TALK 3: POWER OPERATIONS ON E-THEORY

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1. Introduction

This talk is about Burklund-Schlank-Yuan's theorem that $\pi_0 E$ is cofree as a ring with an action of E-theory power operations. I spent a lot of it talking about power operations in general, as well as the fundamental work on E-theory power operations by Ando, Hopkins, Strickland, and Rezk. At tne end, I outlined the proof and the ingredients that went into it.

In these notes, I'm being a bit loose with $K(n)$-localizations. When I talk about CAlg_E in general, I mean the category of $\mathcal{E}_\infty$-E-algebras, but when I talk about it *for E-theory*, I mean the category of $K(n)$-local $\mathcal{E}_\infty$-E-algebras. There are some technical obstacles to porting the unlocalized power operations story to the $K(n)$-local realm, but to my knowledge they were dealt with by [BF15]. Additionally, let me point out that, since we're primarily interested in how the power operations act on $\pi_0 E$, we don't have to worry about the bugbears presented by derived completeness or odd degrees.

I write CRing_A for discrete commutative rings over A, and CAlg_A for $\mathcal{E}_\infty$ algebras over A.

2. What are power operations?

Fix an $\mathcal{E}_\infty$ ring spectrum E. Eventually, we'll take this to be E-theory.

Definition 2.1. A **power operation** (on $\mathcal{E}_\infty$-E-algebras) is a natural transformation

$$\pi_m \Rightarrow \pi_n$$

of the functors

$$\pi_n : \mathsf{CAlg}_E \to \mathsf{Sets}.$$

An **additive power operation** is a natural transformation of the functors $\pi_n : \mathsf{CAlg}_E \to \mathsf{Ab}$.

The functors π_n are representable:

$$\pi_n R = \pi_0 \operatorname{Maps}_{\mathsf{CAlg}_E}(E \otimes \mathbb{P}(S^m), R),$$

where

$$\mathbb{P}(S^m) = \bigvee_r S^{mr}_{h\Sigma_r}$$

is the free $\mathcal{E}_\infty$ algebra on S^m (so that $E \otimes \mathbb{P}(S^m)$ is the free $\mathcal{E}_\infty$-E-algebra on S^m). By the Yoneda lemma,

$$\begin{aligned}
\operatorname{Maps}(\pi_m, \pi_n) &= \pi_0 \operatorname{Maps}_{\mathsf{CAlg}_E}(E \otimes \mathbb{P}(S^m), E \otimes \mathbb{P}(S^n)) \\
&= [S^m, E \otimes \mathbb{P}(S^n)] \\
&= \pi_m(E \otimes \mathbb{P}(S^n)) \\
&= E_m\left(\bigvee_{r \geq 0} S^{nr}_{h\Sigma_r}\right).
\end{aligned}$$

The set of power operations has a huge amount of structure. It has three gradings: the input and output degrees m and n, and the **weight** r that appeared in the above splitting. Because we're always mapping into the homotopy groups of an $\mathcal{E}_\infty$-E-algebra, which form a graded commutative E_*-algebra, the set of all power operations is also a graded commutative E_*-algebra. Because we're mapping *out* of the homotopy groups of an $\mathcal{E}_\infty$-E-algebra, there is also a coalgebraic structure on power operations. Finally, power operations can be composed.

We'll immediately use this structure to simplify the situation. If E is even periodic, then we can use the periodicity to only think about power operations between degrees 0 and 1. For the purposes of this talk, we'll only talk about degree 0.

Definition 2.2. When E is E-theory, write

$$\mathbb{T}(E_0) = \mathrm{End}(\pi_0 : \mathsf{CAlg}_E \to \mathsf{Sets}) = E_0\mathbb{P}(S^0) = E_0\left(\bigvee \Sigma^\infty_+ B\Sigma_r\right).$$

(The reason for the notation will be made apparent later.)

Remark 2.3. When E is E-theory, the theory of even-degree power operations is a retract of the theory of power operations in all degrees. This isn't completely formal, but boils down to the fact that $E_*B\Sigma_n$ is even, for all n.

Power operations act on the homotopy of an $R \in \mathsf{CAlg}_E$ as follows. Given $x \in \pi_0 R$, we can represent x as a map

$$\mathbb{S} \xrightarrow{x} R.$$

Taking an extended power of this map and using the $\mathcal{E}_\infty$ structure on R gives

$$\Sigma^\infty_+ B\Sigma_r = \mathbb{S}^{\otimes r}_{h\Sigma_r} \xrightarrow{x^{\otimes r}_{h\Sigma_r}} R^{\otimes r}_{h\Sigma_r} \xrightarrow{\text{mult}} R.$$

This construction defines a map

$$\mathrm{Pow}_r : \pi_0 R \to R^0 B\Sigma_r,$$

called the **total power operation** of weight r. Now given a class $\alpha \in E_0 B\Sigma_r$, we can pair it with $R^0 B\Sigma_r$ (using the E-module structure on R) to map back to $\pi_0 R$. Thus, $R^0 B\Sigma_r$ is a sort of ring of functionals on the power operations in $E_0 B\Sigma_r$.

The additive variant of this story uses the transfer maps

$$\mathrm{tr} : \Sigma^\infty_+ B\Sigma_r \to \Sigma^\infty_+(B\Sigma_i \times B\Sigma_{r+i}).$$

Write I_{tr} for the ideal in $R^0 B\Sigma_r$ generated by the images of the transfers from $R^0(B\Sigma_i \times B\Sigma_{r-i})$, where $1 \le i \le r-1$. The importance of this ideal is that it contains the failure of power operations to preserve addition. Thus, the **total additive power operation** is the composition

$$\tau_r : \pi_0 R \xrightarrow{\mathrm{Pow}_r} R^0 B\Sigma_r \twoheadrightarrow R^0 B\Sigma_r / I_{\mathrm{tr}}.$$

(My notation differs from Burklund-Schlank-Yuan: what they call τ_r, I call τ_{p^r}.) We can pair this with classes in

$$\ker\left(\mathrm{tr} : E_0 B\Sigma_r \to \bigoplus_{i=1}^{r-1} E_0(B\Sigma_i \times B\Sigma_{r-i})\right)$$

to get individual additive power operations valued in $\pi_0 R$.

2.1. **Plethories.** We can formally summarize the algebraic structure on $\mathbb{T}(E_0)$ by saying that it is a **plethory** on CRing_{E_0}. In other words,

- it's a commutative E_0-algebra,
- it represents a functor $\mathsf{CRing}_{E_0} \to \mathsf{CRing}_{E_0}$ (which means that it has some additional coalgebraic structure), and
- this functor is a comonad (via composition of power operations).

An **algebra over a plethory** is a coalgebra over the corresponding comonad. Many sorts of things we'd consider "commutative rings with extra algebraic structure", including λ-rings, θ-algebras/δ-rings, and group and Lie algebra actions on commutative rings, can be defined in this language.

It turns out that for formal reasons [BW05], algebras over a plethory are both comonadic and monadic over the underlying category of commutative rings. Thus, we have adjunctions

$$\mathsf{Alg}_\mathbb{T} \underset{W_\mathbb{T}}{\overset{F_\mathbb{T}}{\underset{U_\mathbb{T}}{\leftrightarrows}}} \mathsf{CRing}_{E_0}.$$

Explicitly,

$$W_\mathbb{T}(R) = \mathrm{Hom}_{\mathsf{CRing}_{E_0}}(\mathbb{T}(E_0), R).$$

The left adjoint $F_{\mathbb{T}}$ is slightly harder to describe explicitly, but is determined by what it does to the free commutative E_0-ring on one generator:

$$F_{\mathbb{T}}(E_0[x]) = \mathbb{T}(E_0).$$

(The functor $\mathbb{T}$ itself is a monad on Mod_{E_0} which was defined by Rezk [Rez09]. In this language, it's $F_{\mathbb{T}} \circ \mathrm{Sym}_{E_0}$.)

Exercise 2.4. "Taking a logarithm" of the previous paragraph, let $A \in \mathsf{CRing}$, and show that a corepresentable comonad on Mod_A is precisely an A-algebra (which is not necessarily commutative, and where A is not necessarily central). Given such an A-algebra B, describe the corresponding monad.

Example 2.5 (Key example)**.** At height 1, $E_0 = \mathbb{Z}_p$, and $\mathsf{Alg}_{\mathbb{T}}$ is the category of **θ-algebras** (also called **δ-rings**). A θ-algebra is equipped with operations ψ and θ such that

$$\psi(x) = x^p + p\theta(x),$$

$$\theta(x+y) = \theta(x) + \theta(y) + \sum_{i=1}^{p-1} \frac{1}{p}\binom{p}{i} x^i y^{p-i},$$

and

$$\theta(xy) = x^p\theta(y) + y^p\theta(x) + p\theta(x)\theta(y).$$

The way to remember this is that these are exactly the identities on θ required to make ψ a ring homomorphism. In fact, if the underlying ring is torsion-free, then ψ determines θ and it suffices to say that ψ is a ring homomorphism lifting the mod p Frobenius. But in general, θ determines ψ, so we have to state the algebraic identities on θ.

As power operations, both θ and ψ are in weight p. In fact, $E_0 B\Sigma_p$ is free on the two generators θ, ψ.

One can also show [Bou96] that $\mathbb{T}(E_0)$, the free θ-algebra on one generator, is polynomial:

$$\mathbb{T}(E_0) \cong \mathbb{Z}_p[x, \theta x, \theta^2 x, \dots].$$

We have

$$W_{\mathbb{T}}(R) = \mathrm{Hom}_{\mathsf{CRing}_{\mathbb{Z}_p}}(\mathbb{T}(E_0), R) \cong W(R),$$

the p-typical Witt vectors of R (explaining the notaion $W_{\mathbb{T}}$).

In particular, if $R \in \mathsf{Perf}_{\mathbb{F}_p}$, then $W_{\mathbb{T}}(R) = W(R) = \pi_0 E(R)$. That is,

$$\pi_0 E : \mathsf{Perf}_{\mathbb{F}_p} \to \mathsf{Alg}_{\mathbb{T}}$$

is precisely the cofree functor. In other words, we have an adjunction

$$(U_{\mathbb{T}}(\cdot)/p)^\sharp : \mathsf{Alg}_{\mathbb{T}} \leftrightarrows \mathsf{Perf}_{\mathbb{F}_p} : \pi_0 E(\cdot), \tag{1}$$

obtained by composing the $U_{\mathbb{T}} \dashv W_{\mathbb{T}}$ adjunction with the adjunction

$$((\cdot)/p)^\sharp : \mathsf{CRing}_{\mathbb{Z}_p} \leftrightarrows \mathsf{Perf}_{\mathbb{F}_p} : \mathrm{inc}.$$

2.2. **The theorem.** The main theorem of this section of the paper is a generalization of (1) to higher heights.

Theorem 2.6. *Let E be a height n E-theory over a perfect field k. Then there is an adjunction*

$$(U_{\mathbb{T}}(\cdot)/\mathfrak{m})^\sharp : \mathsf{Alg}_{\mathbb{T}} \leftrightarrows \mathsf{Perf}_k : \pi_0 E(\cdot),$$

and the right adjoint is fully faithful.

For full faithfulness, observe that the counit is

$$(\pi_0 E(A)/\mathfrak{m})^\sharp \to A,$$

which is an isomorphism for $A \in \mathsf{Perf}_k$. Thus, it remains to establish the adjunction, or equivalently that

$$\pi_0 E(\cdot) \cong W_{\mathbb{T}}(\cdot).$$

Also, we have a natural map $\pi_0 E(A) \to W_{\mathbb{T}}(A)$, given by the composite

$$\pi_0 E(A) \xrightarrow{\text{unit}} W_{\mathbb{T}} U_{\mathbb{T}} \pi_0 E(A) \xrightarrow{W_{\mathbb{T}}(\cdot/\mathfrak{m})} W_{\mathbb{T}}(A).$$

BSY call this the **reduced evaluation map**, $\overline{\text{ev}}_A$. (I think it's more of a reduced coevaluation map.) Remembering that

$$W_{\mathbb{T}} U_{\mathbb{T}} \pi_0 E(A) = \text{Hom}_{\mathsf{CRing}_{E_0}}(\mathbb{T}(E_0), \pi_0 E(A)),$$

what $\overline{\text{ev}}_A$ is doing is saying, for each element in $\pi_0 E(A)$, how each power operation in $\mathbb{T}(E_0)$ is acting on it, mod the ideal $\mathfrak{m}\pi_0 E(A)$. The proof will show that this map is an isomorphism.

3. More on power operations for E-theory

Now let's state some specific details on how power operations work for E-theory. I'll try to compare things to the familiar height 1 case. This is all work of Strickland [Str98], Ando-Hopkins-Strickland [AHS04], and Rezk [Rez09].

First, the theory is algebraically well-behaved because:

Proposition 3.1 (Strickland)**.** *$E_*^\wedge B\Sigma_r$ is finite free and even as an E_*-module.*

Strickland also calculates the ranks of these free modules. One way to say this is to write $\overline{d}(r)$ for the number of subgroups of order p^r in $(\mathbb{Q}_p/\mathbb{Z}_p)^n$. Then:

Theorem 3.2 (Strickland)**.** *$\mathbb{T}(E_0)$ is polynomial on generators in p-power weights, and the indecomposables of weight p^r are free of rank $\overline{d}(r)$.*

For example, at height 1, $\mathbb{T}(E_0) = \mathbb{Z}_p[x, \theta(x), \theta^{\circ 2}(x), \ldots]$. This has a rank 1 module of indecomposables in weight p^r, generated by $\theta^r(x)$.

Next, write

$$\Gamma \subseteq \mathbb{T}(E_o)$$

for the subalgebra of additive power operations. This is a ring, though *not with the same multiplication as $\mathbb{T}(E_0)$!* Of course, if we multiply two additive operations, the product may not preserve addition. Instead, the *composition* on $\mathbb{T}(E_0)$ makes Γ a (not necessarily commutative) ring. Recall that

$$\Gamma_r = \ker\left(\text{tr} : E_* B\Sigma_r \to \bigoplus_{i=1}^{r-1} E_*(B\Sigma_i \times B\Sigma_{r-i}).\right)$$

We also have:

Theorem 3.3 (Strickland)**.** *Each Γ_r is finite free and even as an E_*-module (so we're justified in calling it an E_0-module). The inclusions $\Gamma_r \to E_0 B\Sigma_r$ split. Finally, Γ_r is only nonzero if r is a power of p.*

At height 1,

$$\Gamma = \mathbb{Z}_p[\psi],$$

generated by ψ (under *composition*, not multiplication). As ψ has weight p, $\psi^{\circ r}$ has weight p^r; thus, Γ_{p^r} is a free rank 1 E_0-module for each r.

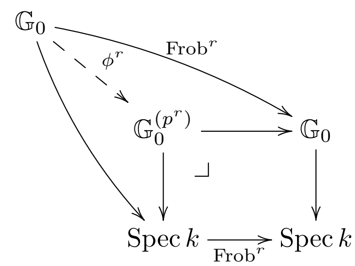
The next thing to know is that the *additive* power operations carry a modular interpretation in terms of formal groups. Writing $\mathbb{G}_0$ for the formal group over k used to define E, recall that we can identify

$$\text{Spf}\, E_0 \cong \mathsf{Def}(\mathbb{G}_0),$$

the moduli of deformations of $\mathbb{G}_0$.

Definition 3.4. Write $\mathbb{G}_0^{(p^r)}$, the r-fold **Frobenius twist** of $\mathbb{G}_0$, for the base change of $\mathbb{G}_0$ along $\text{Frob}^r : k \to k$.

The r-fold **relative Frobenius isogeny** is the map ϕ^r of formal groups over k defined by the following diagram:



Definition 3.5. The moduli of **deformations of** ϕ^r is the functor $\mathsf{Def}(\phi^r)$ assigning to each complete local ring R the set of the following data:

- A deformation $\mathbb{G}$ of $\mathbb{G}_0$ over R;
- A deformation $\mathbb{G}'$ of $\mathbb{G}_0^{(p^r)}$ over R;
- An isogeny $\psi : \mathbb{G} \to \mathbb{G}'$ of formal groups over R, that reduces over the special fiber to ϕ^r.

(See [BSY22, Definition 3.37] for a more precise definition. As they point out, this definition makes sense with ϕ^r replaced by any isogeny of formal groups over k.)

Observe that $\mathsf{Def}(\phi^r)$ has a functor to $\mathsf{Def}(\mathbb{G}_0)$ given by forgetting to the source $\mathbb{G}$, and a functor to $\mathsf{Def}(\mathbb{G}_0^{(p^r)})$ given by forgetting to the target $\mathbb{G}'$. Since k is perfect, we in fact have $\mathsf{Def}(\mathbb{G}_0^{(p^r)}) \cong \mathsf{Def}(\mathbb{G}_0)$: roughly speaking, we can always untwist a deformation of $\mathbb{G}_0^{(p^r)}$ by precomposing with the r-fold Frobenius of k.

Theorem 3.6 (Ando-Hopkins-Strickland)**.** *The functor* $\mathsf{Def}(\phi^r)$ *is a formal scheme, isomorphic to* $\operatorname{Spf} E^0 B\Sigma_{p^r}/I_{\mathrm{tr}}$*. The source and target maps* $\mathsf{Def}(\phi^r) \rightrightarrows \mathsf{Def}(\mathbb{G}_0)$ *correspond, respectively, to the* E_0*-algebra unit* $E_0 \to E^0 B\Sigma_{p^r}/I_{\mathrm{tr}}$*, and to the total* p^r*th additive power operation,* $\tau_{p^r} : E_0 \to E^0 B\Sigma_{p^r}/I_{\mathrm{tr}}$*. Composition of power operations corresponds to composition of isogenies.*

As a result, we can treat the collection of all $\mathsf{Def}(\phi^r)$ as a “lax formal stack”, i.e., a category object in formal schemes, describing deformations of $\mathbb{G}_0$ and isogenies between them that deform powers of the relative Frobenius. Then an E_0-ring with an action of Γ is precisely a quasicoherent sheaf on this lax formal stack. In other words, additive power operations correspond to descent data for certain isogenies of formal groups.

Remark 3.7. For a height 1 formal group $\mathbb{G}$, there is a unique isogeny of degree p^r for every r (given by formal multiplication $[p^r]$), and thus $\mathsf{Def}(\phi^r) \cong \operatorname{Spf} E_0$ for each r. Correspondingly, there's a unique additive power operation of weight p^r for each r, up to multiplication by $\mathbb{Z}_p$, which is $\psi^{\circ r}$. We recover the formula $\Gamma = \mathbb{Z}_p[\psi]$, which we now see is related to the subgroup structure of height 1 formal groups.

The final result, the main idea of [Rez09], relates Γ-algebras to $\mathbb{T}$-algebras.

Theorem 3.8 (Rezk)**.** *There is an operation* $\psi \in \Gamma$ *such that a torsion-free* Γ*-algebra* R *extends to a* $\mathbb{T}$*-algebra iff*

$$\psi(x) \equiv x^p \pmod{p}.$$

This determines the monad $\mathbb{T}$, even on non-torsion-free rings, in a completely analogous way to the case of θ-algebras. More precisely, this theorem forces $\mathbb{T}$ to have an operation θ such that

$$\psi(x) = x^p + p\theta(x),$$

and θ must satisfy certain identities determined by ψ. In particular, because ψ is additive, we must have

$$\theta(x+y) = \theta(x) + \theta(y) + \sum_{i=1}^{p-1} \frac{1}{p} \binom{p}{i} x^i y^{p-i} \tag{2}$$

(θ is a p**-derivation**). This is a key ingredient of the proof.

4. Outline of the proof

Now let's outline the proof.

(1) We need to show that the reduced evaluation map $\pi_0 E(A) \to W_{\mathbb{T}}(A)$ is an isomorphism for all $A \in \mathsf{Perf}_k$. First, we reduce to just proving this for $A = k$. This is done by comparing two filtrations: the $\mathfrak{m}$-adic filtration on $\pi_0 E(A)$, and the **Witt filtration** on $W_{\mathbb{T}}(A)$.

(2) To show that

$$\pi_0 E(k) \to W_{\mathbb{T}}(k) = \mathrm{Hom}_{\mathsf{CRing}_{E_0}}(\mathbb{T}(E_0), k)$$

is injective, we have to find, for each $x \in E_0$, an operation Ψ such that $\Psi(x) \not\equiv 0 \bmod \mathfrak{m}$. If x is very divisible by $\mathfrak{m}$, this means that power operations must be able to lower this divisibility. The argument for this uses some earlier technical work of Jeremy Hahn, together with a transchromatic argument which is inductive on the height. (More incredibly yet, this argument uses the main theorem of the paper and makes the *whole paper* inductive on the height!)

(3) Finally, to show the map is surjective, we return to comparing the two filtrations. Here one is able to establish an inequality between the sizes of the filtration quotients. This ultimately relies on the fact that

$$\tau_{p^r} : E_0 \to E^0 B\Sigma_{p^r}/(I_{\mathrm{tr}} + \mathfrak{m})$$

is a surjection, which is proved by using the deformation-theoretic description of power operations.

I only talked about #2 in the talk, but I've included a summary of the other steps.

5. Comparing filtrations

(I didn't talk about this in the talk. Apologies if it's a bit sketchy.)

Let $A \in \mathsf{Perf}_k$. Recall that

$$W_{\mathbb{T}}(A) = \mathrm{Hom}_{\mathsf{CRing}_{E_0}}(\mathbb{T}(E_0), A).$$

Since $\mathbb{T}(E_0)$ is graded by weight, we have a filtration on $W_{\mathbb{T}}(A)$, in which

$$W_{\mathbb{T}}^{\geq r}(A) = \{f : \mathbb{T}(E_0) \to A | f \text{ vanishes on } \mathbb{T}_k(E_0) \text{ for } k < p^r\}.$$

Each $W_{\mathbb{T}}^{\geq r}(A)$ is an ideal. This is dual to the statement that

$$\mathbb{T}_{<p^r}(E_0) = \bigoplus_{k<p^r} E^0 B\Sigma_k$$

is a coideal. This is true because the comultiplication comes from the diagonal maps of spaces

$$\Delta : B\Sigma_k \to B\Sigma_k \times B\Sigma_k,$$

while the coaddition comes from the transfer maps

$$\Sigma^\infty_+ B\Sigma_k \to \bigoplus_{i=0}^{k} \Sigma^\infty_+(B\Sigma_i \times B\Sigma_{k-i}).$$

At height 1, $W_{\mathbb{T}}(A)$ is the p-typical Witt vectors of A, and the filtration thus defined is the p-adic viltration. (If A is not perfect, it is the V-adic filtration, where V is the Verschiebung.)

Let

$$W_{\mathbb{T}}^{=r}(A) = W_{\mathbb{T}}^{\geq r}(A)/W_{\mathbb{T}}^{\geq (r+1)}(A)$$

be the associated graded. The main idea here is that each $W_{\mathbb{T}}^{=r}(A)$ is a free A-module. First, we can identify

$$W_{\mathbb{T}}^{=r}(A) = \mathrm{Hom}_k(Q\mathbb{T}(E_0)_{p^r}, A),$$

where Q is the indecomposables. Strickland proved that $Q\mathbb{T}(E_0)_{p^r}$ is a k-vector space of dimension $\overline{d}(r)$, so

$$W_{\mathbb{T}}^{=r}(A) \cong A^{\overline{d}(r)}. \tag{3}$$

But there is a small subtlety here. The natural A-module structure comes from

$$A \xrightarrow{[\cdot]} W(A) \to \pi_0 E(A) = W(A)[[u_1, \ldots, u_{n-1}]] \xrightarrow{\overline{\mathrm{ev}}_A} W_{\mathbb{T}}(A),$$

where the first map is the multiplicative lift. It turns out that this makes A act on $W_{\mathbb{T}}^{=r}(A) \cong A^{\overline{d}(r)}$ via the r-fold Frobenius. Nevertheless, since A is perfect, we still get $W_{\mathbb{T}}^{=r}(A) \cong A^{\overline{d}(r)}$ as an A-module – just with a different isomorphism than the one in (3).

Example 5.1. At height 1, $\overline{d}(r) = 1$ for each r, and we're saying that the associated graded of $W(A)$ for the p-adic filtration is isomorphic to A in each degree. A typical element in $W^{=r}(A)$ can be represented in the form

$$V^r[y], \quad y \in A,$$

and the A-module structure comes from

$$[x]V^r[y] = V^r[x^{p^r} y] + \ldots .$$

As stated, A acts through the r-fold Frobenius. (Another useful thing to note here is that the multiplicative lift isn't additive – but it *is* additive mod higher filtration, and so we still get an A-module structure on the associated graded).

The important point is that the exponent $\overline{d}(r)$ is invariant of A. Thus one can prove:

Proposition 5.2. *The map*

$$W_{\mathbb{T}}(k) \otimes_{W(k)} W(A) \to W_{\mathbb{T}}(A)$$

maps the filtration on $W_{\mathbb{T}}(k)$ to the filtration on $W_{\mathbb{T}}(A)$, and is an isomorphism on the associated graded.

Finally, consider the diagram

$$\begin{array}{ccc} \pi_0 E(k) \otimes_{W(k)} W(A) & \xrightarrow{\overline{\mathrm{ev}}_k \otimes 1} & W_{\mathbb{T}}(k) \otimes_{W(k)} W(A) \\ \downarrow & & \downarrow \\ \pi_0 E(A) & \xrightarrow[\overline{\mathrm{ev}}_A]{} & W_{\mathbb{T}}(A) \end{array}$$

The proposition above implies that the right-hand map is an isomorphism after completing with respect to the filtrations on $W_{\mathbb{T}}$. The left-hand map is observed to be an isomorphism after $\mathfrak{m}$-adic completion. The final claim is that the $\mathfrak{m}$-adic filtration and the Witt filtration induce the same topology on $W_{\mathbb{T}}(k)$. Unfortunately, I didn't understand the proof of this claim.

In any case, we get:

Proposition 5.3. *If $\overline{\mathrm{ev}}_k$ is an isomorphism, then so is $\overline{\mathrm{ev}}_A$, for any $A \in \mathsf{Perf}_k$.*

6. Injectivity

(This is the only part of the proof I had time to talk about.)

We need to show that:

Proposition 6.1. *If $x \neq 0$ in E_0, then there exists an operation $\Psi \in \mathbb{T}(E_0)$ such that $\Psi(x) \neq 0$ mod $\mathfrak{m}$.*

Intuitively, this means that power operations are able to lower $\mathfrak{m}$-divisibility. First let's deal with divisibility by p. Let θ be the operation defined above.

Lemma 6.2. *If $x \neq 0$ mod p in E_0, then $v_p(\theta(x)) < v_p(x)$.*

Proof. This follows from the p-derivation equation Equation (2), and is left as an exercise. □

Thus, we can assume that x is not divisible by p. We will prove the following:

Proposition 6.3. *If $x \neq 0$ mod p in E_0, then there exists $r > 0$ such that $\tau_{p^r}(x) \neq 0$ mod $\mathfrak{m}$.*

This says that we can decrease divisibility by $u_1, \ldots, u_{n-1}$ using *additive* power operations.

The main step here is a lemma of Jeremy Hahn [Hah17]:

Lemma 6.4 (Hahn)**.** *Suppose that $n \geq 2$ and $x \neq 0$ mod $\mathfrak{m}_{n-1} := (p, u_1, \ldots, u_{n-2})$. Then there exists $r > 0$ such that $\tau_r(x) \neq 0$ mod $\mathfrak{m}$.*

Proof of Proposition 6.3. At height 1, this is trivial because $\mathfrak{m} = (p)$. We induct on the height n, assuming the main result of the paper (that E-theories are nullstellensatzian) at heights $< n$.

In particular, the main theorem gives us the starred map in the composition:

$$f : E = E(k) \to L_{K(n-1)}E \xrightarrow{(*)} E_{n-1}(K)$$

in which K is some perfect field of characteristic p. On π_0, f looks like

$$Wk[[u_1, \ldots, u_{n-1}]] \to Wk((u_{n-1}))_p^\wedge[[u_1, \ldots, u_{n-2}]] \to WK[[w_1, \ldots, w_{n-2}]].$$

I claim that this composition is injective. It suffices to check this mod the regular sequence $(p, u_1, \ldots, u_{n-2})$, which generates the same ideal as $(p, w_1, \ldots, w_{n-2})$ (because f was a map of complex oriented ring spectra). Mod this ideal, we have

$$k[[u_{n-1}]] \to k((u_{n-1})) \to K.$$

The first map is inverting a non-zero-divisor, and the second map is a map of fields. Thus f is injective, and the same argument shows that f is also injective mod p.

Because f is $\mathcal{E}_\infty$, it commutes with the total power operations. Thus we have a diagram

$$\begin{array}{ccccc}
\pi_0 E/p & \xrightarrow{\tau_{p^r}} & E^0 B\Sigma_{p^r}/(I_{tr}, p) & \twoheadrightarrow & E^0 B\Sigma_{p^r}/(I_{tr} + \mathfrak{m}_{n-1}) \\
{\scriptstyle f}\big\downarrow & & \big\downarrow & & \big\downarrow \\
\pi_0 E_{n-1}(K)/p & \longrightarrow & E_{n-1}(K)^0 B\Sigma_{p^r}/(I_{tr}, p) & \longrightarrow & E_{n-1}(K)^0 B\Sigma_{p^r}/(I_{tr} + \mathfrak{m}_{n-1})
\end{array}$$

Now, if $x \neq 0$ in $\pi_0 E$ mod p, then because f is injective mod p, $f(x)$ is nonzero in $\pi_0 E_{n-1}(K)$ mod p. By the induction hypothesis, for some r, $\tau_r(f(x)) \neq 0$ mod the maximal ideal of $E_{n-1}(K)$, which is $\mathfrak{m}_{n-1}$. Thus, $\tau_r(x) \neq 0$ mod $\mathfrak{m}_{n-1}$. By Hahn's lemma, there is some further τ_s we can apply to get something nonzero mod $\mathfrak{m}$. Then $\tau_{r+s} \neq 0$ mod $\mathfrak{m}$. □

7. Surjectivity

(I also didn't talk about this, and will be even sketchier. In my mind this is the hardest part of this section of the paper.)

There are two steps. The first step is:

Proposition 7.1. *The map*

$$\tau_{p^r} : \pi_0 E \to E^0 B\Sigma_{p^r}/(I_{tr} + \mathfrak{m})$$

is surjective.

Proof sketch. This uses Strickland's modular interpretation of the additive power operations. We need to show that the map of formal schemes

$$\operatorname{Spec} k \times^{\text{source}}_{\mathsf{Def}(\mathbb{G}_0)} \mathsf{Def}(\text{Frob}^r) \to \operatorname{Spec} k \times_{\operatorname{Spf} Wk} \mathsf{Def}(\text{Frob}^r) \xrightarrow{\text{target}} \operatorname{Spec} k \times_{\operatorname{Spf} Wk} \mathsf{Def}(\mathbb{G}_0^{(p^r)})$$

is a closed immersion. This is pulled back from the map

$$\mathsf{Def}(\text{Frob}^r) \xrightarrow{(\text{source},\text{target})} \mathsf{Def}(\mathbb{G}_0) \times_{\operatorname{Spf} Wk} \mathsf{Def}(\mathbb{G}_0^{(p^r)}).$$

It suffices to show that this map is a closed immersion. More generally, the analogous statement is true with

$$\mathbb{G}_0 \xrightarrow{\text{Frob}^r} \mathbb{G}_0^{(p^r)}$$

replaced with any isogeny

$$\mathbb{G}_1 \xrightarrow{q_0} \mathbb{G}_2$$

of formal groups over k. Informally, this means that given a deformation of $\mathbb{G}_1$ and a deformation of $\mathbb{G}_2$ over some complete local ring R, there's *at most one way* to deform the isogeny q_0 to an isogeny between these deformations. In other words, if q is an isogeny between formal groups over R, and $q \equiv 0$ mod the maximal ideal of R, then $q = 0$.

This is a computation using formal group laws which I won't repeat. Note that the assumption of finite height is essential: the statement is definitely not true for the additive formal group law. □

The second step uses the filtrations again. Writing

$$W_{\mathbb{T}}^{\leq r}(k) = W_{\mathbb{T}}(k)/W_{\mathbb{T}}^{\geq (r+1)}(k),$$

we have that $W_{\mathbb{T}}^{\leq r}(k)$ is a finite Wk-module of length

$$\sum_{i=0}^{r} \overline{d}(i).$$

Let $E^{\leq r}(k)$ be the image of

$$\pi_0 E(k) \to W_{\mathbb{T}}(k) \twoheadrightarrow W_{\mathbb{T}}^{\leq r}(k).$$

The total p^rth additive power operation factors as

$$\pi_0 E(k) \to E^{\leq r}(k) \to W_{\mathbb{T}}^{\leq r}(k) \to E(k)^0 B\Sigma_{p^r}/(I_{tr}, \mathfrak{m}).$$

With a bit of algebra, one can use this to get a *lower* bound $E^{\leq r}(k)$: it has length at least $\sum_{i=0}^{r} \overline{d}(i)$. But we already proved that $\pi_0 E(k) \to W_{\mathbb{T}}(k)$ is injective, so this is enough to get surjectivity.

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