SCHEDULE FOR CHROMATIC NULLSTELLENSATZ SEMINAR

Sometime Soon: Introduction and outline of results: (Sections 1 and 9.) Describe what's in the paper [BSY22] and the techniques and background material used. If possible, talk about the history of the redshift problem and earlier results.

October 25: Spherical Witt vectors and the construction of E-theory: (Section 2.) Construct the spherical Witt vectors $\mathbb{W}$, as well as the adjunction

$$\mathbb{W}(\cdot) : \mathsf{Perf} \leftrightarrows \mathsf{CAlg}(\mathsf{Sp}_p) : (\cdot)^\flat.$$

Construct the E-theory functor and associated adjunction

$$E(\cdot) : \mathsf{Perf}_k \leftrightarrows \mathsf{CAlg}^\wedge_{E(k)} : (\cdot)^\flat.$$

As indicated in section 2.5, this is exactly the spherical Witt vectors base changed to $E(k)$, so that the adjunction has many nice properties (Theorem 2.38). This section is largely a rehash of section 5 of [Lur18], and you should especially look at section 5.2 of that paper for the underlying deformation theory. However, no need to go into all of Lurie's arguments on the construction of E-theory.

November 1: Power operations on E-theory: (Section 3.) Recall Rezk's [Rez09] and Strickland's [Str97; Str98] work on power operations on E-theory, and define Rezk's monad $\mathbb{T}$. *(Note that BSY only need the even part of this monad.)* Walk through some of the additional structure related to $\mathbb{T}$ used to prove Theorem 3.4: the right adjoint $W_{\mathbb{T}}$ to the forgetful functor, the reduced evaluation map, and the ring of additive power operations. *(This is probably the most technical part of the paper, so I've broken it up into two talks. You should coordinate with whoever's giving the other talk on how to split up the section.)*

November 8: Cofreeness of E_*: (Section 3.) Finish the proof that E_* is a cofree $W_{\mathbb{T}}$-coalgebra. *(See note on previous talk.)*

November 15: Nilpotence-detecting maps: (Section 4.) Describe the basic properties of nilpotence-detecting maps. Show how to construct nilpotence-detecting maps using the small object argument (Corollary 4.37). Prove Theorem 4.49, which gives a criterion for when a map $A \to B$ of perfect $\mathbb{F}_p$-algebras induces a nilpotence-detecting map $\mathbb{W}_{\mathcal{C}}(A) \to \mathbb{W}_{\mathcal{C}}(B)$ on spherical Witt vectors functors valued in locally rigid ∞-categories $\mathcal{C}$.

November 22: Nilpotence-detecting maps to E-theory: (Section 5.) Prove that any $T(n)$-local $\mathcal{E}_\infty$ ring spectrum has a nilpotence-detecting map to an E-theory (Theorem 5.1). An important step is to define the three "basic maps" f, g, and h in Definition 5.10, and prove that they detect nilpotence.

November 29: E-theories are Nullstellensatzian: (Section 6.) In fact, they are the only Nullstellensatzian $T(n)$-local $\mathcal{E}_\infty$ ring spectra. Now would be an appropriate time to say what this has to do with the classical Nullstellensatz, if we didn't already to that.

December 6: The constructible spectrum: (Section 7 and Appendix A.) Define the constructible spectrum, and prove that a map $R \to S$ detects nilpotence iff the associated map on constructible spectra, $\mathrm{Spec}^c(S) \to \mathrm{Spec}^c(R)$, is surjection (Theorem 7.4). Discuss the calculations of constructible spectra in Subsection 7.1.

December 13: Orientability and Picard spectra: (Section 8.) Prove that, for $E(L) \to R$ a map of $T(n)$-local $\mathcal{E}_\infty$ rings, the map $\mathrm{pic}(\mathsf{Mod}^\wedge_{E(L)}) \to \mathrm{pic}(\mathsf{Mod}^\wedge_R)$ admits a retract (Theorem 8.1.) Use this to prove the criterion for orientability of E-theories by Thom spectra (Corollary 8.13.) Calculate the strict Picard spectrum (Proposition 8.14) of $E(L)$. The discrepancy spectrum is mentioned in Proposition 8.15, but the proof is in a paper which I don't think has appeared yet.

December 20?: Redshift: (Section 9 and the references therein.) Define the chromatic height of an $\mathcal{E}_\infty$ ring, using Hahn's theorem (Theorem 9.4). Prove redshift (Theorem 9.10). This proof relies on work by [Cla+20] and [Yua21], which you should also try to say something about (in particular, the main result follows from Yuan's result for E-theories by descent).

References

[BSY22] Robert Burklund, Tomer M. Schlank, and Allen Yuan. *The Chromatic Nullstellensatz.* July 20, 2022. arXiv: `2207.09929 [math]`.

[Cla+20] Dustin Clausen, Akhil Mathew, Niko Naumann, and Justin Noel. *Descent and vanishing in chromatic algebraic K-theory via group actions.* Nov. 16, 2020. arXiv: `2011.08233 [math]`.

[Lur18] Jacob Lurie. "Elliptic Cohomology II: Orientations". 2018.

[Rez09] Charles Rezk. "The congruence criterion for power operations in Morava E-theory". In: *Homology, Homotopy and Applications* 11.2 (2009), pp. 327–379.

[Str97] Neil P. Strickland. "Finite subgroups of formal groups". In: *Journal of Pure and Applied Algebra* 121.2 (Sept. 25, 1997), pp. 161–208.

[Str98] Neil P. Strickland. *Morava E-theory of symmetric groups.* Jan. 28, 1998. arXiv: `math/9801125`.

[Yua21] Allen Yuan. *Examples of chromatic redshift in algebraic K-theory.* Nov. 21, 2021. arXiv: `2111.10837 [math]`.