The construction of the $\widehat{A}$-genus, continued

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1 Introduction

Recall that we're trying to realize the $\widehat{A}$-genus as an E_∞ map $MSpin \to KO$; more generally, we'd like to compute $\pi_0 E_\infty(MSpin, KO)$. The arithmetic square exhibits KO has a homotopy pullback

$$\begin{array}{ccc} KO & \longrightarrow & \prod_p KO_p \\ \downarrow & & \downarrow \\ KO_{\mathbb{Q}} & \longrightarrow & \left(\prod_p KO_p\right)_{\mathbb{Q}}, \end{array}$$

(where KO_p is and forever shall be p-complete real K-theory), and thus we have a homotopy pushout

$$\begin{array}{ccc} E_\infty(MSpin, KO) & \longrightarrow & \prod_p E_\infty(MSpin, KO_p) \\ \downarrow & & \downarrow \\ E_\infty(MSpin, KO_{\mathbb{Q}}) & \longrightarrow & \left(\prod_p E_\infty(MSpin, KO_p)\right)_{\mathbb{Q}}. \end{array}$$

By the general nonsense of [4], the space of E_∞ maps $MSpin \to X$ is the space of lifts in the diagram

$$\begin{array}{ccccccc} spin & \longrightarrow & gl_1 S & \longrightarrow & gl_1 S/spin & \longrightarrow & bspin \\ & & \downarrow & \swarrow & & & \\ & & gl_1 X; & & & & \end{array}$$

in particular, it's nonempty iff the composition $spin \to gl_1 X$ is trivial, in which case it's given by $[bspin, gl_1 X]$. We're thus reduced to finding $[spin, gl_1 KO]$ and $[bspin, gl_1 KO]$. By the above arithmetic square argument, this reduces to finding these mapping spaces for $gl_1 KO_{\mathbb{Q}}$, $\prod_p gl_1 KO_p$, and $\left(\prod_p gl_1 KO_p\right)_{\mathbb{Q}}$. Ben's done this for $KO_{\mathbb{Q}}$, and in these notes, we'll finish up by doing the other two steps.

2 The Bousfield-Kuhn functor and finite localization

If you've ever looked at Bousfield's construction of localization [5], you'll know that in its original form, it uses cell structures for the spectra involved and a cardinality argument: the fiber of $L_E X$ is given by killing all the maps to X from E-acyclic spectra with less than κ cells, for a certain cardinal κ depending on E. If we'd like to, we can instead build this cardinal κ into the construction, at the cost of losing some E-localness.

Definition 1. A spectrum X is **finitely E-local** if $[A, X]_* = 0$ whenever A is a finite E-acyclic spectrum. X is **finitely E-acyclic** if $[X, B]_* = 0$ for any finitely E-local spectrum B. A map $X \to L_E^f X$ is a **finite E-localization** if $L_E^f X$ is finitely E-local and the cofiber of the map is finitely E-acyclic.

Theorem 2 ([8], [7])**.** *For any E, there's a finite E-localization functor L_E^f with a natural transformation from the identity.*

A good exposition is in [8]; the proof of the above theorem, though, is a fairly obvious generalization of Bousfield's original. There are a few things worth saying about this functor:

- An E-local spectrum is already finitely E-local; conversely, there's a natural map $L_E^f \to L_E$.
- By a Spanier-Whitehead duality argument, $L_E^f X \simeq X \wedge L_E^f S$. Thus, in particular, finite localization is Bousfield localization with respect to the spectrum $L_E^f S$.
- Fracture squares for finite localizations exist in exactly the same way as those for general localizations.
- When $E = K(n)$ and X is type n, $L_E^f X$ is the telescope $\operatorname{hocolim} \Sigma^{-nd} X$ of any map v_n-self map $g : \Sigma^d X \to X$. Thus the telescope conjecture at level n is equivalent to the statement that $L_{K(n)}^f = L_{K(n)}$; we can therefore generalize the telescope conjecture to any spectrum E other than $K(n)$, and in some cases it's even true.

(If you're looking for a summer project on the interactions between set theory and homotopy theory, why not consider localizations of bounded cardinality for larger cardinals than $\aleph_0$?)

Since the computation we're doing involves a fair bit of bouncing around between spectra, their connective covers, and their infinite loop spaces, the following theorem is useful.

Theorem 3 ([6])**.** *For any $n \geq 1$, there is a functor $\Phi_n : \mathsf{HoSpaces}_* \to \mathsf{HoSpectra}$ such that $\Phi_n \Omega^\infty \cong L_{K(n)}$. Likewise, there's a functor $\Phi_n^f : \mathsf{HoSpaces}_* \to \mathsf{HoSpectra}$ such that $\Phi_n^f \Omega^\infty \cong L_{K(n)}^f$, and we can take $\Phi_n = L_{K(n)} \Phi_n^f$.*

These are called the **Bousfield-Kuhn functors**: Bousfield constructed them for $n = 1$, and Kuhn extended this to all n. The point of this is that $K(n)$-localization of spectra factors through spaces; alternatively, you can get back from spaces to spectra at the cost of $K(n)$-localizing.

We'll see the use of this as we keep going. One immediate conclusion is the following. If R is an E_∞ spectrum and X a pointed space, then the units of R are defined so that the *pointed unstable maps* $[X, \Omega^\infty gl_1 R]_+$ are

$$[X, \Omega^\infty gl_1 R]_+ \cong (1 + \widetilde{R}^0(X_+))^\times \subseteq R^0(*) \oplus \widetilde{R}^0(X_+) = R^0(X_+) \cong [X, \Omega^\infty R]_+.$$

When $X = S^k$ for $k \geq 1$, this map is clearly an isomorphism. Thus we have a weak equivalence of spaces $\Omega^\infty gl_1 R\langle 1 \rangle \xrightarrow{\sim} \Omega^\infty R\langle 1 \rangle$. The Bousfield-Kuhn functor lets us lose the connectivity and move to spectra as long as we $K(n)$-localize. Thus, there's a weak equivalence

$$L_{K(n)} gl_1 R \xrightarrow{\sim} L_{K(n)} R,$$

and the composition

$$gl_1 R \to L_{K(n)} gl_1 R \to L_{K(n)} R$$

is Rezk's logarithm functor. Similar statements hold for finite localizations.

3 How to $K(1)$-localize friends and dualize people

We're looking for maps $[spin, gl_1 KO_p]$ and $[bspin, gl_1 KO_p]$. In this section, we'll show that it suffices to deal with these $K(1)$-locally. First, we deal with those pesky gl_1's, which don't generally commute with localization.

Theorem 4. *If R is an E_n-local E_∞-ring spectrum, then $\pi_* gl_1 R \to \pi_* L_n gl_1 R$ is an isomorphism in degrees greater than or equal to $n + 2$.*

Proof. The first step is to replace localization with finite localization. Taking $0 \le m \le n$ and using the logarithm, we have $L_{K(m)} gl_1 R \simeq L_{K(m)} R$ and $L^f_{K(m)} gl_1 R \simeq L^f_{K(m)} R$. Unfortunately, R isn't necessarily $K(m)$-local, but using the fracture squares for $L_n R$ and $L^f_n R$ and the fact that $L_n R \simeq L^f_n R$ (they're both just R), we get $L_{K(m)} R \simeq L^f_{K(m)} R$, and thus that $L_{K(m)} gl_1 R \simeq L^f_{K(m)} gl_1 R$. Reversing the argument of the previous sentence then gives

$$L_n gl_1 R \simeq L^f_n gl_1 R.$$

Thus, the fiber F of $gl_1 R \to L_n gl_1 R$ is also the fiber of $gl_1 R \to L^f_n gl_1 R$. We want to prove that $\pi_* F = 0$ in degrees greater than or equal to $n + 1$.

Now, $gl_1 R \to L^f_n gl_1 R$ is given by smashing $gl_1 R$ with $S \to L^f_n S$. By construction, the fiber of this map is a filtered colimit of finite E_n-acyclic (type $n+1$) spectra Z_α. Thus $F = \operatorname{hocolim}_k F_\alpha$, where $F_\alpha = gl_1 R \wedge Z_\alpha$.

Since each Z_α is type $n + 1$, it's acyclic with respect to $K(0) = H\mathbb{Q}$, and thus its homotopy groups are torsion. The same obviously applies for F_α.

Since each Z_α is finite, we can form its Spanier-Whitehead dual DZ_α, which is also finite, and thus for some q, $\Sigma^q DZ_\alpha = \Sigma^\infty K_\alpha$ for a connected finite complex (that is, a *space*) K_α. Then

$$\Omega^\infty \Sigma^{-q} F_\alpha \simeq \Omega^\infty F(\Sigma^q DZ_\alpha, gl_1 R),$$

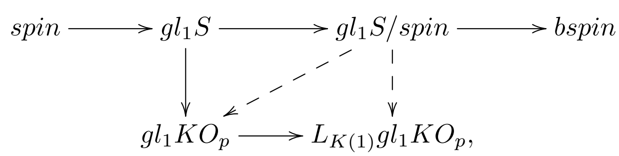
where F denotes the function spectrum; of course, this is just the function *space* $\mathsf{Spectra}(\Sigma^\infty K_\alpha, gl_1 R)$, which is the function space $\mathsf{Spaces}_*(K_\alpha, \Omega^\infty gl_1 R)$. Since K_α is connected, we can use the unstable logarithm to replace this with $\mathsf{Spaces}_*(K_\alpha, \Omega^\infty R) = \mathsf{Spectra}(\Sigma^\infty K_\alpha, R)$; but R is E_n-local and K_α is E_n-acyclic, so this is trivial. Thus, $\pi_n F_\alpha = 0$ for $n > q$. We say that F_α is **coconnected**.

Since F_α is p-local, torsion and coconnected, the fiber of $F_\alpha \to P_n F_\alpha$ can be factored into finitely many wedges of $\Sigma^q H\mathbb{F}_p$ with $q > n$. Again using the Ravenel-Wilson calculation $K(m)_* K(\mathbb{F}_p, q) = 0$ for $q > m$, we get that for $q > n$,

$$[\Sigma^q H\mathbb{F}_p, gl_1 R] = \pi_0 E_\infty(\Sigma^\infty_+ K(\mathbb{F}_p, q), R) \subseteq [\Sigma^\infty_+ K(\mathbb{F}_p, q), R] = 0,$$

since R is E_n-local. Likewise, $[\Sigma^q H\mathbb{F}_p, L_n gl_1 R] = 0$ with even fewer steps. Thus, $[\Sigma^q H\mathbb{F}_p, F] = 0$ for $q > n$, and so $[F_\alpha, F] \cong [P_n F_\alpha, F]$, proving that $F = P_n F$, as desired. ☐

In the case $R = KO_p$, R is E_1-local, so $gl_1 KO_p \to L_{K(1)} gl_1 KO_p$ is an isomorphism on homotopy groups in degrees at least 3. And wouldn't you know it, it just so happens this is where *spin* has its first homotopy group! What this means is that, as spin orientations correspond to dotted maps in the diagram



and maps from *spin* are the obstructions to the existence of these maps, a dotted map exists to $gl_1 KO_p$ iff it does to $L_{K(1)} gl_1 KO_p$. Moreover, *bspin* has its first homotopy group in degree 4, so clearly $[bspin, gl_1 KO_p] \cong [bspin, L_{K(1)} gl_1 KO_p]$. Thus we have proved that

$$E_\infty(MSpin, gl_1 KO_p) \simeq E_\infty(MSpin, L_{K(1)} gl_1 KO_p)$$

in that one is nonempty iff the other is in which case they're indeed weakly equivalent.

Remark 5. Since *bspin* is 3-connected, we just scraped by here. I'm pretty sure this is why you can't spin-orient tmf, but only string-orient it: you now need to deal with both $K(1)$-local and $K(2)$-local stuff, and $K(2)$-localization on $gl_1 tmf_p$ is now only an isomorphism on homotopy groups in degrees 4 and above.

We thus are able to replace our codomain with a $K(1)$-local spectrum. Now we go about replacing the domain.

Lemma 6. $L_{K(1)} bspin \simeq KO_p$.

Proof. Clearly K_p is $K(1)$-local; KO_p is a fiber of a self-map of K_p, so it's also $K(1)$-local; using the Bousfield-Kuhn functor, we get that $L_{K(1)}bo = KO_p$. Finally, $bspin \to bo$ is an isomorphism on $K(1)_*$. Since $E(1) = K_p$ and the Bousfield class of $K(1)$ is a summand of that of $E(1)$, it suffices to show that this map is an isomorphism on (completed, complex) K-theory. For $p > 2$, this map is already an isomorphism; for $p = 2$, its fiber has two nonzero homotopy groups, both of which are $\mathbb{F}_2$'s, and we decompose the map as a composition of two maps, each of whose fibers is an $H\mathbb{F}_2$. There's probably an easier way of proving that $K(1)_*H\mathbb{F}_2 = 0$, but one way to go about it is to use the Ravenel-Wilson computation of the Morava K-theories of Eilenberg-Mac Lane spaces, which all vanish stably [9]. □

Thus, at last, we have

$$\begin{aligned} \pi_0 E_\infty(MSpin, gl_1 KO_p) &\cong [bspin, gl_1 KO_p] \\ &\cong [bspin, L_{K(1)} gl_1 KO_p] \\ &\cong [KO_p, L_{K(1)} gl_1 KO_p] && \text{(by the above lemma)} \\ &\cong [KO_p, L_{K(1)} KO_p] && \text{(via the logarithm)} \\ &\cong [KO_p, KO_p]. && \text{(since } KO_p \text{ is } K(1)\text{-local)} \end{aligned}$$

This is the algebra of operations on p-complete real K-theory. Onwards to its computation!

4 Measure for measure

We're trying to compute $[KO_p, KO_p] = KO_p^0 KO_p$. It turns out to be best to study this cohomology group via its duality with homology, and to work in the $K(1)$-local category. To that end, we define

Definition 7. If E and X are $K(1)$-local spectra, the **($K(1)$)-completed homology** of X with coefficients in E is $E_*^\wedge X = \pi_* L_{K(1)} E \wedge X$.

There's a pairing at work between degree 0 completed homology and degree 0 cohomology, which is familiar but worth carefully recalling. If E is a $K(1)$-local ring spectrum, $f : S \to L_{K(1)} E \wedge E$ is a map representing a completed homology class, and $\alpha : E \to E$ represents a cohomology class, then there's a map

$$S \xrightarrow{f} L_{K(1)} E \wedge E \xrightarrow{1\wedge\alpha} L_{K(1)} E \wedge E \to E$$

giving an element $\langle f, \alpha \rangle$ in $\pi_0 E$. If $\langle f, \alpha \rangle = 0$ for fixed α and all f, then α induces the zero map after $K(1)$-locally smashing with E; since E is E-local in the $K(1)$-local category, this means $\alpha = 0$. Likewise, if $\langle f, \alpha \rangle = 0$ for fixed f and all α, then $f = 0$. Thus we have a nondegenerate pairing between cohomology and completed homology in degree zero.

In particular, if $E = K_p$ and $\alpha = \psi^\lambda$ for $\lambda \in \mathbb{Z}_p^\times$, then $f(\lambda) = \langle f, \lambda \rangle \in \pi_0 K_p = \mathbb{Z}_p$. Thus f defines a function $\mathbb{Z}_p^\times \to \mathbb{Z}_p$. Likewise, $f \in (KO_p)_0^\wedge KO_p$ induces a map $\mathbb{Z}_p^\times/\{\pm 1\} \to \mathbb{Z}_p$, since KO_p is fixed by complex conjugation ψ^{-1}.

Proposition 8. *Under the above map,*

$$(KO_p)_0^\wedge KO_p \cong \mathrm{Hom}_c(\mathbb{Z}_p^\times/\{\pm 1\}, \mathbb{Z}_p)$$

and thus

$$KO_p^0 KO_p \cong \mathrm{Hom}_c(\mathrm{Hom}_c(\mathbb{Z}_p^\times/\{\pm 1\}, \mathbb{Z}_p)),$$

where a subscript c indicates continuous maps.

Proof. If $v_p(\lambda - \mu) = r$ and λ and μ are units, then $v_p(\lambda/\mu) = r$ as well The Adams-Harris-Switzer computation of the cooperations on KO [3] is

$$KO_0 KO \cong \{f \in \mathbb{Q}[x, x^{-1}] : f(-x) = f(x), f(k) \in \mathbb{Z}[1/k] \text{ for all } k\},$$

with this isomorphism given by sending $f : S \to KO \wedge KO$ to the composition

$$S \xrightarrow{f} KO \wedge KO \xrightarrow{1\wedge\psi^k} KO \wedge KO[1/k] \to KO[1/k].$$

Now, the $K(1)$-localization of $KO \wedge KO$ is just its p-adic completion, which we can write as $\varprojlim (KO/p^r) \wedge KO$. Indeed, by the standard arguments, the $K(1)$-localization in this case can be computed by completing at p and inverting v_1, but v_1 is already invertible. Now, for any $\overline{f} \in (KO/p^r)_0 KO$ and any $c \in \mathbb{Z}_p^\times$, let k be an integer congruent to c mod p^r; then k is prime to p, so $\mathbb{Z}[1/k]/p^r \cong \mathbb{Z}/p^r$, and $f(k)$ is well-defined mod p^r, and depends only on c and f mod p^r. Thus we get a map $(KO/p^r)_0 KO \to \mathrm{Hom}((\mathbb{Z}/p^r)^\times/\{\pm 1\}, \mathbb{Z}/p^r)$, and taking the limit gives

$$(KO_p)_0^\wedge KO_p \to \mathrm{Hom}_c(\mathbb{Z}_p^\times/\{\pm 1\}, \mathbb{Z}_p).$$

Conversely, given any continuous map $g : \mathbb{Z}_p^\times/\{\pm 1\} \to \mathbb{Z}_p$, the reduction mod p^r $\overline{g}$ can be written as a polynomial which sends each $k \in (\mathbb{Z}/p^r)^\times$ to an integer in $\mathbb{Z}/p^r$, and we can clearly extend this to the non-units by declaring it to be zero there. Taking the limit gives an element of $(KO_p)_0^\wedge KO_p$.

To turn this into a statement about cohomology, we need something universal coefficient-y. For this we go back to [1], where it's proved that, under a condition on a ring spectrum E, if $E_* X$ is projective over $\pi_* E$, then we have a canonical isomorphism

$$E^* X \xrightarrow{\cong} \mathrm{Hom}^*_{\pi_* E}(E_* X, \pi_* E).$$

The condition is that E is a homotopy colimit of finite spectra E_α such that each dual DE_α satisfies both the assumption and the conclusion of the previous sentence. Adams proves that these conditions are satisfied for KO in particular – if (E_α) are a system of finite spectra as in the condition, then (E_α/p^r) exhibit the condition for KO/p^r, and thus we get

$$(KO/p^r)^* KO_p \cong (KO/p^r)^* KO \cong \mathrm{Hom}^*_{\pi_*(KO/p^r)}((KO/p^r)_* KO, \pi_*(KO/p^r)).$$

Taking the limit, we get that

$$(KO_p)^* KO_p \cong \mathrm{Hom}^*_{\pi_* KO_p}((KO_p)_*^\wedge KO_p, \pi_* KO_p).$$

Finally, taking the degree zero part and doing something mysterious, we get the desired result. □

The point of this is that homology elements are *functionals* on $\mathbb{Z}_p^\times$, and cohomology elements are *measures* on the space of such functionals. We write

$$\int f \, d\alpha = \langle f, \alpha \rangle.$$

In particular, $\int f \, d\psi^\lambda = \langle f, \psi^\lambda \rangle = f(\lambda)$ by definition, so ψ^λ is the Dirac measure at λ.

We'd like to use this viewpoint to understand the effect of a cohomology class on $\pi_* KO_p$. After all, Ben's construction of the rational genus spat out a sequence of rational numbers coming from elements of these homotopy groups, so we need something of the same format if we have any hope of reconciling the two. If $\alpha \in KO_p^0 KO_p$, then the effect of α on π_{4k} is given by $(\alpha_* v^{2k})/v^{2k}$, where v is the image of the complex Bott element. In particular, if $\alpha = \psi^\lambda$, then this is λ^{2k}, proving that $\pi_{4k}\alpha = \int x^{2k} \, d\alpha$. These numbers are called the **moments** of the measure $d\alpha$. Another way of seeing this is that we're pairing α with the homology class

$$S \simeq S^{-4k} \wedge S^{4k} \xrightarrow{v^{-2k} \wedge v^{2k}} L_{K(1)} KO_p \wedge KO_p.$$

The point is that every cohomology class gives us a sequence of p-adic integers given by what it does to the homotopy groups of KO_p; conversely, by the above proposition, if you have a sequence of numbers arising as the moments of some measure, then that measure will correspond to a cohomology class. So we've finally reduced our task to a question in p-adic analysis: which sequences of p-adic integers arise as the (even) moments of some measure on $\mathbb{Z}_p^\times$?

Remark 9. In the real case, this is called the 'moment problem,' and was solved by Riesz of representation-theorem fame in the 20s.

The secret is that a sequence of moments on $\mathbb{Z}_p^\times$ is just a sequence that behaves like the sequence of powers of some p-adic unit when any 'test' numerical polynomial is evaluated on it. Moreover, we can take our test polynomials to be divisible by an arbitrarily high power of their variable x.

Definition 10. Fix $n \geq 0$. Let A_n be the set of polynomials

$$h(x) = a_n x^n + a_{n+1} x^{n+1} + \cdots + a_m x^m,$$

with coefficients in $\mathbb{Q}_p$, such that $h(c) \in \mathbb{Z}_p$ for all $c \in \mathbb{Z}_p^\times$. We say that a sequence $(z_k)_{k \geq n}$ satisfies the **generalized Kummer congruences** if, for all $h(x) = \sum_{k \geq n} a_k x^k \in A_n$, we have $h * (z_k) := \sum_{k \geq n} a_k z_k \in \mathbb{Z}_p$.

Proposition 11. *For any $n \geq 0$, the map*

$$KO_p^0 KO_p \to \prod_{k \geq n} \mathbb{Z}_p \quad : \quad \alpha \mapsto (\pi_{2k}\alpha)_{k \geq n}$$

is injective, with image the set of sequences $(z_k)_{k \geq n}$ (with $z_k = 0$ for k odd) satisfying the generalized Kummer congruences.

Proof. Let $h(x) = \sum a_k x^k \in A_n$. Then since each $\pi_{2k}\alpha = \int x^k \, d\alpha$, we have $h * (\pi_{2k}\alpha) = \int h \, d\alpha$, which is a p-adic integer since it's the integral of an integer-valued function with respect to an integer-valued measure. Thus, every sequence $(\pi_{2k}\alpha)_{k \geq n}$ satisfies the generalized Kummer congruences. Conversely, if (z_k) satisfies the generalized Kummer congruences, then $h * (z_k) \in \mathbb{Z}_p$ for all $h \in A_n$; but A_n is dense in $\mathrm{Hom}_c(\mathbb{Z}_p^\times, \mathbb{Z}_p)$, so we can define a measure α with $\int h \, d\alpha = h * (z_k)$ for all $h \in A_n$. □

Remark 12. By the same argument, we can identify $K_p^0 K_p$ with the set of all sequences $(z_k)_{k \geq n}$ in $\prod \mathbb{Z}_p$ satisfying the generalized Kummer congruences.

5 Crossing the streams

Now, we still need to turn the above result on operations on KO_p into some more understandable statement about spin orientations of KO_p, but it'll be easier to first discuss the bottom right corner of the square, which is the product of $\pi_0 E_\infty(MSpin, KO_p) \otimes \mathbb{Q} \cong [bspin, KO_p] \otimes \mathbb{Q}$.

Remark 13. Strictly speaking, tensor products don't distribute over direct products of modules – in this product of $\mathbb{Z}_{(p)}$-modules tensored with $\mathbb{Q}$, we only get tuples (α_p) in which a finite number of the α_p have negative p-adic valuations. However, as will soon become clear, these are the only tuples in the image of $\pi_0 E_\infty(MSpin, KO_\mathbb{Q})$, so this algebraic wrinkle is unimportant in finding the pullback.

Ben constructed the rational $\widehat{A}$-genus as a lift

$$\begin{array}{ccccccc} spin & \longrightarrow & gl_1 S & \longrightarrow & gl_1 S/spin & \longrightarrow & bspin \\ & & \downarrow & & \downarrow & & \\ & & gl_1 KO & \longrightarrow & gl_1 KO_\mathbb{Q} & & \end{array}$$

Since $gl_1 S$ is torsion, though, it's killed by the map from $gl_1 KO$ to its rationalization. Thus a rational orientation is equivalent to a map of cofiber sequences

$$\begin{array}{ccccccc} gl_1 S & \longrightarrow & gl_1 S/spin & \longrightarrow & bspin & \longrightarrow & \Sigma gl_1 S \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \Sigma^{-1} gl_1 KO \otimes \mathbb{Q}/\mathbb{Z} & \longrightarrow & gl_1 KO & \longrightarrow & gl_1 KO_\mathbb{Q} & \longrightarrow & gl_1 KO \otimes \mathbb{Q}/\mathbb{Z} \end{array}$$

where I've written $gl_1 KO \otimes \mathbb{Q}/\mathbb{Z}$ for the cofiber of the rationalization map. In particular, the composition $\mathbf{m}_{gl_1 KO} : bspin \to gl_1 KO\mathbb{Q}/\mathbb{Z}$ is fixed (it's called the **stable Miller invariant** of $gl_1 KO$). On homotopy, this map is just the characteristic series mod $\mathbb{Z}$ – of *any* rational orientation – and so, in particular, taking our orientation to be the rational $\widehat{A}$-genus, we get

$$(\mathbf{m}_{KO})_* v^k = -\frac{B_k}{2k} v^k \mod \mathbb{Z}$$

for even k.

The same argument applies to the rationalization of the p-adic case. We're looking for maps filling in a diagram

$$\begin{array}{ccccccc}
gl_1S & \longrightarrow & gl_1S/spin & \longrightarrow & bspin & \longrightarrow & \Sigma gl_1S \\
\downarrow & & \vdots & & \vdots & \searrow^{\mathbf{m}_{gl_1KO_p}} & \downarrow \\
\Sigma^{-1}gl_1KO_p \otimes \mathbb{Q}/\mathbb{Z} & \longrightarrow & gl_1KO_p & \longrightarrow & (gl_1KO_p)_{\mathbb{Q}} & \longrightarrow & gl_1KO_p \otimes \mathbb{Q}/\mathbb{Z};
\end{array}$$

the arguments of section 2 let us $K(1)$-localize and post-compose with the logarithm, giving a diagram

$$\begin{array}{ccccccc}
L_{K(1)}gl_1S & \longrightarrow & L_{K(1)}gl_1S/spin & \longrightarrow & KO_p & \longrightarrow & \Sigma L_{K(1)}gl_1S \\
\downarrow & & \vdots & & \vdots & \searrow^{\mathbf{m}_{gl_1KO_p}} & \downarrow \\
\Sigma^{-1}L_{K(1)}gl_1KO_p \otimes \mathbb{Q}/\mathbb{Z} & \longrightarrow & L_{K(1)}gl_1KO_p & \longrightarrow & L_{K(1)}(gl_1KO_p)_{\mathbb{Q}} & \longrightarrow & L_{K(1)}gl_1KO_p \otimes \mathbb{Q}/\mathbb{Z} \\
\ell_1\downarrow & & \ell_1\downarrow & & \ell_1\downarrow & & \ell_1\downarrow \\
\Sigma^{-1}KO_p \otimes \mathbb{Q}/Z & \longrightarrow & KO_p & \longrightarrow & (KO_p)_{\mathbb{Q}} & \longrightarrow & KO_p \otimes \mathbb{Q}/\mathbb{Z}.
\end{array}$$

The logarithm map ℓ_1 is a weak equivalence $L_{K(1)}gl_1KO_p \xrightarrow{\sim} KO_p$, so it also is rationally, and thus the bottom row of vertical maps are all weak equivalences. Meanwhile, using the logarithm map, the computation $L_{K(1)}S \simeq j_p$ (where j is the spectrum representing the image of the J-homomorphism), and the Adams conjecture, we can replace the top row to get

$$\begin{array}{ccccccc}
j_p & \longrightarrow & KO_p & \xrightarrow{1-\psi^c} & KO_p & \longrightarrow & \Sigma j_p \\
\downarrow & & \sim\downarrow b_c & & \downarrow 1 & & \downarrow \\
L_{K(1)}gl_1S & \longrightarrow & L_{K(1)}gl_1S/spin & \longrightarrow & KO_p & \longrightarrow & \Sigma L_{K(1)}gl_1S \\
\downarrow & & \vdots\,\alpha & & \vdots & \searrow^{\mathbf{m}_{gl_1KO_p}} & \downarrow \\
\Sigma^{-1}L_{K(1)}gl_1KO_p \otimes \mathbb{Q}/\mathbb{Z} & \longrightarrow & L_{K(1)}gl_1KO_p & \longrightarrow & L_{K(1)}(gl_1KO_p)_{\mathbb{Q}} & \longrightarrow & L_{K(1)}gl_1KO_p \otimes \mathbb{Q}/\mathbb{Z} \\
\sim\downarrow & & \sim\downarrow \ell_1 & & \sim\downarrow & & \sim\downarrow \\
\Sigma^{-1}KO_p \otimes \mathbb{Q}/Z & \longrightarrow & KO_p & \longrightarrow & (KO_p)_{\mathbb{Q}} & \longrightarrow & KO_p \otimes \mathbb{Q}/\mathbb{Z}.
\end{array}$$

Here c is a generator of $\mathbb{Z}_p/\{\pm 1\}$.

Sending a dotted map α as in the diagram to the composition $\ell_1 \alpha b_c \in [KO_p, KO_p]$, and then to the effect of this on homotopy groups $\pi_{2k}KO_p$ for $k \geq 2$, gives an element of $\prod_{k\geq 2}\mathbb{Z}_p$, which we write $(t_k(\alpha))$. Moving one column to the right, we get an element of $[KO_p, L_{K(1)}(gl_1KO_p)_{\mathbb{Q}}]$, whose effect on homotopy is a sequence $(b_k(\alpha)) \in \prod_{k\geq 2}\pi_{2k}L_{K(1)}(gl_1KO_p)_{\mathbb{Q}}$. Of course, this lifts to a map to $(gl_1KO_p)_{\mathbb{Q}}$, and so we in fact get a sequence in $\prod_{k\geq 2}\mathbb{Q}_p$. Using Rezk's work on the logarithm [10], we have that $(\ell_1)_* b_k(\alpha) = (1-p^{k-1})b_k(\alpha)$, and thus,

$$t_k(\alpha) = (1-\psi^c)^*(\ell_1)_* b_k(\alpha) = (1-c^k)(1-p^{k-1})b_k(\alpha).$$

We have thus proved that

Proposition 14. *The set $\pi_0 E_\infty(MSpin, KO_p)$ can be identified (by applying $b_c^*(\ell_1)_*$ and looking at the effect on homotopy groups) with the set of sequences $(b_k)_{k\geq 2} \in \prod_{k\geq 2}\mathbb{Q}_p$ such that*

- *$b_k = 0$ for k odd,*
- *the sequence $((1-c^k)(1-p^{k-1})b_k)_{k\geq 2}$ satisfies the generalized Kummer congruences,*
- *and $b_k \equiv -B_k/2k$ mod $\mathbb{Z}_p$.*

Remark 15. As shown in [2], for $p > 2$, the p-adic valuation of $-B_k/2k$ is $-v_p(k) - 1$. On the other hand, let $c = 1 + pa$ for $a \in \mathbb{Z}_p$. Then $1 - c^k = \pm \sum_{s=1}^{k} \binom{k}{s} p^s a^s$, so that

$$v_p(1-c^k) \geq \inf_{1\leq s\leq k} s+v_p\left(\binom{k}{s}\right) = s-v_p(s!)+\sum_{i=0}^{s-1} v_p(k-i) \geq s-\sum_{k=1}^{\infty}\left\lfloor \frac{s}{p^k}\right\rfloor+v_p(k) \geq s+v_p(k)-\frac{s}{p-1} \geq v_p(k)+1.$$

Thus $(1 - c^k)(1 - p^{k-1})b_k$ is indeed an integer, for b_k satisfying condition 3 above. For $p = 2$, we instead have $v_p\left(-\frac{B_k}{2k}\right) = -2 - v_2(k)$ for k even (it's just -1 for $k > 1$ odd), but now c is of the form $\pm(1 + 4a)$, so the same argument shows that the desired number is an integer.

Combining everything we've done so far gives us the desired theorem.

Theorem 16. *The set $\pi_0 E_\infty(MSpin, KO)$ can be identified with the set of sequences $(b_k)_{k\geq 2} \in \prod_{k\geq 2} \mathbb{Q}$ such that*

- *$b_k = 0$ for k odd,*
- *for each prime p and p-adic unit c, the sequence$((1 - c^k)(1 - p^{k-1})b_k)_{k\geq 2}$ satisfies the generalized Kummer congruences,*
- *and $b_k \equiv -B_k/2k \bmod \mathbb{Z}$.*

6 Analysis prelim practice

It's still not clear that such a sequence exists! As it turns out, though, $b_k = -B_k/2k$ works just fine. To prove this requires a little more p-adic analysis. We want to construct a measure on $\mathbb{Z}_p/\{\pm 1\}$ whose kth moment, for k even, is $-(1-p^{k-1})(1-c^k)\frac{B_k}{2k}$. This is called the (half) **Mazur measure**. We first construct it on $\mathbb{Z}_p^\times$; we then prove it has the required moments; there's finally a technical detail required to 'halve' this measure when $p = 2$.

As before, let A_1 be the set of polynomials $h \in \mathbb{Q}_p[x]$ with zero constant term such that $h(c) \in \mathbb{Z}_p$ for all $c \in \mathbb{Z}_p^\times$.

Theorem 17. *Let $c \in \mathbb{Z}_p^\times$. There's a $\mathbb{Z}_p$-valued measure μ_c on $\mathbb{Z}_p^\times$ uniquely characterized by the property that, for all $h \in A_1$,*

$$\int_{\mathbb{Z}_p^\times} h\, d\mu_c = \lim_{r\to\infty} \frac{1}{p^r} \sum_{\substack{0\leq i<p^r \\ p\nmid i}} \int_i^{ci} \frac{h(t)}{t}\, dt,$$

the inner integral being a formal antiderivative. The moments of this measure are

$$\int_{\mathbb{Z}_p^\times} x^k\, d\mu_c = -\frac{B_k}{k}(1 - p^{k-1})(1 - c^k),$$

for $k \geq 1$, and

$$\int_{\mathbb{Z}_p^\times} d\mu_c = \frac{1}{p}\log(c^{p-1}).$$

Finally, there's a $\mathbb{Z}_p$-valued measure μ_c' on $\mathbb{Z}_p^\times/\{\pm 1\}$ given by

$$\int_{\mathbb{Z}_p^\times/\{\pm 1\}} f\, d\mu_c' = \frac{1}{2}\int_{\mathbb{Z}_p} f\, d\mu_c.$$

Proof. Since A_1 is dense in $\mathrm{Hom}_c(\mathbb{Z}_p^\times, \mathbb{Z}_p)$, the only thing needed to do to construct the measure is to show that the limit exists and is integral. To show the limit exists, it suffices by linearity to take $h(x) = x^k$ – that is, to calculate the moments.

Recall that the **Bernoulli polynomials** are the coefficients $B_k(t)$ in

$$\frac{xe^{tx}}{e^x-1} = \sum_k B_k(t)\frac{x^k}{k!}.$$

We have $B_k = B_k(0)$. Define $F_k(t) = B_k(t) - B_k$ – these are the coefficients of the power series expansion in x of $\frac{x(e^{tx}-1)}{e^x-1}$. Taking $t = n$, we have

$$\sum_k F_{k+1}(n)\frac{x^k}{(k+1)!} = \frac{e^{nx}-1}{e^x-1} = \sum_{m=0}^{n-1} e^{mx}$$

Thus, if $S_k(n)$ is the power sum $\sum_{m=0}^n m^k$, then

$$S_k(n) = \frac{F_{k+1}(n)}{k+1}.$$

Let $S_k^*(n)$ be the sum of the m^k over all m prime to p between 0 and n. Then

$$S_k^*(p^r) = S_k(p^r) - p^k S_k(p^{r-1}) = \frac{1}{k+1}(F_{k+1}(p^r) - p^k F_{k+1}(p^{r-1})).$$

One can check that as $r \to \infty$, $S_k^*(p^r)/p^r$ approaches $(1-p^{k-1})B_k$.

If $h(x) = x^k$ for $k \geq 1$, we now have

$$\frac{1}{p^r}\sum_{\substack{0\leq i<p^r \\ p\nmid i}} \int_i^{ci} \frac{h(t)}{t}\,dt = \frac{1}{p^r}\sum_{\substack{0\leq i<p^r \\ p\nmid i}} \frac{i^k(c^k-1)}{k} = \frac{c^k-1}{kp^r}S_k^*(p^r).$$

Taking the limit as $r \to \infty$, we get the moment $-(1-p^{k-1})(1-c^k)\frac{B_k}{k}$. This proves that the Mazur measure exists and is well-defined.

To show that it's integral, let $h(x) = \sum a_k x^k \in A_1$, and let $r_0 = -\min_k v_p(a_k/k)$. Let $r \geq r_0$. If i is an integer between 0 and $p^r - 1$ that is a unit in $\mathbb{Z}/p^r$, then ci is also a unit mod p^r, and thus of the form

$$ci = j(1 + m_r(j)p^r)$$

for some $0 \leq j < p^r - 1$ and some $m_r(j) \in \mathbb{Z}_p$. We now calculate that

$$\sum_{\substack{0\leq i<p^r \\ p\nmid i}} (ci)^k = \sum_{\substack{0\leq j<p^r \\ p\nmid j}} j^k(1+m_r(j)p^r)^k \equiv \sum_{\substack{0\leq j<p^r \\ p\nmid j}} j^k + kj^k m_r(j)p^r \mod p^{2r},$$

so that

$$\sum_{\substack{0\leq i<p^r \\ p\nmid i}} (ci)^k - i^k = \sum_{\substack{0\leq j<p^r \\ p\nmid j}} kj^k m_r(j)p^r \mod p^{2r}.$$

The term we want, for our polynomial $h \in A_1$, is then

$$\frac{1}{p^r}\sum_{\substack{0\leq i<p^r \\ p\nmid i}}\sum_k \frac{a_k}{k}((ci)^k - i^k) = \sum_{\substack{0\leq j<p^r \\ p\nmid j}}\sum_k a_k m_r(j) j^k = \sum_{\substack{0\leq j<p^r \\ p\nmid j}} m_r(j)h(j) \mod p^{r-r_0},$$

since we've divided by p^r and by each k occuring in the sum. Since $h(j)$ and $m_r(j)$ are integral, this sum is integral for $r \geq r_0$, and thus the limit, since it exists, is integral.

The computation of the measure's volume just uses the approximation $1 = \lim_{k\to\infty} x^{(p-1)p^k}$.

Construction of the half-measure is obvious when $p > 2$. When $p = 2$, we're saying that whenever $f : \mathbb{Z}_2^\times \to \mathbb{Z}_2$ has $f(x) = f(-x)$, then $\int f\,d\mu_c \in 2\mathbb{Z}_2$. Let A_1' be the set of even polynomials in A_1, which is clearly dense in the set of continuous even maps $\mathbb{Z}_2^\times \to \mathbb{Z}_2$. Obviously, such polynomials only have terms of even degree. Let $h(x) = \sum a_k x^k \in A_1'$, and again let $r_0 = -\min_k v_2(a_k/k)$. We'll show that for $r \geq r_0 - 1$,

$$\frac{1}{2^{r+1}} \sum_{\substack{0 \leq i < 2^r \\ 2 \nmid i}} \frac{(ci)^k - i^k}{k} \in \mathbb{Z}_2.$$

As before, if i is an odd integer between 0 and $2^r - 1$, there's a unique expression

$$ci = \pm j(1 + m_r(j) 2^{r+1})$$

where j is an odd integer between 0 and $2^r - 1$ and $m_r(j) \in \mathbb{Z}_2$. Then if $k \geq 2$ is an even integer, we again have

$$\sum_{\substack{0 \leq i < 2^r \\ 2 \nmid i}} (ci)^k - i^k = \sum_{\substack{0 \leq j < 2^r \\ 2 \nmid j}} k j^k m_r(j) 2^{r+1} \mod 2^{2r+2}$$

and thus

$$\frac{1}{2^{r+1}} \sum_{\substack{0 \leq i < 2^r \\ 2 \nmid i}} \frac{(ci)^k - i^k}{k} \equiv \sum_{\substack{0 \leq j < 2^r \\ 2 \nmid j}} m_r(j) h(j) \mod 2^{r - r_0 + 1},$$

proving the result. □

Remark 18. The same analysis allows us to realize the $\widehat{A}$-genus as a string orientation – the only difference is that we're looking at sequences in $\prod_{k \geq 4} \mathbb{Q}$ rather than $\prod_{k \geq 2} \mathbb{Q}$. The rest of the Ando-Hopkins-Rezk paper is focused on string-orienting tmf. It can no longer be spin-oriented because a connectivity argument fails, and we now have to $K(2)$-localize things as well as $K(1)$-localizing them. It looks like the number theory involved is significantly harder. One hopes that we could likewise string-orient taf for $3 \leq n \leq 5$, at which point we'll hit the same connectivity hurdle.

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