Single Differential Equations: Analysis

MATH 469, Texas A&M University

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Non-dimensionalization

As with difference equations, we can non-dimensionalize differential equations to obtain more convenient forms of the equations we're working with.

Suppose we want to non-dimensionalize the logistic equation

$$\frac{dy}{dt} = ry(1 - \frac{y}{K}).$$

In this case, we'll replace both t and y with dimensionless variables. For this, we set

$$\tau = \frac{t}{A}, \quad Y(\tau) = \frac{y(t)}{B},$$

where A must be chosen as a constant with dimension time and B must be chosen as a constant with dimension biomass.

Non-dimensionalization

Writing $y(t) = BY(\tau)$, we can use the chain rule to compute

$$\frac{dy}{dt} = B\frac{d}{dt}Y(\tau) = B\frac{dY}{d\tau}\frac{d\tau}{dt} = \frac{B}{A}Y'(\tau).$$

This allows us to express the logistic equation as

$$\frac{B}{A}Y' = rBY(1 - \frac{BY}{K}) \implies Y' = rAY(1 - \frac{BY}{K}).$$

We can choose $B = K$ and $A = 1/r$ to obtain the non-dimensionalized equation

$$Y' = Y(1 - Y).$$

Notice that by introducing two dimensionless constants we were able to eliminate two parameters. We see that in contrast to the non-dimensionalized discrete logistic model, there can't be any interesting bifurcation analysis for this model. I.e., solutions behave qualitatively the same for all values of the parameters r and K.

Equilibrium Points

For a single autonomous differential equation

$$\frac{dy}{dt} = f(y), \qquad (*)$$

we say that a value $\hat{y}$ is an equilibrium point if

$$f(\hat{y}) = 0.$$

Notice particularly that $y(t) \equiv \hat{y}$ solves (*) for all t. I.e., we have

$$\frac{d\hat{y}}{dt} = 0 \quad \text{and} \quad f(\hat{y}) = 0,$$

so (*) is always satisfied.

Equilibrium points for differential equations are the analogues of fixed points for difference equations.

Stability

Definition. Suppose $\hat{y}$ is an equilibrium point for the autonomous differential equation

$$\frac{dy}{dt} = f(y), \quad y(0) = y_0.$$

(i) We say that $\hat{y}$ is stable if given any $\epsilon > 0$ there exists $\delta > 0$ so that

$$|y_0 - \hat{y}| < \delta \implies |y(t) - \hat{y}| < \epsilon$$

for all $t \geq 0$.

(ii) We say that $\hat{y}$ is asymptotically stable if $\hat{y}$ is stable, and there exists some $\delta_0 > 0$ so that

$$|y_0 - \hat{y}| < \delta_0 \implies \lim_{t \to +\infty} y(t) = \hat{y}.$$

(iii) If $\hat{y}$ is not stable, then we say that $\hat{y}$ is unstable.

The Phase Line and Stability

The *phase variables* for an equation are those that determine all future behavior. For example, for a pendulum the phase variables would be position and velocity. For a single first-order autonomous differential equation, the phase variable is simply y. I.e., if you know the value of y at some time then you can determine the value of y at all other times.

Example 1. Find the equilibrium point for our respiration model

$$\frac{dy}{dt} = .86 - 15.14y,$$

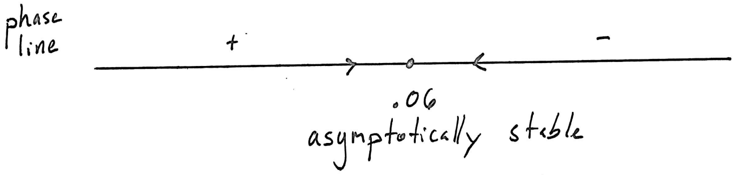
and classify it as asymptotically stable, stable (and not asymptotically stable), or unstable.

As noted in a previous lecture, the equilibrium point $\hat{y}$ solves

$$0 = .86 - 15.14\hat{y} \implies \hat{y} = .06 \text{ L}.$$

The Phase Line and Stability

Notice that if $y < \hat{y}$ then $\frac{dy}{dt} > 0$, while if $y > \hat{y}$ then $\frac{dy}{dt} < 0$. We can record this information on a "phase line."



We conclude that $\hat{y} = .06$ is asymptotically stable. In fact, we can say something stronger. Given any value $y_0 \in \mathbb{R}$, the solution $y(t)$ to

$$\frac{dy}{dt} = .86 - 15.14y, \quad y(0) = y_0$$

will satisfy

$$\lim_{t \to \infty} y(t) = \hat{y}.$$

The Phase Line and Stability

Example 2. Find all equilibrium points for the non-dimensionalized logistic model

$$\frac{dy}{dt} = y(1-y),$$

and classify the stability of each.

First, the equilibrium points satisfy the equation

$$0 = \hat{y}(1-\hat{y}) \implies \hat{y} = 0, 1.$$

In this case, we have the following:

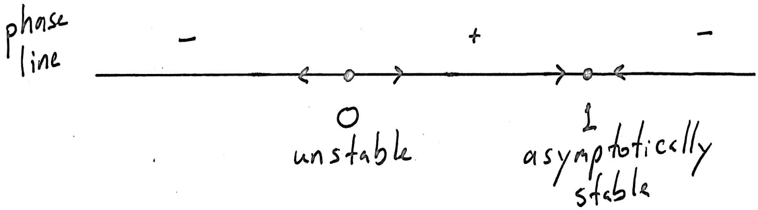
$$y < 0 \implies \frac{dy}{dt} < 0$$

$$0 < y < 1 \implies \frac{dy}{dt} > 0$$

$$y > 1 \implies \frac{dy}{dt} < 0.$$

The Phase Line and Stability

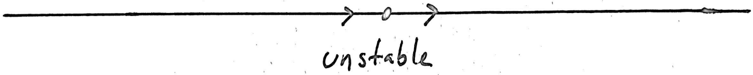
We can summarize this information on a phase line.

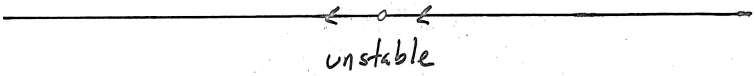


We conclude that $\hat{y} = 0$ is unstable and $\hat{y} = 1$ is asymptotically stable.

The Phase Line and Stability

In general, two other cases are possible, both unstable:





Linearization

As with fixed points, we can also classify the stability of equilibrium points by a criterion on $f'(\hat{y})$. In order to identify this criterion, let's suppose $\hat{y}$ is an equilibrium point for the autonomous differential equation

$$\frac{dy}{dt} = f(y), \quad y(0) = y_0.$$

If we substitute

$$y(t) = \hat{y} + z(t)$$

into this equation, we find

$$z_t = f(\hat{y} + z) = f(\hat{y}) + f'(\hat{y})z + \epsilon(z; \hat{y}).$$

Here, $f(\hat{y}) = 0$ and $\epsilon(z; \hat{y})$ is small, so we approximately have

$$z_t = f'(\hat{y})z, \quad z(0) = z_0 = y_0 - \hat{y}.$$

Linearization

From the previous slide,

$$z_t = f'(\hat{y})z, \quad z(0) = z_0 = y_0 - \hat{y}.$$

This equation is precisely the Malthusian model with $r = f'(\hat{y})$, and we know its solution is

$$z(t) = z_0 e^{f'(\hat{y})t}.$$

We see that if $f'(\hat{y}) < 0$, then

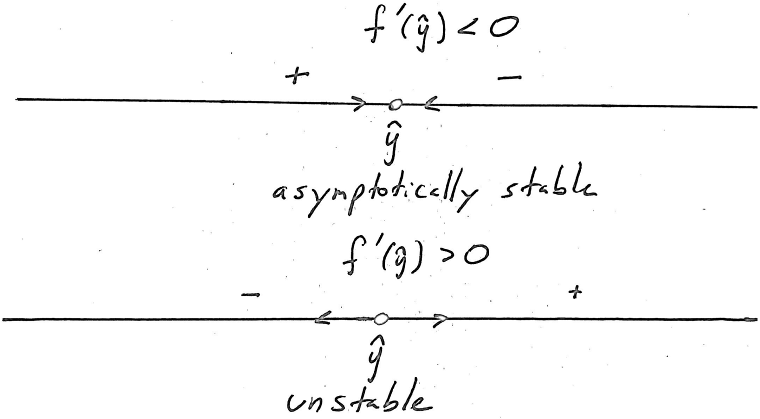
$$\lim_{t\to+\infty} z(t) = 0 \implies \lim_{t\to+\infty} y(t) = \hat{y}.$$

In this case, we have asymptotic stability.

On the other hand, if $f'(\hat{y}) > 0$, then $z(t)$ will grow as t increases, and we can conclude instability. If $f'(\hat{y}) = 0$, stability will be determined by higher order terms, so the criterion is inconclusive.

Linearization

In fact, this criterion is clear from the phase line.



Linearization

Example. Let's use the derivative criterion to classify stability of the equilibrium points $\hat{y} = 0, 1$ for the non-dimensionalized logistic model,

$$\frac{dy}{dt} = y(1 - y).$$

We see that

$$f(y) = y(1 - y) = y - y^2 \implies f'(y) = 1 - 2y.$$

We now check: For $\hat{y} = 0$, we have $f'(0) = 1 > 0$, so $\hat{y} = 0$ is unstable. For $\hat{y} = 1$, $f'(1) = -1$, so $\hat{y} = 1$ is asymptotically stable.

Periodic Solutions

We've already observed that there can't be any interesting bifurcation analysis for the logistic model, and that this is in contrast to the discrete logistic model. A key difference between the two cases is that while the discrete logistic model has 2-cycle solutions, 4-cycle solutions, etc., there are no periodic solutions to the logistic model.

In fact, the following is true: Single autonomous first-order ODE cannot have non-constant periodic solutions. To see this, suppose the autonomous ODE

$$\frac{dy}{dt} = f(y)$$

has a periodic solution $y(t)$ so that

$$y(t) = y(t + P),$$

where P is the solution's period.

Periodic Solutions

I.e., if $y(t_0) = y_0$, then $y(t_0 + P) = y(t_0) = y_0$, $y(t_0 + 2P) = y(t_0 + P) = y_0$ etc.

We'll show that in this case, $y(t)$ must be constant for all t. To this end, we fix any $T \in \mathbb{R}$, multiply our equation $y' = f(y)$ by y', and integrate the resulting equation from T to $T + P$. That is, we write

$$y'(t)^2 = f(y(t))y'(t) \implies \int_T^{T+P} y'(t)^2 dt = \int_T^{T+P} f(y(t))y'(t)dt.$$

If we set

$$F(y) = \int_0^y f(x)dx,$$

then $F'(y) = f(y)$. This allows us to write

$$\int_T^{T+P} f(y(t))y'(t)dt = \int_T^{T+P} \frac{d}{dt}F(y(t))dt.$$

Periodic Solutions

We see that

$$\int_T^{T+P} y'(t)^2 dt = \int_T^{T+P} \frac{d}{dt} F(y(t)) dt$$
$$= F(y(t))\Big|_{t=T}^{t=T+P} = F(y(T+P)) - F(y(T)) = 0.$$

But if

$$\int_T^{T+P} y'(t)^2 dt = 0$$

for all $T \in \mathbb{R}$, then we must have that $y'(t) \equiv 0$ for all $t \in \mathbb{R}$. I.e., $y(t)$ is constant for all t.

Recovery Times

Consider an autonomous ODE

$$\frac{dy}{dt} = f(y), \quad y(0) = y_0,$$

with equilibrium point $\hat{y}$. Suppose $\hat{y}$ is asymptotically stable, but y has been moved away from $\hat{y}$ to y_0. (E.g., this might model a population after harvesting or a resource that's been depleted.)

The recovery time T is the amount of time required for $y(t)$ to reduce its distance to $\hat{y}$ by a factor of $\frac{1}{e}$. Precisely, T is defined so that

$$(y(T) - \hat{y}) = \frac{1}{e}(y_0 - \hat{y}) \implies y(T) = \hat{y} + \frac{1}{e}(y_0 - \hat{y}).$$

(Recall that to four decimal places, $e = 2.7183$.) In order to find a value for T, we use the method of separation of variables.

Recovery Times

That is, we (formally) write our ODE as

$$\frac{dy}{f(y)} = dt,$$

and then we integrate both sides

$$\int_{y(0)}^{y(T)} \frac{dy}{f(y)} = \int_0^T dt = T.$$

We see that

$$T = \int_{y_0}^{\hat{y}+\frac{1}{e}(y_0-\hat{y})} \frac{dy}{f(y)}.$$

Recovery Times

Example. For the respiration model

$$\frac{dy}{dt} = .86 - 15.14y, \quad y(0) = y_0,$$

suppose O_2 drops from the equilibrium value $\hat{y} = .06$ L to $y_0 = .01$ L, and compute the associated recovery time.

In this case,

$$y(T) = \hat{y} + \frac{1}{e}(y_0 - \hat{y}) = .06 + \frac{1}{e}(.01 - .06) = .04$$

(rounding to two decimal places). We need to compute

$$T = \int_{.01}^{.04} \frac{dy}{.86 - 15.14y} = -\frac{1}{15.14} \ln|.86 - 15.14y| \Big|_{.01}^{.04}$$
$$= -\frac{1}{15.14} \ln\left|\frac{.86 - 15.14 * .04}{.86 - 15.14 * .01}\right| = .07 \text{ min.}$$

I.e., about 4.2 seconds.