Single Differential Equations: compartment models and population models

MATH 469, Texas A&M University

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Overview of Compartment Models

Suppose $y(t)$ denotes the amount of a substance in some compartment at time t, and that we can determine the rate (amount per unit time) at which the substance enters the compartment and the rate at which it exits the compartment.

Then we can express the rate of change of $y(t)$ as an ODE

$$\frac{dy}{dt} = \text{incoming rate} - \text{outgoing rate}.$$

In some cases, we can have generation or consumption within the compartment, and this leads to equations of the form

$$\begin{aligned}\frac{dy}{dt} =& \text{incoming rate} - \text{outgoing rate} \\ &+ \text{generation rate} - \text{consumption rate}.\end{aligned}$$

Respiration

Consider the respiration process (a.k.a., breathing). Let

$$y(t) = \text{liters of oxygen } (O_2) \text{ in the lungs at time } t.$$

Let $r_I(t)$ L/min denote the rate at which air enters the lungs at time t, and denote by $r_O(t)$ L/min the rate at which air exits the lungs at time t.

Let C_{O_2} denote the volume fraction of O_2 in the atmosphere, and let $V(t)$ denote the volume of air in the lungs at time t.

Finally, let $A_{O_2}(t)$ denote the rate at which O_2 is consumed by the body. Then:

$$\frac{dy}{dt} = r_I(t)C_{O_2} - r_O(t)\frac{y(t)}{V(t)} - A_{O_2}(t).$$

Respiration

Here,

$$V(t) = V(0) + \int_0^t r_I(s) - r_O(s)ds.$$

Some standard values from our reference by Saltzman for a human are as follows (all constant in time):

$$\begin{aligned} r_I(t), r_O(t) &= 5.3 \text{ L/min} \\ C_{O_2} &= .21 \text{ (ratio of volumes, so no units)} \\ V(0) &= .35 \text{ L} \\ A_{O_2}(t) &= .25 \text{ L/min.} \end{aligned}$$

Our equation for $y(t)$ becomes

$$\begin{aligned} \frac{dy}{dt} &= 5.3 \cdot .21 - \frac{5.3}{.35}y - .25 \\ &= .86 - 15.14y. \end{aligned}$$

Respiration

Let's denote by $\hat{y}$ the liters of O_2 in the lungs at equilibrium. I.e., $\hat{y}$ denotes the value of y so that

$$\frac{dy}{dt} = 0.$$

We see that $\hat{y}$ solves

$$0 = .86 - 15.14\hat{y} \implies \hat{y} = .06 \text{ L}.$$

We often refer to a value such as $\hat{y}$ as an equilibrium point. It's the analogue to a fixed point in difference equations.

Medication

Suppose M grams of a certain heart medication are injected into a patient at time 0, and that whenever the drug is present in the heart its absorption rate out of the bloodstream (and into the heart tissue) is proportional to the concentration in the heart with proportionality constant r_A L/s. (For this example, we'll assume that none of the drug is lost to other body tissue.)

Also suppose blood flows into the patient's heart with variable rate $r_I(t)$ L/s and out with variable rate $r_O(t)$ L/s, and that the volume of blood in the heart at time 0 is $V_H(0)$, and the volume of blood in the patient's body (excluding the heart) at time 0 is $V_B(0)$.

We'll develop a model for the amount of drug absorbed into the heart tissue by time t.

Medication

Let's define the variables

$$\begin{aligned} y(t) =& \text{ grams of drug in the blood in the heart at time } t, \\ & \text{ not yet absorbed into the heart tissue} \\ A(t) =& \text{ grams of drug absorbed into the heart tissue by time } t. \end{aligned}$$

Let's start with A, for which we have

$$\frac{dA}{dt} = r_A \frac{y(t)}{V_H(t)},$$

where

$$V_H(t) = V_H(0) + \int_0^t r_I(s) - r_O(s) ds.$$

Medication

For $y(t)$, we can write

$$
\begin{aligned}
\frac{dy}{dt} &= r_I(t)\frac{M - y(t) - A(t)}{V_B(t)} - r_O(t)\frac{y(t)}{V_H(t)} - r_A\frac{y(t)}{V_H(t)} \\
&= r_I(t)\frac{M - y(t) - A(t)}{V_B(t)} - (r_O(t) + r_A)\frac{y(t)}{V_H(t)},
\end{aligned}
$$

where

$$
V_B(t) = V_B(0) - \int_0^t r_I(s) - r_O(s)ds.
$$

We've derived a system of ODE

$$
\begin{aligned}
\frac{dA}{dt} &= r_A\frac{y(t)}{V_H(t)}; \quad A(0) = 0 \\
\frac{dy}{dt} &= r_I(t)\frac{M - y(t) - A(t)}{V_B(t)} - (r_O(t) + r_A)\frac{y(t)}{V_H(t)}; \quad y(0) = 0.
\end{aligned}
$$

Medication

Alternatively, we can eliminate $y(t)$ and express this as a single second-order equation for $A(t)$. For this, we write

$$\frac{dA}{dt} = r_A \frac{y(t)}{V_H(t)},$$

as

$$y(t) = \frac{V_H(t)}{r_A}\frac{dA}{dt}, \tag{*}$$

and compute

$$\frac{dy}{dt} = \frac{V_H'(t)}{r_A}\frac{dA}{dt} + \frac{V_H(t)}{r_A}\frac{d^2A}{dt^2}.$$

If we substitute this relation into the equation for y, we obtain

$$\frac{V_H'(t)}{r_A}\frac{dA}{dt}+\frac{V_H(t)}{r_A}\frac{d^2A}{dt^2} = r_I(t)\frac{M - y(t) - A(t)}{V_B(t)} - (r_O(t)+r_A)\frac{y(t)}{V_H(t)}.$$

Last, we'll use (*) to eliminate all appearances of $y(t)$.

Medication

This gets us to the system

$$\frac{V_H'(t)}{r_A}A'(t)+\frac{V_H(t)}{r_A}A''(t) = r_I(t)\frac{M-\frac{V_H(t)}{r_A}A'(t)-A(t)}{V_B(t)}$$
$$-(r_O(t)+r_A)\frac{\frac{V_H(t)}{r_A}A'(t)}{V_H(t)}.$$

The initial conditions are $A(0)=0$ and $A'(0)=r_A\frac{y(0)}{V_H(0)}=0$. Some standard values are as follows (constant in time):

$$\begin{aligned} r_I(t), r_O(t) &= .07 \text{ L/s} \\ V_H(t) &= .28 \text{ L} \\ V_B(t) &= 4.72 \text{ L}. \end{aligned}$$

The values of M and r_A depend on the medication.

Population Models

In this case, we'll set

$$y(t) = \text{ \# of individuals in a population at time } t.$$

1. The Malthusian model is

$$\frac{dy}{dt} = ry; \quad y(0) = y_0,$$

with solution

$$y(t) = y_0 e^{rt}.$$

2. The logistic model is

$$\frac{dy}{dt} = ry(1 - \frac{y}{K}); \quad y(0) = y_0,$$

with solution

$$y(t) = \frac{y_0 K}{y_0 + (K - y_0)e^{-rt}}.$$

Population Models

3. The Gompertz model is

$$\frac{dy}{dt} = -ry\ln(y/K); \quad y(0) = y_0,$$

with solution

$$y(t) = K(\frac{y_0}{K})^{e^{-rt}}.$$

4. The General Single-Species Model (GSSM) is

$$\frac{dy}{dt} = \frac{r}{a}y(1 - (\frac{y}{K})^a),$$

with solution

$$y(t) = \frac{Ky_0}{(y_0^a + (K^a - y_0^a)e^{-rt})^{1/a}}.$$

Relations Among the Models

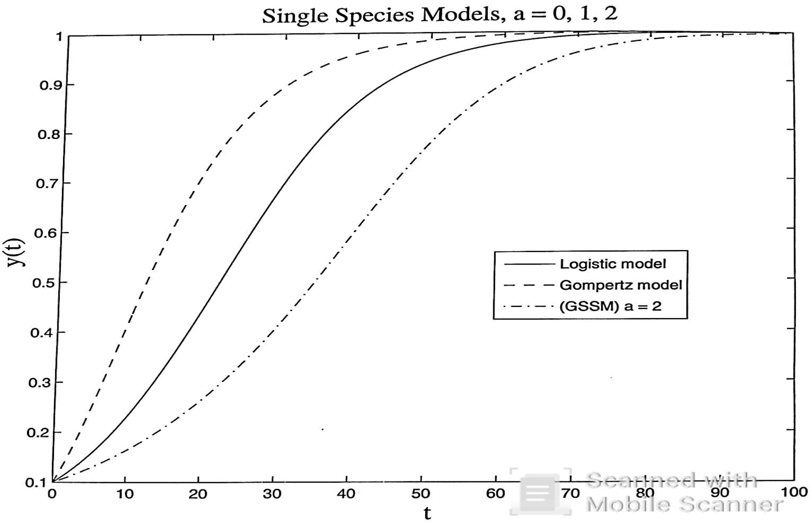
1. The Malthusian model can be viewed as a special case of the logistic model with $K = \infty$.

2. The logistic model is a special case of the (GSSM) with $a = 1$.

3. The Gompertz model can be viewed as a special case of the (GSSM) obtained in the limit $a \to 0$. I.e.,

$$\lim_{a \to 0} \frac{r}{a} y(1 - (\frac{y}{K})^a) = ry \lim_{a \to 0} \frac{(1 - (\frac{y}{K})^a)}{a}$$
$$\overset{\text{L'Hopital}}{=} ry \lim_{a \to 0} \frac{-(\frac{y}{K})^a \ln(\frac{y}{K})}{1} = -ry \ln(y/K).$$

We see that these models are all special cases of the (GSSM). Physically, a is a gauge of how rapidly the population approaches its carrying capacity: the smaller a is, the faster the population approaches its carrying capacity.


Single Species Models, a = 0, 1, 2
y(t)
t
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
0
10
20
30
40
50
60
70
80
90
100
Logistic model
Gompertz model
(GSSM) a = 2