.)

Stability of M-Cycles

We've already seen that this system doesn't have any stable fixed points for these parameter values, and the natural next step is to look for m-cycles. Our first question is: What value of m should we be working with?

To answer this, let's look at two plots of solutions, initialized by $y_{1_0} = \frac{30}{K}$ and $y_{2_0} = 4b$ (because of the non-dimensionalization). First, we solve the model forward for 100 years, then in the next plot we'll zoom in on the final years.

Stability of M-Cycles

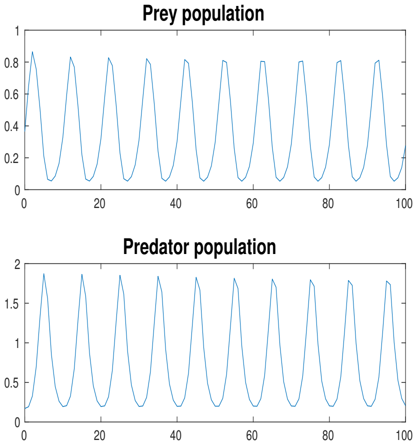

Figure: Predator-prey populations for 100 years.

Stability of M-Cycles

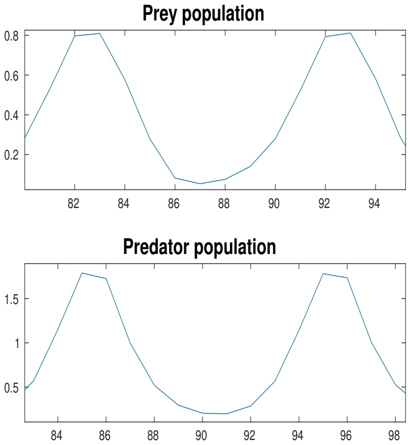

Figure: Predator-prey populatons for the final years.

Stability of M-Cycles

We see that we should be looking for a 10-cycle. Let's think about this.

If we wanted to find a 2-cycle for this system, we would need to solve

$$\hat{y} = \vec{f}^2(\hat{y}) = \vec{f}(\vec{f}(\hat{y})).$$

Here,

$$\vec{f}(\vec{y}) = \begin{pmatrix} (1+a)y_1 - ay_1^2 - y_1 y_2 \\ y_2 - ry_2 + \delta y_1 y_2 \end{pmatrix},$$

so

$$\vec{f}(\vec{f}) = \begin{pmatrix} (1+a)f_1 - af_1^2 - f_1 f_2 \\ f_2 - rf_2 + \delta f_1 f_2 \end{pmatrix}.$$

Stability of M-Cycles

If we substitute our expressions for f_1 and f_2, we obtain

$$\begin{pmatrix} (1+a)((1+a)y_1 - ay_1^2 - y_1 y_2) - a((1+a)y_1 - ay_1^2 - y_1 y_2)^2 - ((1+a)y_1 - ay_1^2 - y_1 y_2)(y_2 - ry_2 + \delta y_1 y_2) \\ (1-r)(y_2 - ry_2 + \delta y_1 y_2) + \delta((1+a)y_1 - ay_1^2 - y_1 y_2)(y_2 - ry_2 + \delta y_1 y_2) \end{pmatrix}$$

This is only for a 2-cycle!

Nonetheless, we can find the 10-cycle numerically by solving

$$\hat{y} = \vec{f}^{10}(\hat{y}).$$

The values are as follows:

prey	.26	.49	.77	.83	.61	.31	.09	.05	.07	.13
pred	.21	.20	.27	.53	1.08	1.73	1.79	1.07	.56	.31

I.e., $\hat{y}_1 = \binom{.26}{.21}$, $\hat{y}_2 = \binom{.49}{.20}$, etc.

Stability of M-Cycles

We've already seen in a previous lecture that the Jacobian matrix in this case is

$$\vec{f}'(y_1, y_2) = \begin{pmatrix} 1 + a - 2ay_1 - y_2 & -y_1 \\ \delta y_2 & 1 - r + \delta y_1 \end{pmatrix}.$$

In order to check the stability of this 10-cycle, we need to compute the eigenvalues of

$$\vec{f}'(\hat{y}_{10})\vec{f}'(\hat{y}_9)\vec{f}'(\hat{y}_8)\vec{f}'(\hat{y}_7)\vec{f}'(\hat{y}_6)\vec{f}'(\hat{y}_5)\vec{f}'(\hat{y}_4)\vec{f}'(\hat{y}_3)\vec{f}'(\hat{y}_2)\vec{f}'(\hat{y}_1).$$

We can compute this numerically, and we find

$$\begin{pmatrix} 1.1829 & .7600 \\ -.4624 & -.3224 \end{pmatrix}.$$

Computing the eigenvalues of this matrix in the usual way, we get $\lambda_1 = -.0335$, $\lambda_2 = .8940$. We can conclude that this 10-cycle is asymptotically stable.

Delay Difference Systems

As with single difference equations, it may be the case with systems that the number of individuals in the next generation of a population is determined by the number of individuals in several previous generations.

In such cases, we can use a delay difference system

$$\vec{y}_{t+1} = \vec{f}(\vec{y}_t, \vec{y}_{t-1}, \dots, \vec{y}_{t-T}), \qquad (*)$$

initialized by $T+1$ vectors, $\vec{y}_0$, $\vec{y}_1$, ..., $\vec{y}_T \in \mathbb{R}^n$.

Similarly as we did with single equations, we can express (*) as a first-order system. We do this by setting

$$\vec{Y}_{1_t} = \vec{y}_t, \vec{Y}_{2_t} = \vec{y}_{t-1}, \dots, \vec{Y}_{T+1_t} = \vec{y}_{t-T}.$$

Delay Difference Systems

From the previous slide,

$$\vec{Y}_{1_t} = \vec{y}_t, \vec{Y}_{2_t} = \vec{y}_{t-1}, \ldots, \vec{Y}_{T+1_t} = \vec{y}_{t-T}.$$

With these choices, we can express (*) as

$$\begin{aligned}
\vec{Y}_{1_{t+1}} &= \vec{y}_{t+1} = \vec{f}(\vec{Y}_{1_t}, \vec{Y}_{2_t}, \ldots, \vec{Y}_{T+1_t}); \quad \vec{Y}_{1_T} = \vec{y}_T \\
\vec{Y}_{2_{t+1}} &= \vec{y}_t = \vec{Y}_{1_1}; \quad \vec{Y}_{2_T} = \vec{y}_{T-1} \\
\vec{Y}_{3_{t+1}} &= \vec{y}_{t-1} = \vec{Y}_{2_t}; \quad \vec{Y}_{3_T} = \vec{y}_{T-2} \\
&\vdots \\
\vec{Y}_{T+1_{t+1}} &= \vec{y}_{t-(T-1)} = \vec{Y}_{T_t}; \quad \vec{Y}_{T+1_T} = \vec{y}_0.
\end{aligned}$$

Each of the vectors $\vec{Y}_{1_t}$, $\vec{Y}_{2_t}$, ..., $\vec{Y}_{T+1_t}$ has length n, so this is a system with $(T+1) \times n$ equations.

Delay Difference Systems

Notice particularly that the system for $\vec{Y}_{1_t}$, $\vec{Y}_{2_t}$, ..., $\vec{Y}_{T+1_t}$ is not a delay system, so it can be analyzed by the techniques we've been discussing in this section.