TALK 1: OVERVIEW OF THE PAPER (BURKLUND)

Disclaimer

These notes were typed by me (Paul VanKoughnett) from my own handwritten notes. It's highly probable that there are mistakes or omissions, which are all my fault. Notes in square brackets are also mine. If in doubt, consult the paper!

1. The Nullstellensatz

Proposition 1.1 (Hilbert's Nullstellensatz)**.** *Let k be an algebraically closed field. If R is a finitely presented k-algebra and $R \neq 0$, then $k \to R$ admits a retraction $R \to k$. (In other words,* $\operatorname{Spec} R$ *has a k-point.)*

We should really think of "finitely presented" as "compact". If we do this, then we can completely abstract away from commutative algebra:

Definition 1.2. Let $\mathcal{C}$ be an ∞-category. An object $X \in \mathcal{C}$ is **nullstellensatzian** if every non-terminal object $R \in (\mathcal{C}_{X/})^{\omega}$ has a retraction $X \to R \to X$.

Thus, Hilbert's Nullstellensatz says that algebraically closed fields are nullstellensatzian in CRing.

Question 1.3. Are there nullstellensatzian objects in other categories?

Answer: Yes! In fact, suppose that $\mathcal{C}$ is presentable, compactly generated, and has a **strict** terminal object 0, meaning that

$$\mathrm{Maps}_{\mathcal{C}}(0, X) = \begin{cases} \emptyset & X \not\simeq 0 \\ * & X \simeq 0 \end{cases}$$

Then every $X \in \mathcal{C}$ maps to some nullstellensatzian object.

Question 1.4. What are the nullstellensatzian objects in other categories?

Answer: Hard to say! In CRing, one can show that nullstellensatzian objects are *precisely* algebraically closed fields. The paper will identify nullstellensatzianobjects in $\mathsf{CAlg}(L_{K(n)}\mathsf{Sp})$[1]. These are currently the only categories for which a full characterization of nullstellensatzianobjects is known.

Remark 1.5. For the category Ring of associative rings, one imagines that the answer is something like "infinite-dimensional matrix algebras over algebraically closed fields." These are objects which are at least admit a lot of maps into them. But proving this would involve some difficult algebra.

Theorem 1.6 (Main theorem)**.** *In $\mathsf{CAlg}(L_{K(n)}\mathsf{Sp})$, the nullstellensatzianobjects are precisely the Lubin-Tate theories $E(k)$ for k an algebraically closed field.*

2. Redshift

The redshift conjecture is that algebraic K-theory increases height – thus, if R is a height n $\mathcal{E}_\infty$ ring, $K(R)$ should be height at least $n+1$. Given the theorem, we can deduce redshift as follows:

Let R be a height n $\mathcal{E}_\infty$ ring. The theorem gives us a map to some $E(k)$ for k algebraically closed. Applying K-theory and localizing, we have an $\mathcal{E}_\infty$ map

$$L_{K(n+1)}K(R) \to L_{K(n+1)}K(E(k)).$$

The target is nonzero by a theorem of Yuan, so the source must be nonzero because this is a ring map. This proves redshift.

[1]Everything discussed here also works for $\mathsf{CAlg}(L_{T(n)}\mathsf{Sp})$, but for simplicity we'll just talk about $K(n)$-local spectra.

Yuan's result is proved by using blueshift, which says that E_n is closely related to the Tate construction $E_{n+1}^{tC_p}$. Then we can understand the K-theory of this Tate construction by doing something with the Tate diagonal [I missed this part].

3. Proving a baby version of the theorem

Theorem 3.1 (Baby version). *If $R \in \mathsf{CAlg}_{K(n)}$, then there is an $\mathcal{E}_\infty$ map $R \to E(k)$ for some algebraically closed field k.*

Example 3.2 (An easy case). Suppose that $R = E(k)$. Then we can pick the identity map.

Of course, this example is a little silly. But the idea of the proof is that we can reduce to this trivial example by a transfinite procedure which transforms R until it is an E-theory. The maps used in this transfinite procedure have to have a certain technical property.

Definition 3.3. A map of $\mathcal{E}_\infty$ rings $A \to B$ **detects nilpotence** if, for all $f : M \to N$ of A-modules such that $f \otimes_A B = 0$, we have that some tensor power

$$f^{\otimes n} : M^{\otimes_A n} \to N^{\otimes_A n}$$

is zero.

Lemma 3.4. *Nilpotence-detecting maps are closed under base change. That is, if $f : A \to B$ detects nilpotence and*

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow & \ulcorner & \downarrow \\ C & \xrightarrow[g]{} & D \end{array}$$

is a pushout diagram of $\mathcal{E}_\infty$ rings, then g also detects nilpotence.

Remark 3.5. As a bit of foreshadowing, if $f : A \to B$ detects nilpotence and A is nonzero, then B is also nonzero. (If B were 0, the identity map on A would be nilpotent.) Thus, this condition will be used to prove that certain objects we construct are nonzero. It should be thought of as a more well-behaved version of the property that the base change functor $\cdot \otimes_A B$ is conservative on module categories – well-behaved in exactly the ways necessary to do a Quillen small object argument.

The proof works by constructing three basic examples of nilpotence-detecting maps. Each of these maps is designed to fix some feature of the original $\mathcal{E}_\infty$ ring R, making it closer to E-theory. Because the maps detect nilpotence, so do their base changes to R, and so the pushout is nonzero if R is.

3.1. **First issue: odd homotopy.** One way in which R could fail to be an E-theory is by having odd homotopy. Suppose that $\alpha \neq 0 \in \pi_1 R$. Without loss of generality (replacing R with $E \otimes R$) we can assume that R is an algebra over an E-theory. Then we can kill α in an $\mathcal{E}_\infty$ way using the following base change:

$$\begin{array}{ccc} E(k)\{x_1\} & \xrightarrow{x_1 \mapsto 0} & E(k) \\ {\scriptstyle x_1 \mapsto \alpha}\downarrow & \ulcorner & \downarrow \\ R & \longrightarrow & \overline{R} = R//^\infty \alpha \end{array}$$

Here the curly braces mean the free $\mathcal{E}_\infty$ ring, and x has degree 1.

Claim 3.6. *$h : E(k)\{x_1\} \to E(k)$ (sending x_1 to 0) detects nilpotence.*

If the claim is true, then $R \to \overline{R}$ also detects nilpotence, and thus $\overline{R} \neq 0$.

We can do this for every element in the odd homotopy of R. Unfortunately, each of these pushouts has the potential to introduce *new* odd homotopy. But by doing it transfinitely often [I think $\aleph_2$ works], we end up with $R \to R'$ where R' is even and nonzero.

How do we prove the claim? Basically, $\pi_* E(k)\{x_1\}$ is exterior on odd degree classes, which are the E-theory power operations on x_1. Odd degree classes square to 2-torsion, and by the May nilpotence conjecture (proved by Mathew-Naumann-Noel) this implies that they are nilpotent in the homotopy of an $\mathcal{E}_\infty$ ring. Thus $E(k)\{x_1\} \to E(k)$ is a quotient by a bunch of nilpotent stuff, and one can prove from this that it's nilpotence-detecting.

3.2. **Second issue: non-perfect residue ring.** Another way that R could fail to be E-theory is if a class $\overline{x} \in \pi_0 R/\mathfrak{m}$ doesn't have all pth power roots. [I think R is still supposed to be an algebra over an E-theory, in which case we can define $\mathfrak{m}$ by pushing forward the maximal ideal of $\pi_0 E$, but I could be wrong.] We fix this by the diagram

$$\begin{array}{ccc} E(k)\{x_0\} & \xrightarrow{g} & E((\pi_0 E(k)\{x\}/\mathfrak{m})^{\mathrm{perf}}) \\ \downarrow & \ulcorner & \downarrow \\ R & \longrightarrow & \overline{R} \end{array}$$

Here A^{perf} is the direct limit perfection of A, i.e., the colimit of A along the Frobenius maps.

How do we construct g? We're mapping out of a free $\mathcal{E}_\infty$ algebra, so it suffices to specify the image of x_0 in π_0 of the target. And we know what the map on residue rings (meaning $\pi_0/\mathfrak{m}$) is supposed to be: it's the direct limit perfection map,

$$\pi_0 E(k)\{x_0\}/\mathfrak{m} \to (\pi_0 E(k)\{x_0\}/\mathfrak{m})^{\mathrm{perf}}.$$

But lifting the image of x_0 in the residue ring to something actually in π_0 of the target amounts to solving some equations between power operations. It turns out that these equations admit a solution, because E_0 is *cofree* as a ring acted on by these power operations. This is one of the major technical hurdles in the paper.

Again, after applying g transfinitely, we can get something with perfect residue ring.

3.3. **Third issue: nilpotence.** Another way in which R could fail to be E-theory is if its residue ring $\pi_0 R/\mathfrak{m}$ has a nilpotent element x. To construct the map dealing with this requires the **spherical Witt vectors**. These fit into an adjunction

$$\mathbb{S}_{\mathbb{W}} : \mathsf{Perf}_{\mathbb{F}_p} \leftrightarrows \mathsf{CAlg}_p^\wedge : \pi_0(\cdot)^\flat,$$

where $(\cdot)^\flat$ is the inverse limit perfection (the limit along the Frobenius maps).

In particular,

$$\mathbb{S}_{\mathbb{W}}(\mathbb{F}_p[t^{1/p^\infty}]) = \mathbb{S}[t^{1/p^\infty}]_p^\wedge.$$

The diagram that will deal with the nilpotent is

$$\begin{array}{ccc} \mathbb{S}[t^{1/p^\infty}]_p^\wedge & \xrightarrow{f} & \mathbb{S}[t^{\pm 1/p^\infty}]_p^\wedge \times \mathbb{S}_p^\wedge \\ \downarrow & & \downarrow \\ R & \longrightarrow & \overline{R} \end{array}$$

The map f inverts t on the first factor and kills all powers of t on the second factor. The left-hand vertical map sends t to x – this exists because we've already made $\pi_0 R/\mathfrak{m}$ perfect by using g.

The map f should geometrically be thought of as (a sort of spherical perfectoid version of) decomposing $\mathbb{A}^1$ into a closed point and its open complement. This detects nilpotence because, more or less, if a map of sheaves over $\mathbb{A}^1$ is zero over a point and zero over its complement, then it's zero.