ERROR ESTIMATES OF A FIRST-ORDER LAGRANGE FINITE ELEMENT TECHNIQUE FOR NONLINEAR SCALAR CONSERVATION EQUATIONS*

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Abstract. This paper establishes a $\mathcal{O}(h^{\frac{1}{4}})$ error estimate in the $L^\infty_t(L^1_x)$-norm for the approximation of scalar conservation equations using an explicit continuous finite element technique. A general a priori error estimate based on entropy inequalities is also given in the Appendix.

Key words. Conservation equations, parabolic regularization, entropy, entropy solutions, finite element method, convergence analysis.

AMS subject classifications. 65M60, 35L65

1. Introduction. The objective of this paper is to derive a priori error estimates for the approximation of nonlinear scalar conservation equations by using an explicit first-order Lagrange finite element technique introduced in Guermond and Nazarov [12]. In particular we prove that the error in the $L^\infty_t(L^1_x)$-norm is at most $\mathcal{O}(h^{\frac{1}{4}})$ under the appropriate CFL condition in any space dimension and for any shape-regular mesh family; the mesh may be composed of an arbitrary combination of simplices, prisms, cuboids, etc. The estimate is established by using the technique of the doubling of the variables introduced by Kružkov [17] and first used by Kuznecov [18] to prove error estimates. We follow the approach of Cockburn et al. [7], Cockburn and Gremaud [6] and Bouchut and Perthame [2] and propose some modifications thereof that makes the methodology slightly easier to apply (see Lemma A.2). To the best our knowledge, this is the first time that a priori error estimates have been established for an explicit method using continuous Lagrange finite elements to approximate nonlinear scalar conservation equations. Similar results have been established by Cockburn and Gremaud [5], but the error estimate therein is $\mathcal{O}(h^{\frac{1}{8}})$ and the algorithm is a shock capturing streamline diffusion method using implicit time stepping and an artificial viscosity scaling like $h^{\frac{3}{4}}$ in the shocks. The finite volume literature is slightly richer in this respect; for instance, $\mathcal{O}(h^{\frac{1}{4}})$ error estimates in the $L^1_t(L^1_x)$-norm have been established for various families of finite volume schemes, see e.g., Eymard et al. [9], Chainais-Hillairet [3].

The paper is organized as follows. The statement of the problem, notation and notions related to finite element meshes are introduced in §2. The description of the approximation method is done in §3. The maximum principle and an L^2-stability estimate are also proved therein. The error analysis is done in §4. We first establish entropy inequalities for the Kružkov entropy family and then deduce a error estimate by using an a priori bound established in the Appendix (see Lemma A.2). The main results of the paper are Theorem 4.5 and, to some extent, some originality is claimed for Lemma A.2.

2. Preliminaries. The objectives of this section is to state the problem, introduce the finite element setting, and establish some preliminary results.

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2.1. Formulation of the problem. Let us consider a scalar conservation equation in a polyhedral domain Ω in $\mathbb{R}^d$,

$$\partial_t u + \nabla\cdot\boldsymbol{f}(u) = 0, \qquad u(\boldsymbol{x},0) = u_0(\boldsymbol{x}), \quad (\boldsymbol{x},t) \in \Omega\times\mathbb{R}_+. \tag{2.1}$$

The initial data u_0 is assumed to be bounded and the flux $\boldsymbol{f}$ is assumed to be Lipschitz,

$$u_0 \in L^\infty(\Omega), \qquad \boldsymbol{f} \in \mathrm{Lip}(\mathbb{R};\mathbb{R}^d). \tag{2.2}$$

We assume that the boundary conditions are either periodic or the initial data is compactly supported. In the second case we are interested in the solution in a time interval $[0,T]$ such that the domain of influence of u_0 over $[0,T]$ does not reach the boundary of Ω. The purpose of these assumptions is to avoid unnecessary technical difficulties induces by boundary conditions. Following the work of Kružkov [17], it is now well understood that this problem has a unique entropy solution; i.e., a weak solution that additionally satisfies the entropy inequalities $\partial_t E(u) + \nabla\cdot\boldsymbol{F}(u) \leq 0$ for all convex entropies $E \in \mathrm{Lip}(\mathbb{R};\mathbb{R})$ and associated entropy fluxes $\boldsymbol{F}_i(u) = \int_0^u E'(v)\boldsymbol{f}_i'(v)\,\mathrm{d}v$, $1 \leq i \leq d$.

2.2. Mesh. The approximation in space of (2.1) will be done by using continuous finite elements. Recall that it is always possible to construct affine finite element meshes over Ω since Ω is assumed to be a polyhedron. We denote by $\{\mathcal{K}_h\}_{h>0}$ an affine shape regular mesh family. The shape regularity is understood in the sense of Ciarlet. The elements in the mesh family $\{\mathcal{K}_h\}_{h>0}$ are assumed to be generated from a finite number of reference elements. The reference elements are denoted $\widehat{K}_1,\ldots,\widehat{K}_\varpi$. For example, the mesh $\mathcal{K}_h$ could be composed of a combination of triangles and parallelograms in two space dimensions ($\varpi = 2$ in this case); it could also be composed of a combination of tetrahedra, parallelepipeds, and triangular prisms in three space dimensions ($\varpi = 3$ in this case). Let K_h be a mesh in the family $\{\mathcal{K}_h\}_{h>0}$. Let K be a cell in the mesh $\mathcal{K}_h$ and let $\widehat{K}_r$, $1 \leq r \leq \varpi$, be the corresponding reference geometric element. The affine diffeomorphism mapping $\widehat{K}_r$ to an arbitrary element $K \in \mathcal{K}_h$ is denoted $\Phi_K : \widehat{K}_r \longrightarrow K$ and its Jacobian matrix is denoted $\mathbb{J}_K$. The assumption that the mapping Φ_K is affine could be removed by proceeding as in Ciarlet and Raviart [4] but this would introduce additional unnecessary technicalities.

We want to approximate the entropy solution of (2.1) with continuous Lagrange finite elements. For this purpose we introduce the set of reference Lagrange finite elements $\{(\widehat{K}_r,\widehat{P}_r,\widehat{\Sigma}_r)\}_{1\leq r\leq\varpi}$. The index $r \in \{1,\ldots,\varpi\}$ will be omitted in the rest of the paper to alleviate the notation. Then we define the scalar-valued Lagrange finite element space

$$X_h = \{v \in \mathcal{C}^0(\Omega;\mathbb{R});\, v|_K\circ\Phi_K \in \widehat{P},\ \forall K \in \mathcal{K}_h\}, \tag{2.3}$$

where $\widehat{P}$ is the reference polynomial space defined on $\widehat{K}$ (note the index r has been omitted). Denoting by $\{\widehat{a}_1,\ldots \widehat{a}_{\widehat{s}}\}$ the Lagrange nodes of $\widehat{K}$, we assume that the space $\widehat{P}$ is such that

$$\min_{\ell\in\mathcal{I}(\widehat{K})} \widehat{v}(\widehat{a}_\ell) \leq \widehat{v}(\widehat{\boldsymbol{x}}) \leq \max_{\ell\in\mathcal{I}(\widehat{K})} \widehat{v}(\widehat{a}_\ell), \quad \forall\widehat{v} \in \widehat{P}, \forall\widehat{\boldsymbol{x}} \in \widehat{K}. \tag{2.4}$$

Let $\mathbb{P}_1$ and $\mathbb{Q}_1$ be the set of multivariate polynomials of total and partial degree at most 1, respectively; then the above assumption holds for $\widehat{P} = \mathbb{P}_1$ when K is a simplex and $\widehat{P} = \mathbb{Q}_1$ when K is a parallelogram or a cuboid. This assumption holds also for first-order prismatic elements in three space dimensions.

Let $\{\boldsymbol{a}_1, \ldots, \boldsymbol{a}_I\}$ be the collection of all the Lagrange nodes in the mesh $\mathcal{K}_h$, and let $\{\varphi_1, \ldots, \varphi_I\}$ be the corresponding global shape functions. Recall that $\{\varphi_1, \ldots, \varphi_I\}$ forms a basis of X_h and $\varphi_i(\boldsymbol{a}_j) = \delta_{ij}$. In the rest of the paper we denote by $\pi_h : \mathcal{C}^0(\Omega) \longrightarrow X_h$ the Lagrange interpolation operator, $\pi_h(v)(\boldsymbol{x}) := \sum_{i=1}^{I} v(\boldsymbol{a}_i)\varphi_i(\boldsymbol{x})$. We define the operator $\mathsf{C} : X_h \longrightarrow \mathbb{R}^I$ so that $\mathsf{C}(v_h)$ is the coordinate vector of v_h in the basis $\{\varphi_1, \ldots, \varphi_I\}$, i.e., $v_h = \sum_{i=1}^{I} \mathsf{C}(v_h)_i \varphi_i$. Note that $\mathsf{C}(v_h)_i = v_h(\boldsymbol{a}_i)$. We are also going to use capital letters for the coordinate vectors to alleviate the notation; for instance we shall write $V = \mathsf{C}(v_h)$ when the context is unambiguous. Note finally that the above assumptions on the mesh and the reference elements imply the following convexity property:

$$\min_{\ell \in \mathcal{I}(K)} \mathsf{C}(v_h)_\ell \leq v(\boldsymbol{x}) \leq \max_{\ell \in \mathcal{I}(K)} \mathsf{C}(v_h)_\ell, \quad \forall v_h \in X_h, \forall \boldsymbol{x} \in K, \forall K \in \mathcal{K}_h. \tag{2.5}$$

Let φ_i be a shape function; the support of φ_i is denoted S_i and the measure of S_i is denoted $|S_i|$, $i = 1, \ldots, I$. We also define $S_{ij} := S_i \cap S_j$ the intersection of the two supports S_i and S_j. For any union of cells in $\mathcal{K}_h$, say E; we define $\mathcal{I}(E)$ to be the collection of the indices of the shape functions whose support on E is of nonzero measure, i.e., $\mathcal{I}(E) := \{j \in \{1, \ldots, I\};\ |S_j \cap E| \neq 0\}$. We are going to regularly invoke $\mathcal{I}(K)$ and $\mathcal{I}(S_i)$ and the partition of unity property: $\sum_{i \in \mathcal{I}(K)} \varphi_i(\boldsymbol{x}) = 1$ for all $\boldsymbol{x} \in K$. Let $M \in \mathbb{R}^{I \times I}$ be the so-called consistent mass matrix with entries $\int_{S_{ij}} \varphi_i(\boldsymbol{x})\varphi_j(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}$. We then define the diagonal lumped mass matrix M^L with diagonal entries

$$m_i := \int_{S_i} \varphi_i(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}. \tag{2.6}$$

The partition of unity property implies that $m_i = \sum_{j \in \mathcal{I}(S_i)} \int \varphi_j(\boldsymbol{x})\varphi_i(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}$, i.e., the entries of M^L are obtained by summing the rows if M. The diagonal matrix M^L is known to be a consistent second-order approximation of M. The two quantities $\|v_h\|_{L^2(\Omega)} = \mathsf{C}(v_h)^\mathsf{T} M \mathsf{C}(v_h)$ and $\|v_h\|_{\ell^2_h}^2 := \mathsf{C}(v_h)^\mathsf{T} M^L \mathsf{C}(v_h)$ are equivalent. This property actually holds for any L^p-norm. More precisely consider the discrete norm $\ell^p_h : X_h \longrightarrow \mathbb{R}^+$, $1 \leq 1 \leq p < +\infty$, defined by

$$\|v_h\|_{\ell^p_h}^p := \sum_{i=1}^{I} m_i |\mathsf{C}(v_h)_i|^p, \qquad \forall v_h \in X_h. \tag{2.7}$$

Lemma 2.1. *The are $m_{\max}, m_{\min} > 0$, depending only on $\{(\widehat{K}_r, \widehat{P}_r, \widehat{\Sigma}_r)\}_{1 \leq r \leq \varpi}$ and $p \in [1, +\infty)$, such that the following holds for all $v_h \in X_h$ and all $\mathcal{K}_h$,*

$$m_{\min} \|v_h\|_{L^p(\Omega)} \leq \|v_h\|_{\ell^p_h} \leq m_{\max} \|v_h\|_{L^p(\Omega)}. \tag{2.8}$$

2.3. Local mesh size. Upon defining $h_K := \mathrm{diam}(K)$ and denoting by ρ_K the diameter of the largest ball that can be inscribed in K, it can be shown that

$$\det \mathbb{J}_K = \frac{|K|}{|\widehat{K}|}, \qquad \frac{\rho_K}{h_{\widehat{K}}} \leq \|\mathbb{J}_K\|_{\ell^2} \leq \frac{h_K}{\rho_{\widehat{K}}}, \qquad \frac{\rho_{\widehat{K}}}{h_K} \leq \|\mathbb{J}_K^{-1}\|_{\ell^2} \leq \frac{h_{\widehat{K}}}{\rho_K}, \tag{2.9}$$

where $\|\mathbb{J}_K\|_{\ell^2}$ is the norm of $\mathbb{J}_K$ subordinated to the Euclidean norm (see e.g., Girault and Raviart [10, (A.2) p. 96]). The shape-regularity assumption of the mesh family $\{\mathcal{K}_h\}_{h>0}$ means that the ratio h_K/ρ_K is bounded uniformly with respect to K and $\mathcal{K}_h$. For further reference we define $\sigma := \sup_{\{\mathcal{K}_h\}} \sup_{K\in\mathcal{K}_h} h_K/\rho_K$. The global maximum mesh size is denoted $h = \max_{K\in\mathcal{K}_h} h_K$. The local minimum mesh size, $\underline{h}_K$, for any $K \in \mathcal{K}_h$ is defined as follows:

$$\underline{h}_K := \frac{1}{\max_{i\neq j\in\mathcal{I}(K)} \|\nabla\varphi_i\|_{L^\infty(S_{ij})}}, \tag{2.10}$$

and the global minimum mesh size is $\underline{h} := \min_{K\in\mathcal{K}_h} \underline{h}_K$. Due to the shape regularity assumption the quantities $\underline{h}_K$ and h_K are uniformly equivalent; it will turn out though that using $\underline{h}_K$ gives a sharper estimate of the CFL number.

2.4. Viscous bilinear form. Let n_K be the number of vertices in K, i.e., $n_K := \text{card}(\mathcal{I}(K))$, and let $\vartheta_K := (n_K - 1)^{-1}$. Note that

$$0 < \vartheta_{\min}(\varpi) := \min_{\{\mathcal{K}_h\}} \min_{K\in\mathcal{K}_h} \vartheta_K, \qquad \vartheta_{\max}(\varpi) := \max_{\{\mathcal{K}_h\}} \max_{K\in\mathcal{K}_h} \vartheta_K < +\infty. \tag{2.11}$$

since there are at most ϖ reference elements defining the mesh family. The artificial viscosity that we are going to introduce to stabilize the Galerkin formulation will be defined locally on each cell, K, by using the following bilinear form:

$$b_K(\varphi_j,\varphi_i) = \begin{cases} -\vartheta_K|K| & \text{if } i\neq j, \quad i,j\in\mathcal{I}(K),\\ |K| & \text{if } i=j, \quad i,j\in\mathcal{I}(K),\\ 0 & \text{if } i\notin\mathcal{I}(K) \text{ or } j\notin\mathcal{I}(K). \end{cases} \tag{2.12}$$

For instance it can be shown that $b_K(\varphi_j,\varphi_i) = \kappa\int_K \mathbb{J}_K^{\mathsf{T}}(\nabla\varphi_j)\cdot\mathbb{J}_K^{\mathsf{T}}(\nabla\varphi_i)\,\mathrm{d}\boldsymbol{x}$ when K is a simplex and $\widehat{K}$ is the regular simplex with all the edges of unit length, i.e., K is the equilateral triangle of side 1 in two space dimension, and K is the regular tetrahedron (all four faces are equilateral triangles) in three space dimensions, see Guermond and Nazarov [12]. In this case $\kappa = \frac{4}{3}$ in two space dimensions and $\kappa = \frac{3}{2}$ in three space dimensions. Note also that $b_K(\varphi_j,\varphi_i) \sim \int_K(\nabla\varphi_j)\cdot(\nabla\varphi_i)\,\mathrm{d}\boldsymbol{x}$ if K is a regular simplex, thereby showing the connection between b_K and the more familiar bilinear form associated with the Laplacian. The properties of b_K we need in this paper on arbitrary meshes can be summarized as follows:

LEMMA 2.2. *There exist constants $b_{\min} > 0$ depending only on the collection $\{(\widehat{K}_r, \widehat{P}_r, \widehat{\Sigma}_r)\}_{1\le r\le\varpi}$ and the shape-regularity constant σ, such that the following identities hold for all $K\in\mathcal{K}_h$ and all $u_h, v_h \in X_h$:*

$$b_K(u_h,v_h) = \vartheta_K|K| \sum_{i\in\mathcal{I}(K)} \sum_{\mathcal{I}(K)\ni j<i} (U_i - U_j)(V_i - V_j) \tag{2.13}$$

$$b_K(u_h,u_h) \ge b_{\min} h_K^2 \|\nabla u_h\|^2_{\boldsymbol{L}^2(K)}. \tag{2.14}$$

Proof. Let us prove (2.13) first. Let $u_h, v_h \in X_h$ and let us set $U := \mathsf{C}(u_h)$ and $V := \mathsf{C}(v_h)$. Let K be a cell in $\mathcal{K}_h$. Up to the abuse of notation that consists of using

u_h instead of $u_h|_K$ to denote the restriction of u_h to K, we have

$$\begin{aligned}|K|^{-1}b_K(u_h,v_h) &= \sum_{i\in\mathcal{I}(K)}\Big(U_iV_i - \sum_{i\neq j\in\mathcal{I}(K)}\vartheta_K U_iV_j\Big) = -\vartheta_K \sum_{i\in\mathcal{I}(K)}\sum_{i\neq j\in\mathcal{I}(K)} U_i(V_j-V_i)\Big)\\ &= -\vartheta_K \sum_{i\in\mathcal{I}(K)}\sum_{\mathcal{I}(K)\ni j<i} U_i(V_j-V_i)+U_j(V_i-V_j)\Big)\\ &= \vartheta_K \sum_{i\in\mathcal{I}(K)}\sum_{\mathcal{I}(K)\ni j<i}(U_i-U_j)(V_i-V_j).\end{aligned}$$

Up to the change of variable $\widehat{u}_h := u_h\circ\Phi_K$, this identity proves that $|K|^{-1}\vartheta_K^{-1}b_K(\cdot,\cdot)^{\frac12}$ is a norm on $\widehat{P}/\mathbb{R}$. Since all the norms are equivalent on $\widehat{P}/\mathbb{R}$ and the collection of reference finite elements is finite, there exist constants c_2, c_2 that depends only of the collection $\{(\widehat{K}_r,\widehat{P}_r,\widehat{\Sigma}_r)\}_{1\le r\le\varpi}$ such that

$$c_1\,\|\nabla\widehat{u}_h\|^2_{L^2(\widehat{K})} \le |K|^{-1}\vartheta_K^{-1}b_K(u_h,u_h) \le c_2\,\|\nabla\widehat{u}_h\|^2_{L^2(\widehat{K})}.$$

After using the change of variable $u_h = \widehat{u}_h\circ\Phi_K^{-1}$, we infer that

$$c_1\,|\mathrm{det}(\mathbb{J}_K^{-1})|\|\mathbb{J}_K^{-1}\|^{-2}\|\nabla u_h\|^2_{L^2(K)} \le \frac{b_K(u_h,u_h)}{|K|\vartheta_K} \le c_2\,|\mathrm{det}(\mathbb{J}_K^{-1})|\|\mathbb{J}_K^{-1}\|^{-2}\|\nabla u_h\|^2_{L^2(K)}.$$

The estimate (2.14) is obtained by using (2.9). ☐

3. Space and time approximation. We introduce the time and space approximation of (2.1) in this section.

3.1. Initial data and CFL number. Let us assume that we have at hand an initial discrete field $u_{0h}\in X_h$ that reasonably approximates u_0 and satisfies the discrete maximum principle, i.e.,

$$u_{\min} := \operatorname*{ess\,inf}_{\boldsymbol{x}\in\Omega} u_0(\boldsymbol{x}) \le \min_{1\le i\le I} u_{0h}(\boldsymbol{a}_i) \le \max_{1\le i\le I} u_{0h}(\boldsymbol{a}_i) \le \operatorname*{ess\,sup}_{\boldsymbol{x}\in\Omega} u_0(\boldsymbol{x}) := u_{\max}. \tag{3.1}$$

There are many ways to construct $u_{0h}\in X_h$ with the above properties, but we are not going to discuss this question for the time being. A more precise statement is made in §4.5.

Since we are going to adopt an explicit time stepping, it is necessary to introduce a notion of CFL number; i.e., we need to estimate the local meshsize and the maximum local wave speed on each mesh cell in $\mathcal{K}_h$. We define the maximum wave speed

$$\beta := \|\boldsymbol{f}\|_{\mathrm{Lip}[u_{\min},u_{\max}]} := \sup_{u_{\min}\le v\neq w\le u_{\max}}\ \sup_{0\neq\boldsymbol{n}\in\mathbb{R}^d} \frac{|(\boldsymbol{f}(v)-\boldsymbol{f}(w))\cdot\boldsymbol{n}|}{\|\boldsymbol{n}\|_{\ell^2}|v-w|}. \tag{3.2}$$

The above definition makes sense as long as $u_{\min} < u_{\max}$. We could extend the definition of wave speed as in (3.6) in the case $u_{\min} = u_{\max}$, but this exercise is useless since in this case the exact and the numerical solutions coincide and, the error being zero, the error estimates are trivial. Let $\Delta t > 0$ be the time step that we assume to be uniform for simplicity. The CFL number, λ, is defined to be

$$\lambda := \max_{K\in\mathcal{K}_h}\frac{\beta\Delta t}{\underline{h}_K}. \tag{3.3}$$

We additionally define $\mu_K := \max_{i\in\mathcal{I}(K)}\frac{1}{|K|}\int_K \varphi_i(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}$ and $\mu_{\max} = \max_{K\in\mathcal{K}_h}\mu_K$, $\mu_{\min} = \min_{K\in\mathcal{K}_h}\mu_K$. Note that $\mu_K = n_K^{-1} = (d+1)^{-1}$ for simplices and $\mu_K = 2^{-d}$ for parallelograms and cuboids.

3.2. Numerical flux. Let $v_h \in X_h$ and set $V := \mathsf{C}(v_h)$. We approximate $\boldsymbol{f}(v_h)$ by introducing $\boldsymbol{f}_{h,v_h} \in \boldsymbol{W}^{1,\infty}(\Omega)$ and $\boldsymbol{f}'_{ij,v_h} \in \boldsymbol{L}^\infty(\Omega)$, $i,j = 1,\ldots,I$, and we assume that these quantities are defined such that the following holds for all $i,j = 1,\ldots,I$ and all $K \in S_{ij}$:

$$\int_{S_i} \nabla\cdot(\boldsymbol{f}_{h,v_h}(\boldsymbol{x}))\varphi_i(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} = \sum_{j\in\mathcal{I}(S_i)} (V_j - V_i)\int_{S_{ij}} \boldsymbol{f}'_{ij,v_h}(\boldsymbol{x})\cdot\nabla\varphi_j(\boldsymbol{x})\varphi_i(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}, \tag{3.4}$$

$$\int_K |\boldsymbol{f}'_{ij,v_h}(\boldsymbol{x})\cdot\nabla\varphi_j(\boldsymbol{x})|\varphi_i(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} \le \int_K \|\boldsymbol{f}'(v_h(\cdot))\cdot\nabla\varphi_j(\boldsymbol{x})\|_{L^\infty(K)}\varphi_i(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}. \tag{3.5}$$

where in the above definition the meaning of $\|\boldsymbol{f}'(v_h(\cdot))\cdot\nabla\varphi_j(\boldsymbol{x})\|_{L^\infty(K)}$ is

$$\|\boldsymbol{f}'(v_h(\cdot))\cdot\nabla\varphi_j(\boldsymbol{x})\|_{L^\infty(K)} := \sup_{\epsilon\to 0}\|\boldsymbol{f}'(\cdot)\cdot\nabla\varphi_j(\boldsymbol{x})\|_{L^\infty(v_h(K)+\epsilon)}. \tag{3.6}$$

Note that this definition is not necessary if $\boldsymbol{f}'$ is continuous.

Example 3.1. (Exact flux) The identity (3.4) holds by setting $\boldsymbol{f}_{h,v_h} = \boldsymbol{f}(v_h)$ and $\boldsymbol{f}'_{ij,v_h}(\boldsymbol{x}) = \boldsymbol{f}'(v_h(\boldsymbol{x}))$ for all $1 \le i,j \le I$. The inequality (3.5) is trivial.

Example 3.2. (Finite element flux) It is possible to set $\boldsymbol{f}_{h,v_h} = \sum_{j=1}^I \boldsymbol{f}(V_j)\varphi_j$, i.e., $\boldsymbol{f}_{h,v_h} = \pi_h\big(\boldsymbol{f}(v_h)\big)$, where recall that π_h is the Lagrange interpolation operator. In this case (3.4) holds with $\boldsymbol{f}'_{ij,v_h}(\boldsymbol{x}) = \frac{\boldsymbol{f}(V_j)-\boldsymbol{f}(V_i)}{V_j-V_i}$ owing to the partition of unity property. Recall that the ratio $\frac{\boldsymbol{f}(V_j)-\boldsymbol{f}(V_i)}{V_j-V_i}$ is well defined since $\boldsymbol{f}$ is Lipschitz continuous by assumption. The inequality (3.5) is a consequence of $\boldsymbol{f}$ being Lipschitz continuous and the property (2.5).

3.3. Time stepping and maximum principle. Let $t^n \ge 0$ be the current time and let $\Delta t > 0$ be the current time step, i.e., $t^{n+1} = t^n + \Delta t$. Let $u_h^n \in X_h$ be the approximation of $u(\cdot,t^n)$ and let us set $U^n := \mathsf{C}(u_h^n)$.

The scheme is defined as follows: the nodal values of $u_h^{n+1} \in X_h$ at time t^{n+1}, i.e., $U^{n+1} := \mathsf{C}(u_h^{n+1})$ are evaluated by

$$U_i^{n+1} = U_i^n - \Delta t m_i^{-1}\sum_{K\subset S_i}\left(\nu_K^n b_K(u_h^n,\varphi_i) + \int_K \nabla\cdot(\boldsymbol{f}_h^n)\varphi_i\,\mathrm{d}\boldsymbol{x}\right), \tag{3.7}$$

where we set $\boldsymbol{f}_h^n := \boldsymbol{f}_{h,u_h^n}$. Note that the mass matrix is lumped and we have set $m_i := \int_{S_i}\varphi_i(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}$. The piecewise constant viscosity field at t^n is defined as follows on each cell $K \in \mathcal{K}_h$:

$$\nu_K^n = \max_{i\ne j\in\mathcal{I}(K)} \frac{\int_{S_{ij}}(\boldsymbol{f}'^n_{ij,h}\cdot\nabla\varphi_j)^+\varphi_i\,\mathrm{d}\boldsymbol{x}}{-\sum_{T\subset S_{ij}} b_T(\varphi_j,\varphi_i)}. \tag{3.8}$$

where $z^+ := \max(0,z)$ is the positive part and we have set $\boldsymbol{f}'^n_{ij,h} := \boldsymbol{f}'_{ij,u_h^n}$.

Theorem 3.1 (Discrete Maximum Principle). *In addition to the above assumptions on the mesh-family and on the flux, assume that the CFL number is such that* $\lambda \le \frac{\mu_{\min}}{\mu_{\max}}\frac{1}{(1+\vartheta_{\min}^{-1})}$. *Then the solution to* (3.7) *satisfies the local discrete maximum principle, i.e.,* $u_{\min} \le \min_{j\in\mathcal{I}(S_i)} U_j^n \le U_i^{n+1} \le \max_{j\in\mathcal{I}(S_i)} U_j^n \le u_{\max}$ *for all* $n \ge 0$.

Proof. See Guermond and Nazarov [12]. ∎

Remark 3.1. (Maximum principle) An immediate consequence of Theorem 3.1 is that $\min_{\boldsymbol{x}\in\Delta_K} u_h^n(\boldsymbol{x}) \le \min_{\boldsymbol{x}\in K} u_h^{n+1}(\boldsymbol{x})$ and $\max_{\boldsymbol{x}\in K} u_h^{n+1}(\boldsymbol{x}) \le \max_{\boldsymbol{x}\in\Delta_K} u_h^n(\boldsymbol{x})$, for all $K \in \mathcal{K}_h$, where $\Delta_K = \cup_{i\in\mathcal{I}(K)} S_i$.

Remark 3.2. (SSP extension) Higher-order in time can be obtained by using a Strong Stability Preserving time stepping (see e.g., Gottlieb et al. [11] for a review), and Theorem 3.1 still holds in this case. The key property of SSP methods is that the solution at the end of each time step is a convex combination of solutions of forward Euler sub-steps. In the rest of the paper we restrict ourselves to the explicit Euler time stepping to simplify the presentation.

Although the definition (3.8) is sufficient for the maximum principle to hold, we are going to need a slightly stronger definition of the viscosity to establish error estimates. In the rest of the paper we redefine ν_K^n to be

$$\nu_K^n = \max_{i\neq j\in\mathcal{I}(K)} \frac{\sum_{K\in S_{ij}} \int_K \|\boldsymbol{f}'(u_h^n(\cdot))\cdot\nabla\varphi_j(\boldsymbol{x})\|_{L^\infty(K)}\varphi_i(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}}{-\sum_{T\subset S_{ij}} b_T(\varphi_j,\varphi_i)}. \tag{3.9}$$

Note that this definition implies that

$$\sum_{K\in S_{ij}} \int_K \|\boldsymbol{f}'(u_h^n(\cdot))\cdot\nabla\varphi_j(\boldsymbol{x})\|_{L^\infty(K)}\varphi_i(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} \le \sum_{K\in S_{ij}} \vartheta_K\nu_K|K|. \tag{3.10}$$

This is proved as follows: let $I_{ij} := \sum_{K\in S_{ij}} \int_K \|\boldsymbol{f}'(u_h^n(\cdot))\cdot\nabla\varphi_j(\boldsymbol{x})\|_{L^\infty(K)}\varphi_i\,\mathrm{d}\boldsymbol{x}$, then

$$I_{ij} = I_{ij}\frac{\sum_{K\in S_{ij}}\vartheta_K|K|}{\sum_{K\in S_{ij}}\vartheta_K|K|} = \sum_{K\in S_{ij}}\vartheta_K|K|\frac{I_{ij}}{\sum_{K\in S_{ij}}\vartheta_K|K|} \le \sum_{K\in S_{ij}}\vartheta_K|K|\nu_K.$$

3.4. Maximum time and boundary conditions. The boundary conditions are assumed to be either periodic or the initial data is assumed to be compactly supported. It the first case, there is no issue with the boundary conditions and the maximum time of existence of the numerical solution is infinite, i.e., we set $T_{\max} = +\infty$. In the second case we are interested in the solution in a time interval $[0, T_{\max}]$ such that the domain of influence of u_0 over $[0, T_{\max}]$ does not reach the boundary of Ω. Let us now estimate $T_{\max}$. The numerical maximum speed of propagation of the information is at most $\frac{h}{\Delta t}$, i.e., nonzero values can propagate over one cell per time step at most since the scheme is explicit and the mass matrix is lumped. Let $R_{\min}$ be the radius of the smallest ball in which the support of u_0 can be inscribed. Up to a translation we assume that 0 is the center of this ball. Let $R_{\max}$ be the radius of the largest ball inscribed in Ω and centered at 0. Then the numerical solution is well defined and compactly supported in Ω for all times $T \le T_{\max} := \frac{\Delta t}{h}(R_{\max} - R_{\min})$.

3.5. L^2-stability. We establish the L^2-stability properties of the method in this section. We start by estimating the viscosity.

LEMMA 3.2 (Viscosity bound). *Under the assumptions of Theorem 3.1, the following bound holds for all $K \in \mathcal{K}_h$ and all $\mathcal{K}_h$,*

$$\nu_K^n \le \beta \underline{h}_K^{-1}\frac{\mu_{\max}}{\vartheta_{\min}}. \tag{3.11}$$

Proof. Owing to the initialization assumption (3.1) and Theorem 3.1, $u_h^n(\boldsymbol{x}) \in [u_{\min}, u_{\max}]$ for all $n \ge 0$ and all $\boldsymbol{x} \in \Omega$; this implies that $\|\boldsymbol{f}'(u_h^n(\cdot))\cdot\nabla\varphi_j\|_{L^\infty(K)} \le \beta\|\nabla\varphi_j\|_{L^\infty(K)}$. Let $K \in \mathcal{K}_h$ and let ν_K^n be the viscosity coefficient defined in either (3.8) or (3.9). The above inequality together with the definition of ϑ in (2.11), the definition of $\underline{h}_K$ and the equality $\int_K \varphi_i(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} = \mu_K|K|$ implies that

$$\nu_K^n \le \beta \max_{i\neq j\in\mathcal{I}(K)} \frac{\|\nabla\varphi_j\|_{L^\infty(S_{ij})}\int_{S_{ij}}\varphi_i\,\mathrm{d}\boldsymbol{x}}{\vartheta_{\min}|S_{ij}|} \le \frac{\mu_{\max}}{\vartheta_{\min}}\frac{\beta}{\underline{h}_K}.$$

This proves the statement. ◻

LEMMA 3.3 (L^2-estimate). *Under the assumptions of Theorem 3.1 and whether the viscosity is defined using* (3.8) *or* (3.9)*, there is a uniform constant $\lambda_0 > 0$ such that the following estimate holds for all $\lambda \le \lambda_0$ and all $N \ge 0$:*

$$\|u_h^{N+1}\|_{\ell_h^2}^2 + \sum_{n=0}^{N}\Delta t \sum_{K\in\mathcal{K}_h} \nu_K^n b_K(u_h^n,u_h^n) \le \|u_h^0\|_{\ell_h^2}^2. \tag{3.12}$$

Proof. Let us multiply (3.7) by $2\Delta t U_i^{n+1}$ and sum over $i = 1,\dots,I$,

$$\begin{aligned}\|u_h^{n+1}\|_{\ell_h^2}^2 &+ \|u_h^{n+1}-u_h^n\|_{\ell_h^2}^2\\ &+ 2\Delta t\sum_{K\in\mathcal{K}_h}\nu_K^n b_K(u_h^n,u_h^n) = \|u_h^n\|_{\ell_h^2}^2 + R_1 + R_2,\end{aligned}$$

where $R_1 = 2\Delta t\sum_{K\in\mathcal{K}_h}\nu_K^n b_K(u_h^n, u_h^n - u_h^{n+1})$ and $R_2 = 2\Delta t\int_\Omega(\nabla\cdot\boldsymbol{f}_h^n)(u_h^n - u_h^{n+1})\,\mathrm{d}\boldsymbol{x}$. Since the mapping $X_h\times X_h \ni (v_h,w_h) \longmapsto b_K(v_h,w_h) \in \mathbb{R}$ is a scalar product (see (2.13)), we can estimate the first term R_1 as follows:

$$\begin{aligned}|R_1| &\le 2\Delta t\sum_{K\in\mathcal{K}_h}\nu_K^n b_K(u_h^n,u_h^n)^{\frac12} b_K(u_h^n-u_h^{n+1},u_h^n-u_h^{n+1})^{\frac12}\\ &\le \epsilon\Delta t\sum_{K\in\mathcal{K}_h}\nu_K^n b_K(u_h^n,u_h^n) + c\,\lambda\vartheta_{\min}^{-1}\epsilon^{-1}\mu_K^{\max}\|u_h^n-u_h^{n+1}\|_{L^2(\Omega)}^2,\end{aligned}$$

where we used (3.11) and $\epsilon > 0$ is an arbitrary positive number. The second term R_2 is estimated by invoking Lemma 3.4

$$|R_2| \le \lambda c\epsilon^{-1}\|u_h^n - u_h^{n+1}\|_{\ell_h^2}^2 + \epsilon\Delta t\sum_{K\in\mathcal{K}_h}\nu_K^n b_K(u_n^n,u_h^n).$$

Collecting the above estimates with $\epsilon = \frac12$ gives

$$\|u_h^{n+1}\|_{\ell_h^2}^2 + (1-c\lambda)\|u_h^{n+1}-u_h^n\|_{\ell_h^2}^2 + \Delta t\sum_{K\in\mathcal{K}_h}\nu_K^n b_K(u_h^n,u_h^n) \le \|u_h^n\|_{\ell_h^2}^2.$$

We conclude by assuming that $\lambda \le \frac{c}{2}$ and by summing the above estimates over n. ◻

LEMMA 3.4. *For all $\epsilon > 0$, there exists a uniform constant c, such that the following holds for all $g \in X_h$:*

$$\Big|\int_\Omega \nabla\cdot(\boldsymbol{f}_h^n(\boldsymbol{x})g(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}\Big| \le \frac{c}{\epsilon}\frac{\beta}{\underline{h}}\|g\|_{\ell_h^2}^2 + \frac{\epsilon}{2}\sum_{K\in\mathcal{K}_h}\nu_K^n b_K(u_h^n,u_h^n). \tag{3.13}$$

Proof. Upon setting $U := \mathsf{C}(u_h^n)$ and $G := \mathsf{C}(g)$, we infer that

$$\int_\Omega \nabla\cdot(\boldsymbol{f}_h^n(\boldsymbol{x}))g(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} = \sum_{i,j=1}^{I}(U_j - U_i)G_i\int_{S_{ij}}\boldsymbol{f}'^n_{ij,h}(\boldsymbol{x})\cdot\nabla\varphi_j(\boldsymbol{x})\varphi_i(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}.$$

Then using the definition of ν_K^n, (2.13), (3.5) and (3.10) we deduce that

$$\begin{aligned}\Big|\int_\Omega \nabla\cdot(\boldsymbol{f}_h^n(\boldsymbol{x}))g(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}\Big| &\le \sum_{i,j=1}^{I} \vartheta_{\max}|U_j-U_i||G_i| \sum_{K\subset S_{ij}} |K|\nu_K^n \\ &\le \vartheta_{\max} \sum_{K\in\mathcal{K}_h} \nu_K^n |K| \sum_{j\in\mathcal{I}(K)} \sum_{j\neq i\in\mathcal{I}(K)} |U_j-U_i||G_i| \\ &\le c \max_{K\in\mathcal{K}_h} (\nu_K^n)^{\frac12} \sum_{K\in\mathcal{K}_h} (\nu_K^n)^{\frac12} b_K(u_h^n,u_h^n)^{\frac12}\Big(m_i \sum_{i\in\mathcal{I}(K)} G_i^2\Big)^{\frac12}.\end{aligned}$$

Using the estimate (3.11) to bound ν_K^n from above, we finally derive

$$\Big|\int_\Omega \nabla\cdot(\boldsymbol{f}_h^n(\boldsymbol{x}))g(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}\Big| \le \frac{c}{\epsilon}\frac{\beta}{\underline{h}}\|g\|_{\ell_h^2}^2 + \frac{\epsilon}{2}\sum_{K\in\mathcal{K}_h} \nu_K^n b_K(u_h^n,u_h^n),$$

where $\epsilon > 0$ is an arbitrary positive number. This completes the proof. □

4. Error analysis. We are going to prove convergence to the entropy solution by establishing an error estimate based on Kružkov's doubling of the variables technique. The argument introduced by Kružkov [17] for proving uniqueness to scalar conservation equations has been modified by Kuznecov [18] to prove error estimates for numerical methods. This powerful, but cumbersome, technique is used for instance in Cockburn and Gremaud [5, 6] to prove convergence of some stabilized finite element techniques. We are going to adopt a variation of this method by reformulating Kuznecov's Lemma (see Lemma 2, p. 1492, in Kuznecov [18]) in the spirit of Bouchut and Perthame [2, Thm 2.1] using a Gronwall type argument from [5, Prop 6.2] and [6, Lemma 5.4]. The approximation result, Lemma A.2, is established in the Appendix. This general result can be used for the analysis of other methods.

In the rest of the paper we restrict ourselves exclusively to the following discrete flux:

$$\boldsymbol{f}_{h,v_h} = \pi_h(\boldsymbol{f}(v_h)), \tag{4.1}$$

since we have not been able to prove entropy estimates with the exact flux $\boldsymbol{f}_{h,v_h} = \boldsymbol{f}(v_h)$. We henceforth denote $\boldsymbol{f}_h^n := \pi_h(\boldsymbol{f}(u_h^n))$, where recall that π_h is the Lagrange interpolation operator. This definition implies that $\boldsymbol{f}'^n_{ij,h}(\boldsymbol{x}) = \frac{\boldsymbol{f}(U_j^n)-\boldsymbol{f}(U_i^n)}{U_j^n-U_i^n}$.

4.1. Global solution. We denote $D := \mathbb{R}^d$ if we solve a Cauchy problem in $\mathbb{R}^d$ (i.e., D is open in this case) and $D := \overline{\Omega}$ if Ω is the $\mathbb{R}^d$-torus and periodic boundary conditions are enforced (i.e., D is closed in this case). To summarize we define

$$D := \begin{cases} \mathbb{R}^d & \text{if Cauchy problem,} \\ \overline{\Omega} & \text{if periodic boundary conditions.}\end{cases} \tag{4.2}$$

Let $T_{\max}$ be the maximal time defined in §3.4. Let $T \in (0, T_{\max}]$ be a fixed time. We denote by $W_c^{1,\infty}(D\times[0,T];\mathbb{R})$ the set of the Lipschitz functions compactly supported in $D\times[0,T]$. We define a global approximation, $\widetilde{u}_h$, of the solution to (2.1) over the domain $\Omega\times[0,T]$ as follows:

$$\widetilde{u}_h(\boldsymbol{x},t) = u_h^n(\boldsymbol{x}), \quad \text{if } t\in[t^n,t^{n+1}), \quad \forall \boldsymbol{x}\in\Omega, \quad \forall t\in[0,T]. \tag{4.3}$$

If a Cauchy problem is solved in $\mathbb{R}^d$, we extend $\widetilde{u}_h$ by zero outside Ω and we abuse the notation by denoting again $\widetilde{u}_h$ the extension in question. If the domain is periodic we are going abuse the notation by using the same symbol to denote a function defined over D and its periodic extension defined over $\mathbb{R}^d$. The rest of the paper consists of estimating $\|\widetilde{u}_h(\cdot,t)-u(\cdot,t)\|_{L^1(D)}$ for all $t\in(0,T_{\max}]$ using Lemma A.2.

4.2. Quasi-interpolations and Kružkov entropies. Let $\bar{\pi}_h : L^1(\Omega) \longrightarrow X_h$ be the quasi-interpolation operator defined as follows:

$$\bar{\pi}_h(\psi)(\boldsymbol{x}) := \sum_{i=1}^{I} \Psi_i\varphi_i(\boldsymbol{x}),\ \forall \boldsymbol{x}\in\Omega, \qquad \Psi_i := m_i^{-1}\int_{S_i} \psi(\boldsymbol{y})\varphi_i(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y}, \tag{4.4}$$

We will use the following standard approximation result.

Lemma 4.1. *There exists a uniform constant c such that the following hods for all $\psi\in W^{1,p}(D)$ and all $p\in[1,+\infty]$:*

$$\|\psi-\bar{\pi}_h(\psi)\|_{L^p(K)} \le ch\|\nabla\psi\|_{L^p(\Delta_K)}, \quad \Delta_K := \cup_{i\in\mathcal{I}(K)} S_i, \quad \forall K\in\mathcal{K}_h. \tag{4.5}$$

Let k be a real number such that $u_{\min}\le k\le u_{\max}$ and let

$$\eta(v) = |v-k|, \qquad \boldsymbol{F}_\eta(v) := \operatorname{sgn}(v-k)(\boldsymbol{f}(v)-\boldsymbol{f}(k)) \tag{4.6}$$

be the associated Kružkov entropy and entropy flux, where $\operatorname{sgn}(z)$ is the sign function with the convention that $\operatorname{sgn}(0)=0$. Note that, using the convention that $\eta'(k)=0$, we have $\eta'(v)=\operatorname{sgn}(v-k)$, i.e., we can also write $\boldsymbol{F}_\eta(v) := \eta'(v)(\boldsymbol{f}(v)-\boldsymbol{f}(k))$.

Lemma 4.2. *Kružkov entropies are such that the following holds for all $a,b\in\mathbb{R}$:*

$$\eta'(a)(a-b) = \eta(a)-\eta(b)+r(b,a), \qquad r(b,a) := \eta(b)(1-\eta'(a)\eta'(b)) \ge 0. \tag{4.7}$$

Proof. If $\eta'(a)=0$ then $\eta(a)=0$ and the statement of the lemma reduces to $0=-\eta(b)+\eta(b)$. Hence, it remains to consider the case $\eta'(a)\neq 0$. The equation (4.7) is equivalent to

$$\eta'(a)(a-b) = \eta(a)-\eta(b)\eta'(a)\eta'(b) = \eta'(a)(\eta(a)\eta'(a)-\eta(b)\eta'(b)). \tag{4.8}$$

Using the definition of $\eta(u)=|u-k|$ and $\eta'(u)=\operatorname{sgn}(u-k)$, we obtain

$$\eta(a)\eta'(a)-\eta(b)\eta'(b) = a-k-(b-k) = a-b \tag{4.9}$$

which proves the result. □

Remark 4.1. (definition of $\pi_h(\eta'(v_h)\psi)$) Let $\psi\in\mathcal{C}^0(\Omega)$ and $v_h\in X_h$ with $V := \mathsf{C}(v_h)$. In the rest of the paper we set $\pi_h(\eta'(v_h)\psi)(\boldsymbol{x}) := \sum_{i=1}^I \operatorname{sgn}(V_i-k)\psi(\boldsymbol{a}_i)\varphi_i(\boldsymbol{x}) = \sum_{i=1}^I \eta'(v_h(\boldsymbol{a}_i))\psi(\boldsymbol{a}_i)\varphi_i(\boldsymbol{x})$.

Remark 4.2. (General entropies) Lemma 4.2 can be reformulated for any smooth entropy: i.e., $\eta'(a)(a-b)=\eta(a)-\eta(b)+r(a,b)$ where $r(a,b)=\int_a^b(b-x)\eta''(\xi)\,\mathrm{d}\xi\ge 0$, for all a,b.

4.3. Discrete entropy inequalities. We first start by establishing entropy inequalities using the Kružkov entropy family defined in (4.6). These inequalities are the premises of Lemma A.2.

LEMMA 4.3. *Let $T \le T_{\max}$ be some positive time. Let ψ be a non-negative Lipschitz function compactly supported in $D\times[0,T]$, $\psi \in W_c^{1,\infty}(D\times[0,T];\mathbb{R}^+)$. Let N be such that $T \in [t^N, t^{N+1})$; then we have*

$$\begin{aligned}(4.10)\quad &\|\pi_h\big(\eta(\widetilde{u}_h(\cdot,T))\bar{\pi}_h\psi(\cdot,t^N)\big)\|_{\ell_h^1} - \|\pi_h\big(\eta(\widetilde{u}_h(\cdot,0))\bar{\pi}_h\psi(\cdot,0)\big)\|_{\ell_h^1} \\ &\qquad - \int_0^T\int_\Omega \big(\eta(\widetilde{u}_h)\partial_t\psi + \boldsymbol{F}_\eta(\widetilde{u}_h)\cdot\nabla\psi\big)\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t = -R_1(\psi) - R_2(\psi) - R_3(\psi),\end{aligned}$$

where R_1, R_2 and R_3 are defined as follows:

$$\begin{aligned}
R_1(\psi) &:= \int_0^T\!\!\int_\Omega \eta(\widetilde{u}_h)\partial_t\psi\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t - \int_0^T\!\!\int_\Omega \pi_h(\eta(\widetilde{u}_h))\partial_t\psi\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t,\\
R_2(\psi) &:= \int_0^T\!\!\int_\Omega \boldsymbol{F}_\eta(\widetilde{u}_h)\cdot\nabla\psi\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t + \sum_{n=0}^{N-1}\Delta t\int_\Omega (\nabla\cdot\boldsymbol{f}_h^n)\pi_h(\eta'(u_h^{n+1})\bar{\pi}_h(\psi^{n+1}))\,\mathrm{d}\boldsymbol{x},\\
R_3(\psi) &:= \sum_{n=0}^{N-1}\left[\sum_{i=1}^{I} m_i\Psi_i^{n+1} r(U_i^n,U_i^{n+1}) + \Delta t\sum_{K\in\mathcal{K}_h}\nu_K b_K\big(u_h^n, \pi_h(\eta'(u_h^{n+1})\bar{\pi}_h(\psi^{n+1}))\big)\right].
\end{aligned}$$

Proof. Let $\bar{\pi}_h$ be the quasi-interpolation operator defined in (4.4) and let us set $\Psi_i(\tau) := (\bar{\pi}_h\psi)(\boldsymbol{a}_i,\tau) = \frac{1}{m_i}\int_{S_i}\psi(\boldsymbol{x},\tau)\varphi_i(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}$ (we henceforth denote $\Psi_i^{n+1} := \Psi_i(t^{n+1})$ to alleviate the notation). We multiply (3.7) by $m_i\eta'(U_i^{n+1})\Psi_i^{n+1}$ and upon denoting $\Delta U_i^{n+1} := U_i^{n+1} - U_i^n$ and using Lemma 4.2, the term involving the time increment is re-written as follows:

$$m_i\Psi_i^{n+1}\eta'(U_i^{n+1})\Delta U_i^{n+1} = m_i\Psi_i^{n+1}\Delta\eta(U_i^{n+1}) + m_i\Psi_i^{n+1} r(U_i^n,U_i^{n+1}).$$

We sum over n from 0 to $N-1$ and re-arrange the time summation

$$\begin{aligned}\sum_{n=0}^{N-1} m_i\Psi_i^{n+1}\eta'(U_i^{n+1})\Delta U_i^{n+1} &= m_i\Psi_i^N\eta(U_i^N) - m_i\Psi_i^0\eta(U_i^0)\\ &\quad \sum_{n=0}^{N-1} -m_i\eta(U_i^n)\Delta\Psi_i^{n+1} + \sum_{n=0}^{N-1} m_i\Psi_i^{n+1} r(U_i^n,U_i^{n+1}).\end{aligned}$$

We now sum over i, and upon observing that

$$m_i\Delta\Psi_i^{n+1} = \int_{S_i}\big(\psi(\boldsymbol{x},t^{n+1}) - \psi(\boldsymbol{x},t^n)\big)\varphi_i(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} = \int_{t^n}^{t^{n+1}}\int_{S_i}\partial_t\psi(\boldsymbol{x},t)\varphi_i(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t,$$

which also implies that

$$\begin{aligned}\sum_{i=1}^{I} m_i\eta(U_i^n)\Delta\Psi_i^{n+1} &= \int_{t^n}^{t^{n+1}}\int_\Omega \partial_t\psi(\boldsymbol{x},t)\sum_{i=1}^{I}\eta(U_i^n)\varphi_i(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t\\ &= \int_{t^n}^{t^{n+1}}\int_\Omega \partial_t\psi(\boldsymbol{x},t)\pi_h(\eta(u_h^n))\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t = \int_{t^n}^{t^{n+1}}\int_\Omega \partial_t\psi(\boldsymbol{x},t)\pi_h(\eta(\widetilde{u}_h))\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t,\end{aligned}$$

we obtain

$$\sum_{i=1}^{I}\sum_{n=0}^{N-1} m_i \Psi_i^{n+1}\eta'(U_i^{n+1})\Delta U_i^{n+1} = \sum_{i=1}^{I} m_i(\Psi_i^N\eta(U_i^N) - \Psi_i^0\eta(U_i^0)) \\ - \int_0^{t^N}\!\!\int_\Omega \partial_t\psi(\boldsymbol{x},t)\pi_h(\widetilde{u}_h)\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t + \sum_{n=0}^{N-1}\sum_{i=1}^{I} m_i\Psi_i^{n+1} r(U_i^n, U_i^{n+1}).$$

The rest of the proof consists of realizing that

$$\sum_{i=1}^{I}\eta'(U_i^{n+1})\Psi_i^{n+1}\varphi_i = \sum_{i=1}^{I}\eta'(u_h^{n+1})(\boldsymbol{a}_i)\bar{\pi}(\psi^{n+1})(\boldsymbol{a}_i)\varphi_i = \pi_h\big(\eta'(u_h^{n+1})\bar{\pi}(\psi^{n+1})\big).$$

The conclusion follows readily. □

4.4. Entropy production estimates. We now have to estimate the remainders in the right-hand side of (4.10), $R_1(\psi)$, $R_2(\psi)$ and $R_3(\psi)$, as needed in the a priori estimate (A.6). In the rest of the paper we denote

$$|\psi|_{\Delta_K^n} := |\nabla\psi|_{L^\infty(\Delta_K\times[t^n,t^{n+1}])} + \frac{1}{\beta}|\partial_t\psi|_{L^\infty(K\times[t^n,t^{n+1}])}, \qquad \forall n>0, \tag{4.11}$$

where recall that $\Delta_K := \cup_{i\in\mathcal{I}(K)}S_i$.

LEMMA 4.4. *Assume that the discrete flux is defined by* (4.1) *and the artificial viscosity is defined by* (3.9)*. Then, there are uniform constants $\lambda_0, c>0$ such that the following holds for all $\lambda\le\lambda_0$:*

$$R_1(\psi)+R_2(\psi)+R_3(\psi) \ge -c\,\beta\sum_{n=0}^{N-1}\Delta t\sum_{K\in\mathcal{K}_h} h_K|\psi|_{\Delta_K^n}\|\nabla u_h^n\|_{L^1(K)}. \tag{4.12}$$

Proof. The key observation to establish the statement is to realize that $R_3(\psi)$ produces dissipation in time and space, i.e., it generates two non-negative terms, and these terms are essential to control $R_2(\psi)$. The term $R_1(\psi)$ is harmless and controlled separately.

(1) Control of $R_3(\psi)$: Let us denote by $R_{3,1}(\psi)$ and $R_{3,2}(\psi)$ the two terms composing $R_3(\psi)$. The first term $R_{3,1}(\psi) := \sum_{n=0}^{N-1}\sum_{i=1}^{I} m_i\Psi_i^{n+1}r(U_i^n,U_i^{n+1})$ is clearly non-negative. This is the entropy dissipation created by the Euler time stepping. It will be used later to control time discrepancies arising elsewhere. The second term $R_{3,2}(\psi) := \Delta t\sum_{n=0}^{N-1}\sum_{K\in\mathcal{K}_h}\nu_K b_K\big(u_h^n, \pi_h(\eta'(u_h^{n+1})\bar{\pi}_h(\psi^{n+1}))\big)$ is the source of entropy dissipation induced by the artificial viscosity, up to a time discrepancy. This term needs to be handled carefully to extract the entropy dissipation induced by the artificial viscosity which will then be used to dominate space discrepancies arising elsewhere. *Actually, it is particularly important to realize that this term induces space-time dissipation, meaning that it is not a good idea to try to correct the time discrepancies in $R_{3,2}(\psi)$.* Let $K\in\mathcal{K}_h$ and let $z_h := \pi_h\big(\eta'(u_h^{n+1})\bar{\pi}_h(\psi^{n+1})\big)$, then using (2.13), we infer that

$$b_K(u_h^n, z_h) = \vartheta_K|K|\sum_{i\in\mathcal{I}(K)}\sum_{\mathcal{I}(K)\ni j<i}(U_i^n - U_j^n)(\eta'(U_i^{n+1})\Psi_i^{n+1} - \eta'(U_j^{n+1})\Psi_j^{n+1}).$$

Note here that we did not correct the time discrepancies. We now use (4.7) from Lemma 4.2 to derive

$$
\begin{aligned}
(U_i^n - U_j^n)\eta'(U_i^{n+1}) &= (U_i^n - U_i^{n+1})\eta'(U_i^{n+1}) + (U_i^{n+1} - U_j^n)\eta'(U_i^{n+1}) \\
&= \eta(U_i^n) - \eta(U_i^{n+1}) - r(U_i^n, U_i^{n+1}) + \eta(U_i^{n+1}) - \eta(U_j^n) + r(U_j^n, U_i^{n+1}) \\
&= \eta(U_i^n) - \eta(U_j^n) - r(U_i^n, U_i^{n+1}) + r(U_j^n, U_i^{n+1}),
\end{aligned}
$$

which in turn implies that

$$
\begin{aligned}
(U_i^n - U_j^n)(\eta'(U_i^{n+1})\Psi_i^{n+1} - \eta'(U_j^{n+1})\Psi_j^{n+1}) &= \Psi_i^{n+1} r(U_j^n, U_i^{n+1}) + \Psi_j^{n+1} r(U_i^n, U_j^{n+1}) \\
- \Psi_i^{n+1} r(U_i^n, U_i^{n+1}) - \Psi_j^{n+1} r(U_j^n, U_j^{n+1}) &+ (\Psi_i^{n+1} - \Psi_j^{n+1})\big(\eta(U_i^n) - \eta(U_j^n)\big) \\
\geq \Psi_i^{n+1} r(U_j^n, U_i^{n+1}) &+ \Psi_j^{n+1} r(U_i^n, U_j^{n+1}) - \Psi_i^{n+1} r(U_i^n, U_i^{n+1}) \\
&- \Psi_j^{n+1} r(U_j^n, U_j^{n+1}) - c h_K |\psi|_{\Delta_K^n} |U_i^n - U_j^n|,
\end{aligned}
$$

where we used the shape regularity of the mesh and recall that we denote $|\psi|_{\Delta_K^n} := |\nabla\psi|_{L^\infty(\Delta_K\times[t^n,t^{n+1}])} + \frac{1}{\beta}|\partial_t\psi|_{L^\infty(K\times[t^n,t^{n+1}])}$ to shorten the notation. *This estimate is essential*; it means that, up to time discrepancies $r(U_i^n, U_i^{n+1}) + r(U_j^n, U_j^{n+1})$, which are present in $R_{3,1}(\psi)$, the bilinear form b_K induces space-time dissipation since $r(U_j^n, U_i^{n+1}) + r(U_i^n, U_j^{n+1}) \geq 0$. In conclusion, using the estimate (3.11) we obtain

$$
\begin{aligned}
\nu_K b_K(u_h^n, z_h) &\geq \nu_K \vartheta_K |K| \sum_{i\neq j\in\mathcal{I}(K)} r(U_j^n, U_i^{n+1})\Psi_i^{n+1} \\
- \nu_K \vartheta_K |K| \sum_{i\in\mathcal{I}(K)} 2(n_K - 1) r(U_i^n, U_i^{n+1})\Psi_i^{n+1} &- c h_K |\psi|_{\Delta_K^n} \nu_K |K| \sum_{i\neq j\in\mathcal{I}(K)} |U_i^n - U_j^n| \\
&\geq \nu_K \vartheta_K |K| \sum_{i\neq j\in\mathcal{I}(K)} r(U_j^n, U_i^{n+1})\Psi_i^{n+1} \\
&- c\beta h_K^{-1} \sum_{i\in\mathcal{I}(K)} m_i r(U_i^n, U_i^{n+1})\Psi_i^{n+1} - c'\beta h_K |\psi|_{\Delta_K^n} \|\nabla u_h^n\|_{L^1(K)}
\end{aligned}
$$

Putting together all the above estimates, we infer that

$$
\begin{aligned}
R_3(\psi) \geq -c\beta \sum_{n=0}^{N-1} \Delta t \sum_{K\in\mathcal{K}_h} h_K |\psi|_{\Delta_K^n} \|\nabla u_h^n\|_{L^1(K)} &+ (1 - c'\lambda) \sum_{n=0}^{N-1} \sum_{i=1}^{I} m_i r(U_i^n, U_i^{n+1})\Psi_i^{n+1} \\
&+ \sum_{n=0}^{N-1} \Delta t \sum_{K\in\mathcal{K}_h} \nu_K \vartheta_K |K| \sum_{i\neq j\in\mathcal{I}(K)} r(U_j^n, U_i^{n+1})\Psi_i^{n+1}.
\end{aligned}
\tag{4.13}
$$

(2) Control of $R_2(\psi)$: Recall that

$$
R_2(\psi) := \sum_{n=0}^{N-1} \int_{t^n}^{t^{n+1}} \int_\Omega \boldsymbol{F}_\eta(u_h^n)\cdot\nabla\psi \, \mathrm{d}\boldsymbol{x} \, \mathrm{d}t + \sum_{n=0}^{N-1} \Delta t \sum_{i=1}^{I} I_2(i),
$$

where we have set $I_2(i) := \int_\Omega (\nabla\cdot\boldsymbol{f}_h^n)\varphi_i(\boldsymbol{x})\eta'(U_i^{n+1})\Psi_i^{n+1} \, \mathrm{d}\boldsymbol{x}$. We now define the approximate entropy flux

$$
\boldsymbol{F}_{\eta,h}^n(\boldsymbol{x}) := \pi_h\big(\boldsymbol{F}(u_h^n(\boldsymbol{x}))\big) = \sum_{j=1}^{I} \boldsymbol{F}(U_j^n)\varphi_j(x).
$$

Then, upon introducing $\psi^n(\boldsymbol{x}) = \frac{1}{\Delta t}\int_{t^n}^{t^{n+1}} \psi(\boldsymbol{x},t)\,\mathrm{d}t$, we re-write $I_2(i)$ as follows:

$$
\begin{aligned}
I_2(i) = &\int_\Omega \left(\eta'(U_i^{n+1})\nabla{\cdot}\boldsymbol{f}_h^n - \nabla{\cdot}\boldsymbol{F}_{\eta,h}^n\right)\varphi_i(\boldsymbol{x})\Psi_i^{n+1}\,\mathrm{d}\boldsymbol{x}\\
&+\int_\Omega (\nabla{\cdot}\boldsymbol{F}_{\eta,h}^n)\varphi_i(\boldsymbol{x})(\Psi_i^{n+1} - \psi^n(\boldsymbol{x}))\,\mathrm{d}\boldsymbol{x}\\
&+\int_\Omega \nabla{\cdot}(\boldsymbol{F}_{\eta,h}^n - \boldsymbol{F}_\eta(u_h^n))\psi^n(\boldsymbol{x})\varphi_i(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} + \int_\Omega \nabla{\cdot}(\boldsymbol{F}_\eta(u_h^n))\psi^n(\boldsymbol{x})\varphi_i(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x},
\end{aligned}
$$

which, after using the partition of unity property, proves that $R_2(\psi) = R_{2,1}(\psi) + R_{2,2}(\psi) + R_{2,3}(\psi)$, where

$$
\begin{aligned}
R_{2,1}(\psi) &:= \sum_{n=0}^{N-1}\Delta t\sum_{i=1}^{I}\int_\Omega \left(\eta'(U_i^{n+1})\nabla{\cdot}\boldsymbol{f}_h^n - \nabla{\cdot}\boldsymbol{F}_{\eta,h}^n\right)\varphi_i(\boldsymbol{x})\Psi_i^{n+1}\,\mathrm{d}\boldsymbol{x}\\
R_{2,2}(\psi) &:= \sum_{n=0}^{N-1}\Delta t\sum_{i=1}^{I}\int_\Omega (\nabla{\cdot}\boldsymbol{F}_{\eta,h}^n)\varphi_i(\boldsymbol{x})(\Psi_i^{n+1} - \psi^n(\boldsymbol{x}))\,\mathrm{d}\boldsymbol{x}\\
R_{2,3}(\psi) &:= \sum_{n=0}^{N-1}\Delta t\sum_{i=1}^{I}\int_\Omega \nabla{\cdot}(\boldsymbol{F}_{\eta,h}^n - \boldsymbol{F}_\eta(u_h^n))\psi^n(\boldsymbol{x})\varphi_i(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}
\end{aligned}
$$

Now we estimate $R_{2,1}(\psi)$. Recalling that we have set $\boldsymbol{f}_h^n = \pi_h(\boldsymbol{f}(u_h^n))$ and using again the partition of unity property, we obtain

$$
\begin{aligned}
&\int_\Omega (\nabla{\cdot}\boldsymbol{f}_h^n)\varphi_i(\boldsymbol{x})\eta'(U_i^{n+1})\Psi_i^{n+1}\,\mathrm{d}\boldsymbol{x}\\
&\qquad = \sum_{j\in\mathcal{I}(S_i)}\int_{S_i} (\boldsymbol{f}(U_j^n) - \boldsymbol{f}(k)){\cdot}\nabla\varphi_j(\boldsymbol{x})\varphi_i(\boldsymbol{x})\eta'(U_i^{n+1})\Psi_i^{n+1}\,\mathrm{d}\boldsymbol{x}\\
&\qquad = \sum_{j\in\mathcal{I}(S_i)}\int_{S_i} \frac{\boldsymbol{f}(U_j^n) - \boldsymbol{f}(k)}{U_j^n - k}{\cdot}\nabla\varphi_j(\boldsymbol{x})\varphi_i(\boldsymbol{x})(U_j^n - k)\eta'(U_i^{n+1})\Psi_i^{n+1}\,\mathrm{d}\boldsymbol{x},
\end{aligned}
$$

with the convention that $\frac{\boldsymbol{f}(U_j^n)-\boldsymbol{f}(k)}{U_j^n-k}$ should be replaced by 0 when $U_j^n = k$. This modification is not important because $\frac{\boldsymbol{f}(U_j^n)-\boldsymbol{f}(k)}{U_j^n-k}{\cdot}\nabla\varphi_j(\boldsymbol{x})\varphi_i(\boldsymbol{x})(U_j^n - k) = (\boldsymbol{f}(U_j^n) - \boldsymbol{f}(k)){\cdot}\nabla\varphi_j(\boldsymbol{x})$, and this number is zero if $U_j^n = k$. *Now we evaluate exactly* $(U_j^n - U_i^n)\eta'(U_i^{n+1})$. We use (4.7) from Lemma 4.2 to derive

$$
\begin{aligned}
(U_j^n - k)\eta'(U_i^{n+1}) &= \eta(U_j^n) - \eta(k) - r(U_j^n, U_i^{n+1}) + r(k, U_i^{n+1}),\\
&= \eta(U_j^n) - r(U_j^n, U_i^{n+1}).
\end{aligned}
$$

Recalling that $\eta(U_j^n) = \eta'(U_j^n)(U_j^n - k)$ and $\boldsymbol{F}_\eta(U_j^n) = \eta'(U_j^n)(\boldsymbol{f}(U_j^n) - \boldsymbol{f}(k))$, we conclude that

$$
\begin{aligned}
&\int_\Omega (\nabla{\cdot}\boldsymbol{f}_h^n)\varphi_i(\boldsymbol{x})\eta'(U_i^{n+1})\Psi_i^{n+1}\,\mathrm{d}\boldsymbol{x}\\
&\qquad = \sum_{j\in\mathcal{I}(S_i)}\int_{S_i} \frac{\boldsymbol{f}(U_j^n) - \boldsymbol{f}(k)}{U_j^n - k}{\cdot}\nabla\varphi_j(\boldsymbol{x})\varphi_i(\boldsymbol{x})(\eta(U_j^n) - r(U_j^n, U_i^{n+1}))\Psi_i^{n+1}\,\mathrm{d}\boldsymbol{x}\\
&\qquad = \int_{S_i} (\nabla{\cdot}\boldsymbol{F}_{\eta,h}^n)\varphi_i(\boldsymbol{x})\Psi_i^{n+1}\,\mathrm{d}\boldsymbol{x} + J_2(i),
\end{aligned}
$$

where

$$J_2(i) = -\sum_{K\in S_i}\sum_{j\in\mathcal{I}(K)}\int_K \frac{\boldsymbol{f}(U_j^n)-\boldsymbol{f}(k)}{U_j^n-k}\cdot\nabla\varphi_j(\boldsymbol{x})\varphi_i(\boldsymbol{x})r(U_j^n,U_i^{n+1})\Psi_i^{n+1}\,\mathrm{d}\boldsymbol{x}.$$

This proves that

$$R_{2,1}(\psi) = \sum_{n=0}^{N-1}\Delta t\sum_{i=1}^{I} J_2(i).$$

We now have a little problem since in order to use the last positive term in the estimate of $R_3(\psi)$, see (4.13), we need to produce a local viscosity using a local wave speed. The purpose of the coming developments is to transform the above integral to invoke local speeds only. Let us rewrite $J_2(i)$ as a sum of integrals: $J_2(i) = \sum_{K\in S_i} J_2(i,K)$, with obvious notation. Let K be a cell in S_i. Now we observe that if $k \le \min_{j\in\mathcal{I}(K)}(U_j^n)$ or $\max_{j\in\mathcal{I}(K)}(U_j^n) \le k$ then $\eta'(U_j^n) = \eta'(U_i^n)$ for all $j\in\mathcal{I}(K)$, which means that in this case

$$\begin{aligned}\frac{r(U_j^n,U_i^{n+1})}{U_j^n-k} &= \frac{1}{U_j^n-k}\eta(U_j^n)(1-\eta'(U_j^n)\eta'(U_i^{n+1}))\\ &= \eta'(U_j^n)(1-\eta'(U_j^n)\eta'(U_i^{n+1})) = \eta'(U_i^n)(1-\eta'(U_i^n)\eta'(U_i^{n+1})).\end{aligned}$$

Let us then assume that $k \le \min_{j\in\mathcal{I}(K)}(U_j^n)$ or $\max_{j\in\mathcal{I}(K)}(U_j^n) \le k$, then the partition of unity property together with the above argument implies that

$$\begin{aligned}J_2(i,K) &= -\sum_{j\in\mathcal{I}(K)}\int_K \boldsymbol{f}(U_j^n)\cdot\nabla\varphi_j(\boldsymbol{x})\varphi_i(\boldsymbol{x})\eta'(U_i^n)(1-\eta'(U_i^n)\eta'(U_i^{n+1}))\Psi_i^{n+1}\,\mathrm{d}\boldsymbol{x} =\\ &\sum_{j\in\mathcal{I}(K)}\int_K\frac{\boldsymbol{f}(U_j^n)-\boldsymbol{f}(U_i^n)}{U_i^n-U_j^n}\cdot\nabla\varphi_j(\boldsymbol{x})\varphi_i(\boldsymbol{x})(U_j^n-U_i^n)\eta'(U_i^n)(1-\eta'(U_i^n)\eta'(U_i^{n+1}))\Psi_i^{n+1}\mathrm{d}\boldsymbol{x}\\ &\ge -\sum_{i\ne j\in\mathcal{I}(K)}\int_K\left|\frac{\boldsymbol{f}(U_j^n)-\boldsymbol{f}(U_i^n)}{U_j^n-U_i^n}\cdot\nabla\varphi_j(\boldsymbol{x})\right|\varphi_i(\boldsymbol{x})\big(r(U_i^n,U_i^{n+1})+r(U_j^n,U_i^{n+1})\big)\Psi_i^{n+1}\mathrm{d}\boldsymbol{x},\end{aligned}$$

where we used

$$\begin{aligned}(U_j^n-U_i^n)\eta'(U_i^n)(1-\eta'(U_i^n)\eta'(U_i^{n+1})) &= (U_j^n-k)\eta'(U_i^n)(1-\eta'(U_i^n)\eta'(U_i^{n+1})\\ &\quad+(k-U_i^n)\eta'(U_i^n)(1-\eta'(U_i^n)\eta'(U_i^{n+1})\\ &= r(U_j^n,U_i^{n+1})-r(U_i^n,U_i^{n+1}),\end{aligned}$$

which implies that

$$|(U_j^n-U_i^n)\eta'(U_i^n)(1-\eta'(U_i^n)\eta'(U_i^{n+1}))| \le r(U_i^n,U_i^{n+1})+r(U_j^n,U_i^{n+1}).$$

Otherwise, if $\min_{j\in\mathcal{I}(K)}(U_j^n) \le k \le \max_{j\in\mathcal{I}(K)}(U_j^n)$, then

$$\begin{aligned}&\int_K\frac{\boldsymbol{f}(U_j^n)-\boldsymbol{f}(k)}{U_j^n-k}\cdot\nabla\varphi_j(\boldsymbol{x})\varphi_i(\boldsymbol{x})r(U_j^n,U_i^{n+1})\Psi_i^{n+1}\,\mathrm{d}\boldsymbol{x}\\ &= r(U_j^n,U_i^{n+1})\Psi_i^{n+1}\frac{1}{U_j^n-k}\int_k^{U_j^n}\int_K \boldsymbol{f}'(s)\cdot\nabla\varphi_j(\boldsymbol{x})\varphi_i(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}s\\ &\ge -r(U_j^n,U_i^{n+1})\Psi_i^{n+1}\int_K\|\boldsymbol{f}'(u_h^n(\cdot))\cdot\nabla\varphi_j(\boldsymbol{x})\|_{L^\infty(K)}\varphi_i(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x},\end{aligned}$$

where we used that $[\min_{j\in\mathcal{I}(K)}(U_j^n), \max_{j\in\mathcal{I}(K)}(U_j^n)] \subset u_h^n(K)$. Hence, we proved that the following holds in all the cases

$$J_2(i,K) \ge -\sum_{i\neq j\in\mathcal{I}(K)} \int_K \|\boldsymbol{f}'(u_h^n(\cdot))\cdot\nabla\varphi_j(\boldsymbol{y})\|_{L^\infty(K)}\varphi_i(\boldsymbol{x})\big(r(U_i^n,U_i^{n+1}) \\ + r(U_j^n,U_i^{n+1})\big)\Psi_i^{n+1}\mathrm{d}\boldsymbol{x}.$$

Then upon invoking the bound (3.10), we have

$$\begin{aligned} J_2(\psi) \ge &-\sum_{i\neq j\in\mathcal{I}(S_i)} \sum_{K\in S_{ij}} \int_K \|\boldsymbol{f}'(u_h^n(\cdot))\cdot\nabla\varphi_j(\boldsymbol{y})\|_{L^\infty(K)}\varphi_i(\boldsymbol{x})\big(r(U_i^n,U_i^{n+1}) \\ &\qquad + r(U_j^n,U_i^{n+1})\big)\Psi_i^{n+1}\mathrm{d}\boldsymbol{x} \\ \ge &-\sum_{i\neq j\in\mathcal{I}(S_i)} \big(r(U_i^n,U_i^{n+1}) + r(U_j^n,U_i^{n+1})\big)\Psi_i^{n+1} \sum_{K\in S_{ij}} \vartheta_K\nu_K|K|. \end{aligned}$$

Now we are able to conclude that there is a uniform $c>0$ such that

$$R_{2,1}(\psi) \ge -\sum_{n=0}^{N-1}\Delta t \sum_{K\in\mathcal{K}_h(\psi)} \nu_K\vartheta_K|K| \sum_{i\neq j\in\mathcal{I}(K)} r(U_j^n,U_i^{n+1})\Psi_i^{n+1} \\ - c\,\lambda \sum_{n=0}^{N-1}\sum_{i=1}^{I} m_i r(U_i^n,U_i^{n+1})\Psi_i^{n+1}.$$

The term $R_{2,2}(\psi)$ is controlled by proceeding as in the proof of Lemma 3.4. Namely, we rewrite

$$\begin{aligned} &\int_\Omega (\Psi_i^{n+1} - \psi^n(\boldsymbol{x}))(\nabla\cdot\boldsymbol{F}_{\eta,h}^n)\varphi_i(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} \\ &\qquad = \sum_{j\in\mathcal{I}(S_i)} \int_{S_i} (\Psi_i^{n+1} - \psi^n(\boldsymbol{x}))\eta'(U_j^n)(\boldsymbol{f}(U_j^n) - \boldsymbol{f}(k))\cdot\nabla\varphi_j(\boldsymbol{x})\varphi_i(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}. \end{aligned}$$

Here, again we need to localize the estimate by getting rid of $\boldsymbol{f}(k)$. Let us consider $K\in S_i$. Let us assume first that $k \le \min_{j\in\mathcal{I}(K)}(U_j^n)$ or $\max_{j\in\mathcal{I}(K)}(U_j^n) \le k$, then the partition of unity property implies that we can replace $\boldsymbol{f}(k)$ by $\boldsymbol{f}(U_i^n)$, i.e.,

$$\begin{aligned} &\int_K (\Psi_i^{n+1} - \psi^n(\boldsymbol{x}))(\nabla\cdot\boldsymbol{F}_{\eta,h}^n)\varphi_i(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} \\ &\quad = \sum_{j\in\mathcal{I}(K)} \int_K (\Psi_i^{n+1} - \psi^n(\boldsymbol{x}))\eta'(U_i^n)(\boldsymbol{f}(U_j^n) - \boldsymbol{f}(U_i^n))\cdot\nabla\varphi_j(\boldsymbol{x})\varphi_i(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} \\ &\quad = \sum_{j\in\mathcal{I}(K)} \int_K (\Psi_i^{n+1} - \psi^n(\boldsymbol{x}))\eta'(U_i^n)(U_j^n - U_i^n)\frac{\boldsymbol{f}(U_j^n) - \boldsymbol{f}(U_i^n)}{U_j^n - U_i^n}\cdot\nabla\varphi_j(\boldsymbol{x})\varphi_i(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} \end{aligned}$$

This implies that

$$\begin{aligned} &\int_K (\Psi_i^{n+1} - \psi^n(\boldsymbol{x}))(\nabla\cdot\boldsymbol{F}_{\eta,h}^n)\varphi_i(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} \\ &\qquad \ge -\|\Psi_i^{n+1} - \psi^n\|_{L^\infty(K)} \sum_{j\in\mathcal{I}(K)} |U_j^n - U_i^n| \int_K \|\boldsymbol{f}'(u_h^n(\cdot))\cdot\nabla\varphi_j\|_{L^\infty(K)}\varphi_i\,\mathrm{d}\boldsymbol{x}. \end{aligned}$$

If $\min_{j\in\mathcal{I}(K)}(U_j^n) \le k \le \max_{j\in\mathcal{I}(K)}(U_j^n) \le k$, by proceeding as above we have

$$\begin{aligned}
&\int_K (\Psi_i^{n+1} - \psi^n(\boldsymbol{x}))(\nabla\cdot \boldsymbol{F}_{\eta,h}^n)\varphi_i(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} \\
&\quad \ge -\|\Psi_i^{n+1} - \psi^n\|_{L^\infty(K)} \sum_{j\in\mathcal{I}(K)} |U_j^n - k| \int_K \|\boldsymbol{f}'(u_h^n(\cdot))\cdot\nabla\varphi_j\|_{L^\infty(K)}\varphi_i\,\mathrm{d}\boldsymbol{x} \\
&\quad \ge -c\|\Psi_i^{n+1} - \psi^n\|_{L^\infty(K)} \sum_{j\in\mathcal{I}(K)} |U_j^n - U_i^n| \int_K \|\boldsymbol{f}'(u_h^n(\cdot))\cdot\nabla\varphi_j\|_{L^\infty(K)}\varphi_i\,\mathrm{d}\boldsymbol{x},
\end{aligned}$$

where in the last inequality we used that k is a convex combination of $(U_l^n)_{l\in\mathcal{I}(K)}$ and we used the triangle inequality repeatedly. Upon invoking the bound (3.10), the above argument implies that the following holds independently of the value of k:

$$\begin{aligned}
&\int_\Omega (\Psi_i^{n+1} - \psi^n(\boldsymbol{x}))(\nabla\cdot \boldsymbol{F}_{\eta,h}^n)\varphi_i(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} \ge \\
&-c(1+\lambda)\sum_{K\in S_i} |\psi|_{\Delta_K^n} h_K \nu_K^n |K| \sum_{i\in\mathcal{I}(K)} |U_j^n - U_i^n| \ge -c\,\beta \sum_{K\in\mathcal{K}_h} h_K |\psi|_{\Delta_K^n} \|\nabla u_h^n\|_{L^1(K)}.
\end{aligned}$$

In conclusion, $R_{2,2}(\psi) \ge -c\,\beta \sum_{n=0}^{N-1}\Delta t \sum_{K\in\mathcal{K}_h} h_K |\psi|_{\Delta_K^n} \|\nabla u_h^n\|_{L^1(K)}$. Now we estimate $R_{2,3}(\psi)$. The partition of unity property implies that

$$\begin{aligned}
R_{2,3}(\psi) &:= \sum_{n=0}^{N-1}\Delta t \sum_{i=1}^{I} \int_\Omega \nabla\cdot(\boldsymbol{F}_{\eta,h}^n - \boldsymbol{F}_\eta(u_h^n))\psi^n(\boldsymbol{x})\varphi_i(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} \\
&= -\sum_{n=0}^{N-1}\Delta t \int_\Omega (\boldsymbol{F}_{\eta,h}^n - \boldsymbol{F}_\eta(u_h^n))\cdot\nabla\psi^n(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} \\
&= -\sum_{n=0}^{N-1}\Delta t \sum_{K\in\mathcal{K}_h}\sum_{j\in\mathcal{I}(K)} \int_K (\boldsymbol{F}_\eta(U_j^n) - \boldsymbol{F}_\eta(u_h^n))\cdot\nabla\psi^n(\boldsymbol{x})\varphi_j(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} \\
&\ge -c\,\beta \sum_{n=0}^{N-1}\Delta t \sum_{K\in\mathcal{K}_h} |\psi|_{\Delta_K^n} \sum_{j\in\mathcal{I}(K)} |K||U_j^n - u_h^n|,
\end{aligned}$$

where we used the Lipschitz continuity of the entropy flux. In conclusion, $R_{2,3}(\psi) \ge -c\beta\sum_{n=0}^{N-1}\Delta t\sum_{K\in\mathcal{K}_h} h_K|\psi|_{\Delta_K^n}\|\nabla u_h^n\|_{L^1(K)}$. Putting together the above estimates we obtain,

$$\begin{aligned}
R_2(\psi) \ge &-\sum_{n=0}^{N-1}\Delta t \sum_{K\in\mathcal{K}_h} \nu_K \vartheta_K |K| \sum_{i\ne j\in\mathcal{I}(K)} r(U_j^n, U_i^{n+1})\Psi_i^{n+1} \\
&- c\,\lambda \sum_{n=0}^{N-1}\sum_{i=1}^{I} m_i r(U_i^n, U_i^{n+1})\Psi_i^{n+1} - c'\,\beta \sum_{n=0}^{N-1}\Delta t \sum_{K\in\mathcal{K}_h} h_K |\psi|_{\Delta_K^n} \|\nabla u_h^n\|_{L^1(K)}.
\end{aligned}$$

(3) Control of $R_1(\psi)$: Recall that $R_1(\psi) = \int_0^T\int_\Omega (\eta(\widetilde{u}_h) - \pi_h(\eta(\widetilde{u}_h)))\partial_t\psi\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t$. Using the partition of unity property of the shape functions, we have $\eta(\widetilde{u}_h(\boldsymbol{x},t)) = \sum_{i=1}^{I}\eta(\widetilde{u}_h(\boldsymbol{x},t))\varphi_i(\boldsymbol{x})$ for all $\boldsymbol{x}\in K$, which in turn implies that that

$$\int_\Omega (\eta(\widetilde{u}_h) - \pi_h(\eta(\widetilde{u}_h)))\partial_t\psi\,\mathrm{d}\boldsymbol{x} = \sum_{i=1}^{I}\int_{S_i} (\eta(\widetilde{u}_h) - \eta(\widetilde{u}_h(\boldsymbol{a}_i,t)))\varphi_i\partial_t\psi\,\mathrm{d}\boldsymbol{x}$$

The conclusion follows readily since $|\eta(a) - \eta(b)| \le |a-b|$ and $\widetilde{u}_h$ is a discrete function, i.e., the following inequality holds for all $t \in [t^n, t^{n+1})$: $\int_K |\widetilde{u}_h(\boldsymbol{x},t) - \widetilde{u}_h(\boldsymbol{a}_i,t)||\partial_t\psi(\boldsymbol{x},t)|\,\mathrm{d}\boldsymbol{x} \le c\,h_K \|\nabla u_h^n\|_{L^1(K)} \|\partial_t\psi\|_{L^\infty(K\times(t^n,t^{n+1}))}$ for all $K \in \mathcal{K}_h$. In conclusion we have

$$R_1(\psi) \ge -c\,\beta \sum_{n=0}^{N-1} \Delta t \sum_{K\in\mathcal{K}_h} h_K |\psi|_{\Delta_K^n} \|\nabla u_h^n\|_{L^1(K)}.$$

(4) Now we conclude by combining all the above estimates

$$\begin{aligned} R_1(\psi) + R_2(\psi) + R_3(\psi) \ge (1-c\lambda) \sum_{n=0}^{N-1}\sum_{i=1}^{I} m_i r(U_i^n, U_i^{n+1})\Psi_i^{n+1} \\ -c'\,\beta \sum_{n=0}^{N-1} \Delta t \sum_{K\in\mathcal{K}_h} h_K |\psi|_{\Delta_K^n} \|\nabla u_h^n\|_{L^1(K)}. \end{aligned}$$

The conclusion follows by assuming that the CFL number, λ, is small enough. □

4.5. Convergence estimates. The purpose of this section is to derive an error estimate; this will be done by using Lemma A.2 together with Lemma 4.4. We henceforth assume that $u_0 \in BV(\Omega)$ and that u_h^0 is evaluated so that

$$\|u_0 - u_h^0\|_{L^1(\Omega)} \le ch|u|_{BV(\Omega)}. \tag{4.14}$$

We introduce three mutually exclusive assumptions that we henceforth refer to: (H1), (H2), (H3). In the first case, (H1), we assume that there is a uniform BV bound on the approximate solution u_h, i.e., there is a uniform constant c such that

$$\sum_{n=0}^{N-1} \Delta t \sum_{K\in\mathcal{K}_h} h_K \|\nabla u_h^n\|_{L^1(K)} \le cT|u_0|_{BV(\Omega)}. \tag{H1}$$

The proof of this estimate in one space dimension is standard and can be done by using Harten's Lemma [15, Lemma 2.2] (the details are left to the reader). We conjecture that this estimate is true in every space dimension on fairly general meshes, but this question is still open. In the second case, (H2), we assume that the flux does not degenerate in the sense that there is a uniform constant $\alpha > 0$ so that,

$$\inf_{0\neq\boldsymbol{n}\in\mathbb{R}^d} \frac{\|\boldsymbol{f}'(\cdot)\cdot\boldsymbol{n}\|_{L^\infty([u_{\min},u_{\max}])}}{\|\boldsymbol{n}\|_{\ell^2}} \ge \alpha\beta, \tag{H2}$$

where the L^∞-norm is defined in (3.6). In the third case, (H3), we introduce a parameter $\alpha > 0$ and we change the definition of the viscosity over each cell $K \in \mathcal{K}_h$ so that the new viscosity is equal to $\max(\nu_K, \alpha\frac{\beta}{h_K})$, namely we modify (3.9) as follows:

$$\nu_K^n = \max\left(\frac{\alpha\beta}{h_K}, \max_{i\neq j\in\mathcal{I}(K)} \frac{\sum_{K\in S_{ij}} \int_K \|\boldsymbol{f}'(u_h(\cdot))\cdot\nabla\varphi_j(\boldsymbol{x})\|_{L^\infty(K)}\varphi_i(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}}{-\sum_{T\subset S_{ij}} b_T(\varphi_j,\varphi_i)}\right). \tag{H3}$$

This assumption is pretty standard; for instance it is similar to assumption (2.4b) in Cockburn and Gremaud [5], it is also similar to the fact that ϵ_1 in (2.5) in [5] does not vanish when $\beta = 0$.

We are in measure to state the main result of the paper.

THEOREM 4.5 ($L_t^\infty(L_x^1)$ error estimate). *In addition to* (2.2), *assume also that* $u_0 \in BV(\Omega)$, *the discrete flux is defined by* (4.1) *and the artificial viscosity is defined by* (3.9). *Then, there exists a uniform constant* $\lambda_0 > 0$ *such that the following holds for all* $\lambda \le \lambda_0$*:*
(i) Under assumption (H1), there is a uniform constant c such that

$$\|u(\cdot,T) - \tilde{u}_h(\cdot,T)\|_{L^1(\Omega)} \le c h^{\frac{1}{2}} \sqrt{\beta T} |u_0|_{BV(\Omega)}. \tag{4.15}$$

(ii) Under assumption (H2) or (H3), there is a uniform constant c such that

$$\|u(\cdot,T) - \tilde{u}_h(\cdot,T)\|_{L^1(\Omega)} \le c\, h^{\frac{1}{4}} |\Omega|^{\frac{1}{2}} (\beta T)^{\frac{1}{4}} |u_0|_{BV(\Omega)}^{\frac{1}{2}} |u_h^0|_*^{\frac{1}{2}}. \tag{4.16}$$

where $|u_h^0|_* := (\|u_h^0\|_{L^2(\Omega)}^2 - \|\overline{u}_h^0\|_{L^2(\Omega)}^2)^{\frac{1}{2}}$ *and* $\overline{v} := \frac{1}{|\Omega|}\int_\Omega v(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}$.

Proof. Owing to Lemma 4.3 and Lemma 4.4 it is legitimate to apply Lemma A.2 with $\sigma_h = 0$, $\mathcal{T}_h = t^N$ and

$$\Lambda_T(\psi) = c\,\beta \sum_{n=0}^{N-1} \Delta t \sum_{K \in \mathcal{K}_h} h_K |\psi|_{\Delta_K^n} \|\nabla u_h^n\|_{L^1(K)}, \tag{4.17}$$

where we recall that $N = N(T)$ is defined by $T \in [t^N, t^{N+1})$. Then using Lemma A.2 and the BV bound on u_0, we have $\|u_0 - u_{0h}\|_{L^1(\Omega)} \le ch|u_0|_{BV(\Omega)}$ and obtain

$$\|u(\cdot,T) - \tilde{u}_h(\cdot,T)\|_{L^1(\Omega)} \le c\,((\epsilon + h)|u_0|_{BV(\Omega)} + \Lambda^*).$$

and the rest of the proof consists of estimating

$$\Lambda^* := \sup_{0 \le \widetilde{T} \le T} \frac{\int_0^{\widetilde{T}} \int_D \Lambda_{\widetilde{T}}(\phi)\,\mathrm{d}\boldsymbol{y}\,\mathrm{d}s}{\Gamma_\delta(\widetilde{T})},$$

where $\phi(\boldsymbol{x},\boldsymbol{y},t,s) := \boldsymbol{\omega}_\epsilon(\boldsymbol{x}-\boldsymbol{y})\omega_\delta(t-s)$ has been defined in (A.2) and we denote $\Gamma_\delta(\tau) := \int_0^\tau \omega_\delta(s) ds$ for any $\tau \ge 0$. From now on we assume that $h \le \epsilon$.

Consider $\widetilde{T} \in (0,T]$ and define $\widetilde{N}$ such that $t^{\widetilde{N}} \le \widetilde{T} < t^{\widetilde{N}+1}$. We have that

$$\int_0^{\widetilde{T}}\!\!\int_D \Lambda_{\widetilde{T}}(\phi)\,\mathrm{d}\boldsymbol{y}\,\mathrm{d}s = c\,\beta \sum_{n=0}^{\widetilde{N}-1} \Delta t \sum_{K \in \mathcal{K}_h} h_K \|\nabla u_h^n\|_{L^1(K)} \int_0^{\widetilde{T}}\!\!\int_D |\phi|_{\Delta_K^n}\,\mathrm{d}\boldsymbol{y}\,\mathrm{d}s.$$

Recalling the definition $|\phi|_{\Delta_K^n} := |\nabla_{\boldsymbol{x}}\phi|_{L^\infty(\Delta_K \times [t^n,t^{n+1}])} + \frac{1}{\beta}|\partial_t \phi|_{L^\infty(K\times[t^n,t^{n+1}])}$, and recalling that $h \le \epsilon$, it can be shown that there is a uniform constant $c > 0$ such that

$$|\phi(\cdot,\boldsymbol{y},\cdot,s)|_{\Delta_K^n} \le \frac{c}{\Delta t |K|} \int_{t^n}^{t^{n+1}}\!\!\int_{\Delta_K} (\frac{1}{\beta}|\partial_t \phi(\boldsymbol{x},\boldsymbol{y},t,s)| + \|\nabla_{\boldsymbol{x}}\phi(\boldsymbol{x},\boldsymbol{y},t,s)\|)\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t,$$

for all $0 \le n \le \widetilde{N} - 1$, which implies that

$$\int_0^{\widetilde{T}}\!\!\int_D |\phi(\cdot,\boldsymbol{y},\cdot,s)|_{\Delta_K^n}\,\mathrm{d}\boldsymbol{y}\,\mathrm{d}s \le \\ \frac{c}{\Delta t|K|} \int_{t^n}^{t^{n+1}}\!\!\int_{\Delta_K} \int_0^{\widetilde{T}}\!\!\int_D (\frac{1}{\beta}|\partial_t\phi(\boldsymbol{x},\boldsymbol{y},t,s)| + \|\nabla_{\boldsymbol{x}}\phi(\boldsymbol{x},\boldsymbol{y},t,s)\|)\,\mathrm{d}\boldsymbol{y}\,\mathrm{d}s\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t.$$

Now we evaluate $\int_0^{\widetilde{T}}\int_D(\frac{1}{\beta}|\partial_t\phi(\boldsymbol{x},\boldsymbol{y},t,s)| + \|\nabla_{\boldsymbol{x}}\phi(\boldsymbol{x},\boldsymbol{y},t,s)\|)\,\mathrm{d}\boldsymbol{y}\,\mathrm{d}s$. Using that $n \le \widetilde{N}-1$, since $\Lambda_{\widetilde{T}}$ involves a sum for $n=0$ to $n=\widetilde{N}-1$ (see (4.17)), we infer that $0 \le t^n \le t \le t^{n+1} \le t^{\widetilde{N}} \le \widetilde{T}$, which implies that $0 \le t \le \widetilde{T}$. We then can apply Lemma A.1 for all $t \in [t^n, t^{n+1}]$

$$\int_0^{\widetilde{T}}\int_D(\frac{1}{\beta}|\partial_t\phi(\boldsymbol{x},\boldsymbol{y},t,s)| + \|\nabla_{\boldsymbol{x}}\phi(\boldsymbol{x},\boldsymbol{y},t,s)\|)\,\mathrm{d}\boldsymbol{y}\,\mathrm{d}s \le c\frac{\Gamma_\delta(\widetilde{T})}{\beta\delta} + c'\frac{\Gamma_\delta(\widetilde{T})}{\epsilon} \le c''\frac{\Gamma_\delta(\widetilde{T})}{\epsilon}.$$

This computation in turn implies that

$$\int_0^{\widetilde{T}}\int_D |\phi(\cdot,\boldsymbol{y},\cdot,s)|_{\Delta_K^n}\,\mathrm{d}\boldsymbol{y}\,\mathrm{d}s \le c\frac{\Gamma_\delta(\widetilde{T})}{\epsilon}.$$

Using the above bound, we estimate $\int_0^{\widetilde{T}}\int_D \Lambda_{\widetilde{T}}(\phi)\,\mathrm{d}\boldsymbol{y}\,\mathrm{d}s$ for $\widetilde{T} \in (0,T]$ as follows:

$$\int_0^{\widetilde{T}}\int_D \Lambda_{\widetilde{T}}(\phi)\,\mathrm{d}\boldsymbol{y}\,\mathrm{d}s \le c\,\beta\sum_{n=0}^{N-1}\Delta t\sum_{K\in\mathcal{K}_h} h_K\|\nabla u_h^n\|_{L^1(K)}\frac{\Gamma_\delta(\widetilde{T})}{\epsilon}.$$

Therefore, we obtain that for any $\widetilde{T}$, $0 \le \widetilde{T} \le T$, we have

$$\frac{\int_0^{\widetilde{T}}\int_D \Lambda_{\widetilde{T}}(\phi)\,\mathrm{d}\boldsymbol{y}\,\mathrm{d}s}{\Gamma_\delta(\widetilde{T})} \le \frac{c\beta}{\epsilon}\sum_{n=0}^{N-1}\Delta t\sum_{K\in\mathcal{K}_h} h_K\|\nabla u_h^n\|_{L^1(K)}.$$

In conclusion, taking the supremum over $\widetilde{T} \in [0,T]$, we infer that

$$\Lambda^* \le \frac{c\beta}{\epsilon}\sum_{n=0}^{N-1}\Delta t\sum_{K\in\mathcal{K}_h} h_K\|\nabla u_h^n\|_{L^1(K)}. \tag{4.18}$$

We finish the proof of the theorem by bounding the right-hand side of (4.18) in each of the three cases (H1),(H2) and (H3).
(1) Assumption (H1): Using (H1) we infer that

$$\Lambda^* \le \frac{c\beta}{\epsilon}hT|u_0|_{BV(\Omega)}.$$

Then we have

$$\|u(\cdot,T) - \tilde{u}_h(\cdot,T)\|_{L^1(\Omega)} \le c\,((\epsilon+h)|u_0|_{BV(\Omega)} + \frac{h\beta T}{\epsilon}|u_0|_{BV(\Omega)}).$$

It possible to optimize the choice of ϵ in the above estimate. We choose $\epsilon^2 = \beta hT$ which implies that

$$\|u(\cdot,T) - \tilde{u}_h(\cdot,T)\|_{L^1(\Omega)} \le c\sqrt{h}\sqrt{\beta T}|u_0|_{BV(\Omega)}.$$

This proves the error estimate in the case of assumption (H1), see (4.15).
(2) Assumption (H2) or (H3): The L^2-estimate (3.12) implies that

$$\sum_{n=0}^{N-1}\Delta t\sum_{K\in\mathcal{K}_h}\nu_K b_K(u_h^n,u_h^n) \le \|u_h^0\|_{\ell_h^2}^2 - \|u_h^N\|_{\ell_h^2}^2.$$

Recall that $\overline{u}_h^0 := \frac{1}{|\Omega|}\int_\Omega u_h^0(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}$. Using the mass conservation property of the method, $\int_\Omega u_h^0(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} = \int_\Omega u_h^N(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}$, we have that

$$\|\overline{u}_h^0\|_{\ell_h^2}^2 = |\Omega|(\overline{u}_h^0)^2 \le \|u_h^N\|_{\ell_h^2}^2.$$

Using the above and the fact that $\overline{u}_h^0$ is orthogonal to $u_h^0 - \overline{u}_h^0$ with respect to both the $L^2(\Omega)$ and ℓ_h^2 scalar products, we obtain

$$\begin{aligned}\sum_{n=0}^{N-1}\Delta t\sum_{K\in\mathcal{K}_h}\nu_K b_K(u_h^n,u_h^n) &\le \|u_h^0\|_{\ell_h^2}^2 - \|u_h^N\|_{\ell_h^2}^2 \le \|u_h^0\|_{\ell_h^2}^2 - \|\overline{u}_h^0\|_{\ell_h^2}^2 = \|u_h^0-\overline{u}_h^0\|_{\ell_h^2}^2\\ &\le c\,\|u_h^0-\overline{u}_h^0\|_{L^2(\Omega)}^2 = c\left(\|u_h^0\|_{L^2(\Omega)}^2 - \|\overline{u}_h^0\|_{L^2(\Omega)}^2\right).\end{aligned}$$

This bound together with (2.14) implies that there are uniform constants $c, c' > 0$ such that

$$c'\beta\sum_{n=0}^{N-1}\Delta t\sum_{K\in\mathcal{K}_h}\underline{h}_K\|\nabla u_h^n\|_{L^2(K)}^2 \le \sum_{n=0}^{N-1}\Delta t\sum_{K\in\mathcal{K}_h}\nu_K b_K(u_h^n,u_h^n) \le c\,(\|u_h^0\|_{L^2(\Omega)}^2 - \|\overline{u}_h^0\|_{L^2(\Omega)}^2)$$

where we used that each of the assumptions (H2) and (H3) implies that there is $c'' > 0$ such that $\nu_k \ge c''\frac{\alpha\beta}{\underline{h}_K}$. It is then possible to estimate $\sum_{n=0}^{N-1}\Delta t\sum_{K\in\mathcal{K}_h} h_K\|\nabla u_h^n\|_{L^1(K)}$ in (4.18). We have that

$$\begin{aligned}\sum_{n=0}^{N-1}\Delta t\sum_{K\in\mathcal{K}_h} h_K\|\nabla u_h^n\|_{L^1(K)} &= \sum_{n=0}^{N-1}\Delta t\sum_{K\in\mathcal{K}_h} h_K\int_K|\nabla u_h^n|\,\mathrm{d}\boldsymbol{x}\\ &\le \left(\sum_{n=0}^{N-1}\Delta t\sum_{K\in\mathcal{K}_h} h_K|K|\right)^{\frac12}\left(\sum_{n=0}^{N-1}\Delta t\sum_{K\in\mathcal{K}_h} h_K\int_K|\nabla u_h^n|^2\,\mathrm{d}\boldsymbol{x}\right)^{\frac12}\\ &\le c h^{\frac12}\beta^{-\frac12}T^{\frac12}|\Omega|^{\frac12}(\|u_h^0\|_{L^2(\Omega)}^2 - \|\overline{u}_h^0\|_{L^2(\Omega)}^2)^{\frac12},\end{aligned}$$

which in turn implies that

$$\Lambda^* \le \frac{c}{\epsilon}|\Omega|h^{\frac12}(\beta T)^{\frac12}(\|u_h^0\|_{L^2(\Omega)}^2 - \|\overline{u}_h^0\|_{L^2(\Omega)}^2)^{\frac12}.$$

Then we have

$$\|u(\cdot,T)-\tilde{u}_h(\cdot,T)\|_{L^1(\Omega)} \le c\left((\epsilon+h)|u_0|_{BV(\Omega)} + \frac{|\Omega|h^{\frac12}(\beta T)^{\frac12}}{\epsilon}(\|u_h^0\|_{L^2(\Omega)}^2 - \|\overline{u}_h^0\|_{L^2(\Omega)}^2)^{\frac12}\right),$$

which after optimizing ϵ gives

$$\|u(\cdot,T)-\tilde{u}_h(\cdot,T)\|_{L^1(\Omega)} \le c\,h^{\frac14}|\Omega|^{\frac12}(\beta T)^{\frac14}|u_0|_{BV(\Omega)}^{\frac12}(\|u_h^0\|_{L^2(\Omega)}^2 - \|\overline{u}_h^0\|_{L^2(\Omega)}^2)^{\frac14}$$

Recalling the definition $|u_h^0|_* := (\|u_h^0\|_{L^2(\Omega)}^2 - \|\overline{u}_h^0\|_{L^2(\Omega)}^2)^{\frac12}$, we finally obtain

$$\|u(\cdot,T)-\tilde{u}_h(\cdot,T)\|_{L^1(\Omega)} \le c\,h^{\frac14}|\Omega|^{\frac12}(\beta T)^{\frac14}|u_0|_{BV(\Omega)}^{\frac12}|u_h^0|_*^{\frac12}.$$

This concludes the proof. □

Remark 4.3. (Higher-order approximation) The method described in this paper (see (3.7)) is only first-order, but the proposed methodology can be modified to make it formally second-order as shown in Guermond et al. [14]. The main idea consists of combining the present first-order method and a high-order entropy viscosity method (see Guermond et al. [13]) by using the flux correction technique of Boris–Book–Zalesak, (see Boris and Book [1], Zalesak [19]).

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Appendix A. Kružkov estimates revisited. We revisit general results established in Proposition 3.1 from Cockburn et al. [7], Lemma 3.1 from Cockburn and Gremaud [5], Proposition 5.3 from Cockburn and Gremaud [6], and Proposition 3.18 from Holden and Risebro [16]. The route that we follow consists of reformulating Kuznecov's Lemma (see Lemma 2, p. 1492, in Kuznecov [18]) in the spirit of Bouchut and Perthame [2, Thm 2.1] using a Gronwall type argument from [5, Prop 6.2] and [6, Lemma 5.4]. Our objective is to reduce the establishing of an a priori estimate to that of entropy inequalities using only the Kružkov entropy family, i.e., we do not want to invoke smooth entropies and to deal with the associated loss of symmetry of the entropy flux. Theorem 2.1 from [2] is not sufficient for this purpose since it requires an a priori bound on the BV-norm of the approximate solution. The results [7, Proposition 3.1], [5, Lemma 3.1] and [6, Proposition 5.3] are not appropriate either since they mix the error estimation with the proof of the entropy inequalities, making the technique very difficult to follow and to apply (at least to us). The main result of this section is Lemma A.2.

We introduce $\delta > 0$ and $\epsilon = \beta\delta$, and we define two mollifiers ω_δ and $\boldsymbol{\omega}_\epsilon$

$$\omega_\delta(t) := \begin{cases} \frac{1}{3\delta} & |t| \le \delta, \\ \frac{2\delta - |t|}{3\delta^2} & \delta \le |t| \le 2\delta, \\ 0 & \text{otherwise,} \end{cases} \qquad \boldsymbol{\omega}_\epsilon(\boldsymbol{x}) := \Pi_{l=1}^d \omega_\epsilon(x_l), \quad \boldsymbol{x} := (x_1, \ldots, x_d). \tag{A.1}$$

Now, following an idea of Kružkov [17] we define

$$\phi(\boldsymbol{x}, \boldsymbol{y}, t, s) := \boldsymbol{\omega}_\epsilon(\boldsymbol{x} - \boldsymbol{y})\omega_\delta(t - s), \qquad \forall (\boldsymbol{y}, s) \in D \times [0, T]. \tag{A.2}$$

Moreover, as done in Cockburn and Gremaud [6, 5], we set $\Gamma_\delta(t) := \int_0^t \omega_\delta(s)\,\mathrm{d}s$.

Lemma A.1. *There $c > 0$, uniform, such that the following holds for all $t \in [0, T]$:*

$$\int_0^T |\omega_\delta'(s - t)|\,\mathrm{d}s \le c\,\frac{\Gamma_\delta(T)}{\delta}, \tag{A.3}$$

$$\frac{1}{2}\Gamma_\delta(T) \le \int_0^T \omega_\delta(s - t)\,\mathrm{d}s \le 2\,\Gamma_\delta(T). \tag{A.4}$$

Proof. It can be shown that $\delta \int_0^T |\omega_\delta'(s - t)|\,\mathrm{d}s \le 2 \int_0^T |\omega_{2\delta}(s - t)|\,\mathrm{d}s$, which in turn implies that

$$\begin{aligned} \delta \int_0^T |\omega_\delta'(s - t)|\,\mathrm{d}s &\le 2\left(\int_0^t \omega_{2\delta}(s - t)\,\mathrm{d}s + \int_t^T \omega_{2\delta}(s - t)\,\mathrm{d}s \right) \\ &\le 2(\Gamma_{2\delta}(t) + \Gamma_{2\delta}(T - t)) \le 4\Gamma_{2\delta}(T). \end{aligned}$$

We conclude by showing that there is a uniform constant c so that $\Gamma_{2\delta}(T) \le c\Gamma_\delta(T)$. The details are omitted. This proves (A.3). The two inequalities in (A.4) are a consequence of $\int_0^T \omega_\delta(s - t)\,\mathrm{d}s = \Gamma_\delta(t) + \Gamma_\delta(T - t)$ and $\int_{\frac{T}{2}}^T \omega_\delta(s)\,\mathrm{d}s \le \int_0^{\frac{T}{2}} \omega_\delta(s)\,\mathrm{d}s$. ∎

Lemma A.2. *Assume (2.2) and $u_0 \in BV(\Omega)$. Let $\widetilde{u}_h : D \times [0, T] \longrightarrow \mathbb{R}$ be an approximate solution of (2.1) as defined in §4.1 with $T \in [0, T_{\max}]$. Assume that* *the following holds for all $k \in [u_{\min}, u_{\max}]$ and all non-negative Lipschitz function ψ compactly supported in $D\times[0,T]$:*

$$
\begin{aligned}
\text{(A.5)}\quad & -\int_0^T\!\!\int_D \big(|\widetilde{u}_h - k|\partial_t\psi + \operatorname{sgn}(\widetilde{u}_h - k)(\boldsymbol{f}(\widetilde{u}_h) - \boldsymbol{f}(k))\cdot\nabla\psi\big)\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t \\
&+ \|\pi_h\big((\widetilde{u}_h(\cdot,T) - k)\bar{\pi}_h\psi(\cdot,\mathcal{T}_h)\big)\|_{\ell^1_h} - \|\pi_h\big((\widetilde{u}_h(\cdot,0) - k)\bar{\pi}_h\psi(\cdot,\sigma_h)\big)\|_{\ell^1_h} \le \Lambda_T(\psi),
\end{aligned}
$$

where $\|\cdot\|_{\ell^1_h}$ is defined in (2.7), $|T - \mathcal{T}_h| \le \gamma\Delta t$, $|0 - \sigma_h| \le \gamma\Delta t$, *where $\gamma > 0$ is a uniform constant, and $\Lambda_T(\psi)$ is a bounded functional on Lipschitz functions. Then the following estimate holds*

$$
\text{(A.6)}\quad \|u(\cdot,T) - \tilde{u}_h(\cdot,T)\|_{L^1(\Omega)} \le c\,(\|u_0 - u_h^0\|_{L^1(\Omega)} + (\epsilon + h + \beta\Delta t)|u_0|_{BV(\Omega)} + \Lambda^*).
$$

where $\Lambda^* := \sup_{0\le\widetilde{T}\le T} \frac{\int_0^{\widetilde{T}}\int_D \Lambda_{\widetilde{T}}(\phi)\,\mathrm{d}\boldsymbol{y}\,\mathrm{d}s}{\Gamma_\delta(\widetilde{T})}$ *and ϕ is defined in* (A.2).

Proof. Following the work of Kružkov [17] and Kuznecov [18], we are going to establish the error estimate by using the technique of the doubling of the variables. Let $(\boldsymbol{y}, s) \in D\times[0,T]$ and let us set $k = u(\boldsymbol{y}, s)$ in (A.5), note that this is legitimate since $u_{\min} \le u(\boldsymbol{y}, t) \le u_{\max}$, then (A.5) implies that

$$
\begin{aligned}
&-\int_0^T\!\!\int_D \big(|\widetilde{u}_h - u(\boldsymbol{y},s)|\partial_t\psi + \operatorname{sgn}(\widetilde{u}_h - u(\boldsymbol{y},s))(\boldsymbol{f}(\widetilde{u}_h) - \boldsymbol{f}(u(\boldsymbol{y},s)))\cdot\nabla\psi\big)\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t \\
&\qquad\qquad + \|\pi_h\big((\widetilde{u}_h(\cdot,T)) - u(\boldsymbol{y},s))\bar{\pi}_h\psi(\cdot,\tau)\big)\|_{\ell^1_h}\big|_{\tau=\sigma_h}^{\tau=\mathcal{T}_h} \le \Lambda_T(\psi).
\end{aligned}
$$

Now we introduce $\epsilon > 0$, $\delta := \epsilon/\beta$, and we set $\psi(\boldsymbol{x}, t) = \boldsymbol{\omega}_\epsilon(\boldsymbol{x} - \boldsymbol{y})\omega_\epsilon(t - s)$ where $\boldsymbol{\omega}_\epsilon$ and ω_δ are the two mollifiers introduced in (A.1). We now select ψ so that $\psi(\boldsymbol{x}, t) := \phi(\boldsymbol{x}, \boldsymbol{y}, t, s)$ where the function ϕ has been defined in (A.2). From now on we replace $\Lambda_T(\psi)$ by $\Lambda_T(\phi)$ to account for the presence of the two new parameters $(\boldsymbol{y}, s) \in D\times[0,T]$. We now integrate with respect to $(\boldsymbol{y}, s)$ over $D\times[0,T]$,

$$
\begin{aligned}
&-\int_0^T\!\!\int_D\!\int_0^T\!\!\int_D \big(|\widetilde{u}_h - u|\partial_t\phi + \operatorname{sgn}(\widetilde{u}_h - u)(\boldsymbol{f}(\widetilde{u}_h) - \boldsymbol{f}(u))\cdot\nabla_{\boldsymbol{x}}\phi\big)\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t\,\mathrm{d}\boldsymbol{y}\,\mathrm{d}s \\
&+\int_0^T\!\!\int_D \|\pi_h\big((\widetilde{u}_h(\cdot,T) - u(\boldsymbol{y},s))\bar{\pi}_h\phi(\cdot,\boldsymbol{y},\mathcal{T}_h,s)\big)\|_{\ell^1_h}\mathrm{d}\boldsymbol{y}\,\mathrm{d}s\big|_{\tau=\sigma_h}^{\tau=\mathcal{T}_h} \le \int_0^T\!\!\int_D \Lambda_T(\phi)\,\mathrm{d}\boldsymbol{y}\,\mathrm{d}s.
\end{aligned}
$$

Moreover, u being the entropy solution to (2.1) implies that

$$
\begin{aligned}
&-\int_0^T\!\!\int_D \big(|k - u(\boldsymbol{y},s)|\partial_s\theta + \operatorname{sgn}(k - u(\boldsymbol{y},s))(\boldsymbol{f}(k) - \boldsymbol{f}(u(\boldsymbol{y},s)))\cdot\nabla_{\boldsymbol{y}}\psi\big)\,\mathrm{d}\boldsymbol{y}\,\mathrm{d}s \\
&\qquad\qquad + \|(k - u(\cdot,T))\theta(\cdot,T)\|_{L^1(D)} - \|(k - u(\cdot,0))\theta(\cdot,0)\|_{L^1(D)} \le 0,
\end{aligned}
$$

for any $\theta \in W_c^{1,\infty}(D\times[0,T];\mathbb{R}^+)$. Now we choose $\theta(\boldsymbol{y}, s) := \boldsymbol{\omega}_\epsilon(\boldsymbol{x} - \boldsymbol{y})\omega_\delta(t - s) = \phi(\boldsymbol{x}, \boldsymbol{y}, t, s)$ and $k := \widetilde{u}_h(\boldsymbol{x}, t)$ where $(\boldsymbol{x}, t) \in D\times[0,T]$, and we integrate with respect to $(\boldsymbol{x}, t)$ over $D\times[0,T]$,

$$
\begin{aligned}
&-\int_0^T\!\!\int_D\!\int_0^T\!\!\int_D \big(|\widetilde{u}_h - u|\partial_s\phi + \operatorname{sgn}(\widetilde{u}_h - u)(\boldsymbol{f}(\widetilde{u}_h) - \boldsymbol{f}(u))\cdot\nabla_{\boldsymbol{y}}\phi\big)\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t\,\mathrm{d}\boldsymbol{y}\,\mathrm{d}s \\
&\qquad\qquad \int_0^T\!\!\int_D \|(\widetilde{u}_h(\boldsymbol{x},t) - u(\cdot,\tau))\phi(\boldsymbol{x},\cdot,t,\tau)\|_{L^1(D)}\big|_{\tau=0}^{\tau=T}\mathrm{d}\boldsymbol{x}\,\mathrm{d}t \le 0.
\end{aligned}
$$

Upon observing that $\phi_t = -\phi_s$ and $\nabla_{\boldsymbol{x}}\phi = -\nabla_{\boldsymbol{y}}\phi$, (*this is the decisive observation*), the above arguments imply that

$$E_1(\mathcal{T}_h) - E_1(\sigma_h) + E_2(T) - E_2(0) \le \int_0^T\!\!\int_D \Lambda_T(\phi)\,\mathrm{d}\boldsymbol{y}\,\mathrm{d}s,$$

where $E_1(\sigma_h)$, $E_1(\mathcal{T}_h)$ and $E_2(\tau)$, $\tau\in\{0,T\}$, are defined as follows:

$$\begin{aligned}
E_1(\sigma_h) &:= \int_0^T\!\!\int_D \|\pi_h\big((\widetilde{u}_h(\cdot,0)-u(\boldsymbol{y},s))\bar{\pi}_h\phi(\cdot,\boldsymbol{y},\sigma_h,s)\big)\|_{\ell^1_h}\mathrm{d}\boldsymbol{y}\,\mathrm{d}s,\\
E_1(\mathcal{T}_h) &:= \int_0^T\!\!\int_D \|\pi_h\big((\widetilde{u}_h(\cdot,T)-u(\boldsymbol{y},s))\bar{\pi}_h\phi(\cdot,\boldsymbol{y},\mathcal{T}_h,s)\big)\|_{\ell^1_h}\mathrm{d}\boldsymbol{y}\,\mathrm{d}s,\\
E_2(\tau) &:= \int_0^T\!\!\int_D \|(\widetilde{u}_h(\boldsymbol{x},t)-u(\cdot,\tau))\phi(\boldsymbol{x},\cdot,t,\tau)\|_{L^1(D)}\mathrm{d}\boldsymbol{x}\,\mathrm{d}t.
\end{aligned}$$

We are going to estimate $E_1(\sigma_h)$ and $E_1(\mathcal{T}_h)$ by invoking the decomposition $\widetilde{u}_h(\boldsymbol{a}_i,0)-u(\boldsymbol{y},s) = \widetilde{u}_h(\boldsymbol{a}_i,0)-\bar{\pi}_h u(\boldsymbol{a}_i,0)+\bar{\pi}_h u(\boldsymbol{a}_i,0)-u(\boldsymbol{y},0)+u(\boldsymbol{y},0)-u(\boldsymbol{y},s)$. For $E_1(\sigma_h)$ we are going to use $|\widetilde{u}_h(\boldsymbol{a}_i,0)-u(\boldsymbol{y},s)| \le |\widetilde{u}_h(\boldsymbol{a}_i,0)-\bar{\pi}_h u(\boldsymbol{a}_i,0)| + |\bar{\pi}_h u(\boldsymbol{a}_i,0)-u(\boldsymbol{y},0)|+|u(\boldsymbol{y},0)-u(\boldsymbol{y},s)|$ and for $E_1(\mathcal{T}_h)$ we are going to use $|\widetilde{u}_h(\boldsymbol{a}_i,0)-u(\boldsymbol{y},s)| \ge |\widetilde{u}_h(\boldsymbol{a}_i,0)-\bar{\pi}_h u(\boldsymbol{a}_i,0)| - |\bar{\pi}_h u(\boldsymbol{a}_i,0)-u(\boldsymbol{y},0)|-|u(\boldsymbol{y},0)-u(\boldsymbol{y},s)|$, yielding for both cases

$$E_{11}(\mathcal{T}_h)-E_{12}(\mathcal{T}_h)-E_{13}(\mathcal{T}_h)\le E_1(\mathcal{T}_h),\qquad E_1(\sigma_h)\le E_{11}(\sigma_h)+E_{12}(\sigma_h)+E_{13}(\sigma_h).$$

We start by estimating $E_{11}(\sigma_h)$, $E_{12}(\sigma_h)$ and $E_{13}(\sigma_h)$. Using the definition of the ℓ^1_h-norm and that of the operator $\bar{\pi}_h$, we deduce that

$$\begin{aligned}
E_{11}(\sigma_h) &:= \int_0^T\!\!\int_D \sum_{i=1}^I m_i|\widetilde{u}_h(\boldsymbol{a}_i,0)-\bar{\pi}_h u(\boldsymbol{a}_i,0)|\frac{1}{m_i}\int_D \phi(\boldsymbol{z},\boldsymbol{y},\sigma_h,s)\varphi_i(\boldsymbol{z})\,\mathrm{d}\boldsymbol{z}\,\mathrm{d}\boldsymbol{y}\,\mathrm{d}s\\
&= \sum_{i=1}^I |\widetilde{u}_h(\boldsymbol{a}_i,0)-\bar{\pi}_h u(\boldsymbol{a}_i,0)|\int_0^T\!\!\int_D \left(\int_D \phi(\boldsymbol{z},\boldsymbol{y},\sigma_h,s)\,\mathrm{d}\boldsymbol{y}\right)\varphi_i(\boldsymbol{z})\,\mathrm{d}\boldsymbol{z}\,\mathrm{d}s\\
&= \sum_{i=1}^I m_i|\widetilde{u}_h(\boldsymbol{a}_i,0)-\bar{\pi}_h u(\boldsymbol{a}_i,0)|\int_0^T \omega_\delta(s-\sigma_h)\,\mathrm{d}s = \Gamma_{\delta,\sigma_h}(T)\|e(\cdot,0)\|_{\ell^1_h}.
\end{aligned}$$

where we have defined $e(\boldsymbol{x},\tau):=\widetilde{u}_h(\boldsymbol{x},\tau)-\bar{\pi}_h u(\boldsymbol{x},\tau)$ and $\Gamma_{\delta,\tau}(T):=\int_0^T\omega_\delta(s-\tau)\,\mathrm{d}t$ for any $\tau\ge 0$. Lemma A.1 implies that

$$E_{11}(\sigma_h)\le 2\Gamma_\delta(T)\|e(\cdot,0)\|_{\ell^1_h}.$$

We now estimate $E_{12}(\sigma_h)$ as follows

$$\begin{aligned}
E_{12}(\sigma_h) &:= \int_0^T\!\!\int_D \sum_{i=1}^I m_i|\bar{\pi}_h u(\boldsymbol{a}_i,0)-u(\boldsymbol{y},0)|\frac{1}{m_i}\int_D \phi(\boldsymbol{z},\boldsymbol{y},\sigma_h,s)\varphi_i(\boldsymbol{z})\,\mathrm{d}\boldsymbol{z}\,\mathrm{d}\boldsymbol{y}\,\mathrm{d}s\\
&= \Gamma_{\delta,\sigma_h}(T)\int_D\sum_{i=1}^I\left|\int_D (u(\boldsymbol{w},0)-u(\boldsymbol{y},0))\varphi_i(\boldsymbol{w})\,\mathrm{d}\boldsymbol{w}\right|\frac{1}{m_i}\int_D \boldsymbol{\omega}_\epsilon(\boldsymbol{z}-\boldsymbol{y})\varphi_i(\boldsymbol{z})\,\mathrm{d}\boldsymbol{z}\,\mathrm{d}\boldsymbol{y}\\
&= \Gamma_{\delta,\sigma_h}(T)\int_D\sum_{i=1}^I\frac{1}{m_i}\left|\int_D\int_D (u(\boldsymbol{w},0)-u(\boldsymbol{y},0))\varphi_i(\boldsymbol{w})\boldsymbol{\omega}_\epsilon(\boldsymbol{z}-\boldsymbol{y})\varphi_i(\boldsymbol{z})\,\mathrm{d}\boldsymbol{z}\,\mathrm{d}\boldsymbol{w}\right|\,\mathrm{d}\boldsymbol{y}.
\end{aligned}$$

The triangle inequality yields

$$E_{12}(\sigma_h) \le 2\Gamma_\delta(T)\int_D \sum_{i=1}^{I} \frac{1}{m_i}\int_D\int_D |u(\boldsymbol{w},0)-u(\boldsymbol{z},0)|\,\varphi_i(\boldsymbol{w})\boldsymbol{\omega}_\epsilon(\boldsymbol{z}-\boldsymbol{y})\varphi_i(\boldsymbol{z})\,\mathrm{d}\boldsymbol{z}\,\mathrm{d}\boldsymbol{w}\,\mathrm{d}\boldsymbol{y}$$
$$+2\Gamma_\delta(T)\int_D \sum_{i=1}^{I} \frac{1}{m_i}\int_D\int_D |u(\boldsymbol{z},0)-u(\boldsymbol{y},0)|\,\varphi_i(\boldsymbol{w})\boldsymbol{\omega}_\epsilon(\boldsymbol{z}-\boldsymbol{y})\varphi_i(\boldsymbol{z})\,\mathrm{d}\boldsymbol{z}\,\mathrm{d}\boldsymbol{w}\,\mathrm{d}\boldsymbol{y}.$$

Let us denote by $\Gamma_\delta(T)E^1_{12}(0)$ and $\Gamma_\delta(T)E^2_{12}(0)$ the two terms in the right-hand side of the above inequality. For any fixed $\tau \ge 0$, a standard approximation result on BV functions gives

$$E^1_{12}(\tau) \le \sum_{i=1}^{I}\frac{1}{m_i}\int_{S_i}\int_{S_i}|u(\boldsymbol{w},\tau)-u(\boldsymbol{z},\tau)|\varphi_i(\boldsymbol{w})\varphi_i(\boldsymbol{z})\,\mathrm{d}\boldsymbol{z}\,\mathrm{d}\boldsymbol{w}$$
$$\le \sum_{i=1}^{I}\frac{1}{m_i}\int_{S_i}\int_{S_i}|u(\boldsymbol{w},\tau)-u(\boldsymbol{z},\tau)|\,\mathrm{d}\boldsymbol{z}\,\mathrm{d}\boldsymbol{w} \le \sum_{i=1}^{I}\frac{1}{m_i}\sqrt{d}|Q_i|^{1+\frac{1}{d}}|u(\cdot,\tau)|_{BV(Q_i)},$$

where Q_i is the smallest cube that contains S_i, see Cohen et al. [8, (2.13)]. Moreover, the mesh regularity assumption implies that there is a uniform constant c such that $|Q_i| \le c\,m_i$ and $|Q_i|^{\frac{1}{d}} \le c\,h$, thereby implying that

$$E^1_{12}(\tau) \le ch\sum_{i=1}^{I}|u(\cdot,\tau)|_{BV(Q_i)} \le c'h|u(\cdot,\tau)|_{BV(\Omega)}.$$

We finally obtain $E^1_{12}(\tau) \le ch|u_0|_{BV(\Omega)}$, since it is known that $|u(\cdot,\tau)|_{BV(\Omega)} \le |u_0|_{BV(\Omega)}$ for any $\tau \ge 0$. Let us now estimate $E^2_{12}(\tau)$. We have

$$E^2_{12}(\tau) = \int_D \sum_{i=1}^{I}\left|\int_D (u(\boldsymbol{z},\tau)-u(\boldsymbol{y},\tau))\boldsymbol{\omega}_\epsilon(\boldsymbol{z}-\boldsymbol{y})\varphi_i(\boldsymbol{z})\,\mathrm{d}\boldsymbol{z}\right|\,\mathrm{d}\boldsymbol{y}$$
$$\le \int_D \sum_{i=1}^{I}\int_D |u(\boldsymbol{z},\tau)-u(\boldsymbol{y},\tau)|\boldsymbol{\omega}_\epsilon(\boldsymbol{z}-\boldsymbol{y})\varphi_i(\boldsymbol{z})\,\mathrm{d}\boldsymbol{z}\,\mathrm{d}\boldsymbol{y}$$
$$= \int_{D\times D}|u(\boldsymbol{z},\tau)-u(\boldsymbol{y},\tau)|\boldsymbol{\omega}_\epsilon(\boldsymbol{z}-\boldsymbol{y})\,\mathrm{d}\boldsymbol{z}\,\mathrm{d}\boldsymbol{y} = \int_{D\times D}|u(\boldsymbol{z},\tau)-u(\boldsymbol{z}-\boldsymbol{w},\tau)|\boldsymbol{\omega}_\epsilon(\boldsymbol{w})\,\mathrm{d}\boldsymbol{z}\,\mathrm{d}\boldsymbol{w}$$
$$= \int_D \boldsymbol{\omega}_\epsilon(\boldsymbol{w})\sup_{\|\boldsymbol{y}\|_{\ell^\infty}\le 2\epsilon}\int_D |u(\boldsymbol{z},\tau)-u(\boldsymbol{z}-\boldsymbol{y},\tau)|\,\mathrm{d}\boldsymbol{z}\,\mathrm{d}\boldsymbol{w} \le c\epsilon|u(\cdot,\tau)|_{BV(\Omega)} \le c'\epsilon|u_0|_{BV(\Omega)}.$$

We infer that $E_{12}(\tau) \le \Gamma_\delta(T)c(\epsilon+h)|u_0|_{BV(\Omega)}$ for any $\tau \ge 0$. Next, we estimate $E_{13}(\sigma_h)$ by invoking the Lipschitz continuity in time of the exact solution u, i.e., $\|u(\cdot,t)-u(\cdot,s)\|_{L^1(D)} \le \beta|u_0|_{BV(\Omega)}|t-s|$ (see Holden and Risebro [16, Thm 2.14]):

$$E_{13}(\sigma_h) := \int_0^T\int_D \sum_{i=1}^{I} m_i|u(\boldsymbol{y},0)-u(\boldsymbol{y},s)|\frac{1}{m_i}\int_D \phi(\boldsymbol{z},\boldsymbol{y},\sigma_h,s)\varphi_i(\boldsymbol{z})\,\mathrm{d}\boldsymbol{z}\,\mathrm{d}\boldsymbol{y}\,\mathrm{d}s$$
$$= \int_0^T\int_D |u(\boldsymbol{y},0)-u(\boldsymbol{y},s)|\omega_\delta(\sigma_h-s)\,\mathrm{d}\boldsymbol{y}\,\mathrm{d}s$$
$$= \int_0^T \|u(\cdot,0)-u(\cdot,s)\|_{L^1(D)}\omega_\delta(\sigma_h-s)\,\mathrm{d}s \le 2\Gamma_\delta(T)\beta(2\delta+\gamma\Delta t)|u_0|_{BV(\Omega)}.$$

Note that here we used the assumption $\sigma_h \le \gamma \Delta t$ and $|s - \sigma_h| \le 2\delta$. In conclusion,

$$E_1(\sigma_h) \le E_{11}(\sigma_h) + c\,\Gamma_\delta(T)(h + \beta\Delta t + \epsilon)|u_0|_{BV(\Omega)}, \tag{A.7}$$

where recall that $E_{11}(\sigma_h) \le 2\Gamma_\delta(T)\|e(\cdot,0)\|_{\ell^1_h}$.

We estimate the term $E_1(\mathcal{T}_h)$ in the same way as we did for $E_1(\sigma_h)$ and we obtain the following bound:

$$E_{11}(\mathcal{T}_h) - c\,\Gamma_\delta(T)(h + \beta\Delta t + \epsilon)|u_0|_{BV(\Omega)} \le E_1(\mathcal{T}_h), \tag{A.8}$$

where $E_{11}(\mathcal{T}_h) := \Gamma_{\delta,\mathcal{T}_h}(T)\|e(\cdot,T)\|_{\ell^1_h}$, and $\frac{1}{2}\Gamma_\delta(T)\|e(\cdot,T)\|_{\ell^1_h} \le E_{11}(\mathcal{T}_h)$ owing to Lemma A.1 again.

We now estimate $E_2(\tau)$ for $\tau \in \{0, T\}$ by invoking the decomposition $\widetilde{u}_h(\boldsymbol{x},t) - u(\boldsymbol{y},\tau) = \widetilde{u}_h(\boldsymbol{x},t) - \bar{\pi}_h u(\boldsymbol{x},t) + \bar{\pi}_h u(\boldsymbol{x},t) - u(\boldsymbol{y},t) + u(\boldsymbol{y},t) - u(\boldsymbol{y},\tau)$ and by applying the triangle inequality: $|E_2(\tau) - E_{21}(\tau)| \le E_{22}(\tau) + E_{23}(\tau)$. For $E_2(T)$ we are going to use $E_2(T) \ge E_{21}(T) - E_{22}(T) - E_{23}(T)$ and for $E_2(0)$ we are going to use $E_2(0) \le E_{21}(T) + E_{22}(T) + E_{23}(T)$. The definition of $E_{21}(\tau)$ implies that

$$\begin{aligned} E_{21}(\tau) &:= \int_0^T\!\!\int_D |\widetilde{u}_h(\boldsymbol{x},t) - \bar{\pi}_h u(\boldsymbol{x},t)| \int_D \phi(\boldsymbol{x},\boldsymbol{y},t,\tau)\,\mathrm{d}\boldsymbol{y}\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t \\ &= \int_0^T \|e(\cdot,t)\|_{L^1(D)} w_\delta(\tau - t)\,\mathrm{d}t. \end{aligned}$$

We now estimate $E_{22}(\tau)$

$$\begin{aligned} E_{22}(\tau) &:= \int_0^T\!\!\int_D\int_D |\bar{\pi}_h u(\boldsymbol{x},t) - u(\boldsymbol{y},t)|\phi(\boldsymbol{x},\boldsymbol{y},t,\tau)\,\mathrm{d}\boldsymbol{y}\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t \\ &\le \int_0^T\!\!\int_D\int_D (|\bar{\pi}_h u(\boldsymbol{x},t) - u(\boldsymbol{x},t)| + |u(\boldsymbol{x},t) - u(\boldsymbol{y},t)|)\phi(\boldsymbol{x},\boldsymbol{y},t,\tau)\,\mathrm{d}\boldsymbol{y}\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t \\ &\le c\,\Gamma_\delta(T) h \sup_{0\le t\le T} |u(\cdot,t)|_{BV(D)} + \Gamma_\delta(T) \sup_{0\le t\le T} \int_{D\times D} |u(\boldsymbol{x},t) - u(\boldsymbol{x}-\boldsymbol{w},t)|\boldsymbol{\omega}_\epsilon(\boldsymbol{w})\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}\boldsymbol{w} \\ &\le c\,\Gamma_\delta(T)(\epsilon + h)|u_0|_{BV(\Omega)}. \end{aligned}$$

We finish with $E_{23}(\tau)$, and we have

$$\begin{aligned} E_{23}(\tau) &:= \int_0^T\!\!\int_D\int_D |u(\boldsymbol{y},t) - u(\boldsymbol{y},\tau)|\phi(\boldsymbol{x},\boldsymbol{y},t,\tau)\,\mathrm{d}\boldsymbol{y}\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t \\ &= \int_0^T\!\!\int_D \sum_{i=1}^I m_i |u(\boldsymbol{y},t) - u(\boldsymbol{y},\tau)| \frac{1}{m_i}\int_D \phi(\boldsymbol{z},\boldsymbol{y},\tau,s)\varphi_i(\boldsymbol{z})\,\mathrm{d}\boldsymbol{z}\,\mathrm{d}\boldsymbol{y}\,\mathrm{d}s \\ &= \int_0^T\!\!\int_D |u(\boldsymbol{y},\tau) - u(\boldsymbol{y},s)|\omega_\delta(\tau - s)\,\mathrm{d}\boldsymbol{y}\,\mathrm{d}s \\ &= \int_0^T \|u(\cdot,\tau) - u(\cdot,s)\|_{L^1(D)}\omega_\delta(\tau - s)\,\mathrm{d}s \le 2\Gamma_\delta(T)\beta\delta|u_0|_{BV(\Omega)} \end{aligned}$$

In conclusion $|E_2(\tau) - E_{21}(\tau)| \le c\,\Gamma_\delta(T)(h+\epsilon)|u_0|_{BV(\Omega)}$, thereby implying that

$$\int_0^T \|e(\cdot,t)\|_{L^1(D)} w_\delta(T-t)\,\mathrm{d}t - c\,\Gamma_\delta(T)(h+\epsilon)|u_0|_{BV(\Omega)} \le E_2(T), \tag{A.9}$$

$$E_2(0) \le \int_0^T \|e(\cdot,t)\|_{L^1(D)} w_\delta(t)\,\mathrm{d}t + c\,\Gamma_\delta(T)(h+\epsilon)|u_0|_{BV(\Omega)}. \tag{A.10}$$

We now combine all the above estimates for $E_1(\sigma_h)$, $E_1(\mathcal{T}_h)$, $E_2(0)$, and $E_2(T)$, i.e., (A.7), (A.8), (A.9), (A.10), to infer that

$$\frac{1}{2}\Gamma_\delta(T)\|e(\cdot,T)\|_{\ell^1_h} + \int_0^T \|e(\cdot,t)\|_{L^1(D)}\omega_\delta(T-t)\,\mathrm{d}t \le 2\Gamma_\delta(T)\|e(\cdot,0)\|_{\ell^1_h}$$
$$+ \int_0^T \|e(\cdot,t)\|_{L^1(D)}\omega_\delta(t)\,\mathrm{d}t + c\,\Gamma_\delta(T)(\epsilon+\beta\Delta t+h)|u_0|_{BV(\Omega)} + \int_0^T\!\!\int_D \Lambda_T(\phi)\,\mathrm{d}\boldsymbol{y}\,\mathrm{d}s.$$

Since $e(\cdot,t)\in X_h$, $0\le t\le T$, and the discrete norm $\|\cdot\|_{\ell^1_h}$ is equivalent to the L^1-norm, there are uniform constants $a, a'>0$ such that $a\|e(\cdot,T)\|_{L^1(D)} \le \frac{1}{2}\|e(\cdot,T)\|_{\ell^1_h}$ and $2\|e(\cdot,0)\|_{\ell^1_h} \le a'\|e(\cdot,0)\|_{L^1(D)}$, thereby implying that

$$a\,\Gamma_\delta(T)\|e(\cdot,T)\|_{L^1(D)} + \int_0^T \|e(\cdot,t)\|_{L^1(D)}\omega_\delta(T-t)\,\mathrm{d}t \le \int_0^T \|e(\cdot,t)\|_{L^1(D)}\omega_\delta(t)\,\mathrm{d}t,$$
$$\Gamma_\delta(T)\left(a'\|e(\cdot,0)\|_{L^1(D)} + c(\epsilon+\beta\Delta t+h)|u_0|_{BV(\Omega)} + \Lambda^*\right),$$

where recall that we have defined $\Lambda^* := \sup_{0\le\widetilde{T}\le T} \frac{\int_0^{\widetilde{T}}\int_D \Lambda_{\widetilde{T}}(\phi)\,\mathrm{d}\boldsymbol{y}\,\mathrm{d}s}{\Gamma_\delta(\widetilde{T})}$. Using Lemma A.3 with $\theta(t) = \|e(t)\|_{L^1(D)}$ and $b = a'\,\|e(\cdot,0)\|_{L^1(D)} + c\,(\epsilon+\beta\Delta t+h)|u_0|_{BV(\Omega)} + \Lambda^*$, we finally conclude that

$$e(T) \le \max(1,a',c)\,c(a)(\|e(\cdot,0)\|_{L^1(D)} + (\epsilon+\beta\Delta t+h)|u_0|_{BV(\Omega)} + \Lambda^*).$$

This completes the proof. □

The following lemma, which is a Gronwall-type estimate, is inspired from an argument invoked in Cockburn and Gremaud [5, Prop 6.2], Cockburn and Gremaud [6, Lemma 5.4], (see also Holden and Risebro [16, Lemma 3.17]). This result is essential to complete the proof of Lemma A.2.

Lemma A.3 (Gronwall). *Let $\theta : [0,T_{\max}] \longrightarrow \mathbb{R}_+$ be a non-negative bounded function and assume that there exist $a>0$ and $b>0$ such that the following holds for all $T\in[0,T_{\max}]$:*

$$a\,\Gamma_\delta(T)\theta(T) + \int_0^T \theta(\tau)\omega_\delta(T-\tau)\,\mathrm{d}\tau \le b\,\Gamma_\delta(T) + \int_0^T \theta(\tau)\omega_\delta(\tau)\,\mathrm{d}\tau, \tag{A.11}$$

then there is $c(a)$ such that $\theta(T)\le bc(a)$ for all $T\in[0,T_{\max}]$ and all $\delta>0$.

Proof. We consider three cases: $T\in[0,\delta]$, $T\in(\delta,2\delta]$ and $T>2\delta$. Assume first that $T\in[0,\delta]$. The definition of the kernel ω_δ implies that $\omega(t)=\omega(T-t)$ for all $t\in[0,T]$. As a result, (A.11) implies that $\theta(T)\le\frac{b}{a}$ if $T\in[0,\delta]$. Assume now that $T\in(\delta,2\delta]$. Then observing that $\frac{1}{3}\le\Gamma_\delta(T)\le\frac{1}{2}$, we have

$$\begin{aligned}\frac{a}{3}\theta(T) - \frac{b}{2} &\le \int_0^T \theta(\tau)\left(\omega_\delta(\tau)-\omega_\delta(T-\tau)\right)_+\,\mathrm{d}\tau\\ &\le \int_0^\delta \theta(\tau)\left(\omega_\delta(\tau)\right)_+\,\mathrm{d}\tau + \int_\delta^T \theta(\tau)\left(\omega_\delta(\tau)-\omega_\delta(T-\tau)\right)_+\,\mathrm{d}\tau.\end{aligned}$$

Now we use that $\omega_\delta(\tau)=\frac{1}{3\delta}$ when $0\le t\le\delta$ and $\omega_\delta(\tau)-\omega_\delta(T-\tau)\le 0$ when $\delta\le t\le 2\delta$, and using the bound already established above on $\theta(t)$ for $0\le t\le\delta$ we obtain that

$$\frac{a}{3}\theta(T) - \frac{b}{2} \le \frac{b}{3a}.$$

In conclusion $\theta(T) \leq \frac{3b}{a}(\frac{1}{2} + \frac{1}{3a})$. Finally let assume that $T > 2\delta$; then using (A.11) we infer that

$$\frac{a}{2}\,\theta(T) \leq \frac{b}{2} + \int_0^T \theta(\tau)\omega_\delta(\tau)\,\mathrm{d}\tau = \frac{b}{2} + \int_0^{2\delta} \theta(\tau)\omega_\delta(\tau)\,\mathrm{d}\tau \leq \frac{b}{2} + \frac{3b}{a}(\frac{1}{2} + \frac{1}{3a})\frac{1}{2},$$

giving the estimate $\theta(T) \leq \frac{b}{a}(1 + \frac{3}{2a} + \frac{1}{a^2})$ for all $T > 2\delta$. This completes the proof with $c(a) = \frac{1}{a}\max(1 + \frac{3}{2a} + \frac{1}{a^2}, \frac{3}{2} + \frac{1}{a})$. $\square$

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