The $T(n)$-local splitting of Goodwillie towers

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1 Introduction

The calculus of functors saw many of its early applications in the study of manifolds, which one could think of as phenomena special to the category of spaces. If we move to the stable category, we lose track of manifolds, embeddings, and so on, but we gain new organization, much of which falls under the aegis of chromatic homotopy theory. As it turns out, Goodwillie calculus interacts with chromatic apparatus in deep and surprising ways. In a 2004 paper [7], Nicholas Kuhn inaugurated this program, proving that the Goodwillie tower of *any* homotopy functor $F : \mathsf{Spec} \to \mathsf{Spec}$ splits after 'finite $K(n)$-localization,' also known as '$T(n)$-localization.' In this talk, we'll prove this result, first discussing the necessary chromatic and equivariant background knowledge. We'll also describe a surprising functor constructed by Bousfield and Kuhn, relating the category of spaces to the various $K(n)$-local categories of spectra, that is key to proving results like this.

2 Chromatic homotopy theory

Chromatic homotopy theory is born from the observation, perhaps dating to the work of Miller-Ravenel-Wilson [12] and Devinatz-Hopkins-Smith [3], that p-local spectra tend to split into various 'layers,' each of which has a certain kind of 'periodicity.' Specifically, for each prime p there are p-local spectra called the **Morava K-theories** with coefficient rings

$$\pi_* K(n) = \mathbb{F}_p[v_n^{\pm 1}], \quad |v_n| = 2(p^n - 1).$$

By convention, $v_0 = p$ and $K(0) = H\mathbb{Q}$; also, $K(\infty) = H\mathbb{F}_p$. $K(1)$ is a summand of p-local complex K-theory.

For any spectrum E, Bousfield constructed a localization functor $L_E : \mathsf{Spec} \to \mathsf{Spec}$ which kills all spectra X with $E_* X = 0$. Thus, the E-localization of X could be thought of as the part of X that is entirely described by its E_*-homology. More precisely, the following is true.

Definition 2.1. A spectrum X is **E-acyclic** if $E_* X = 0$. X is **E-local** if $[A, X] = 0$ for all E-acyclic spectra A. An **E-localization** of X is a map $X \to L_E X$ such that

- $L_E X$ is E-local;

- the map has E-acyclic cofiber, meaning it induces an equivalence on E_*-homology.

Bousfield [1] proved that for any E, a functorial E-localization exists. It should be emphasized that this data includes not only the functor $X \mapsto L_E X$, but also a natural transformation from the identity to this localization functor.

The localization functors naturally form a lattice called the **Bousfield lattice**. We say that $\langle E \rangle \leq \langle F \rangle$ if every F-acyclic spectrum is E-acyclic, or equivalently if $L_E L_F \simeq L_E$; obviously, this relation depends not on the spectrum E but only on the localization functor it induces (its so-called **Bousfield class**). In a happy coincidence of notation, the meet and join of this lattice are none other than the smash product ($\wedge$) and wedge sum ($\vee$) of representative spectra. The thick subcategory theorem [4] says that the Bousfield classes $\langle K(n) \rangle$ for various p and n constitute all of the minimal elements of this lattice.[1] In this sense, the Morava K-theories are the primes of stable homotopy theory. The chromatic program thus splits into two steps: first, analyze $K(n)$-localizations of a spectrum, which often carry extra structure (some of which we'll see in this talk!); second, assemble these monochromatic pieces back into the original spectrum, using fracture squares and the chromatic spectral sequence of [12].

We now briefly describe Bousfield's construction of localizations. Though more abstract proofs now exist, in the general context, say, of a combinatorial model category, the original proof is rather topological and direct: one shows that X is E-local iff $[A, X] = 0$ when A is an E-acyclic CW-spectrum with a bounded cardinality on its set of cells; one then finds a set of representatives for the homotopy types of maps from such spectra to X, wedges them together, and takes the cofiber of the resulting map to X. The point of this is that the cardinality bound found can be forced into the proof.

Definition 2.2. A spectrum X is **finitely E-local** if $[A, X] = 0$ for all finite E-acyclic spectra A. X is **finitely E-acyclic** if $[X, B] = 0$ for all finitely E-local spectra A. A **finite E-localization** of X is a map $X \to L_E^f X$ such that

- $L_E^f X$ is finitely E-local;
- the map has finitely E-acyclic cofiber.

Theorem 2.3. *For any E, there is a functorial finite E-localization.*

See [11]. In fact, finite localization is just another kind of Bousfield localization: by a Spanier-Whitehead duality argument,

$$L_E^f X \simeq X \wedge L_E^f S \simeq L_{L_E^f S} X.$$

We now consider the case $E = K(n)$. The Periodicity Theorem of [4] says that any $K(n-1)_*$-acyclic spectrum X has a self-map $\Sigma^d X \to X$ inducing an isomorphism on $K(n)_* X$, with $d > 0$ if $n > 0$. Thus, we can form the mapping telescope

$$X[v_n^{-1}] = \operatorname{hocolim}\left(X \to \Sigma^{-d} X \to \Sigma^{-2d} X \to \cdots\right).$$

[1] The usual statement of this theorem is as follows: the categories $\mathcal{C}_{n,p}$ of $K(n-1)$-acyclic spectra for various p and n constitute all of the subcategories of Spec closed under taking cofibers and retracts, the so-called **thick subcategories**.

We let $T(n)$ be $X[v_n^{-1}]$ for any finite type n spectrum X. Here, X is said to be **type** n if $K(n-1)_*X = 0$ but $K(n)_*X \neq 0$.[2] It's a consequence of the thick subcategory theorem that $L^f_{K(n)} \simeq L_{T(n)}$, independent of choice of X. It's these finite $K(n)$-localizations that we'll be using in the sequel.

By the way, it's clear from the definitions that $\langle T(n)\rangle \leq \langle K(n)\rangle$ in the Bousfield lattice, meaning that $L_{K(n)} \simeq L_{K(n)}L_{T(n)}$, and that finitely $K(n)$-local spectra are also $K(n)$-local. It's an open conjecture, called the telescope conjecture, whether these functors are the same: is every $K(n)$-acyclic spectrum also finitely $K(n)$-acyclic? This is known to be true for $n = 1$, and believed to be false for $n = 2$.

I conclude this section by stating Kuhn's main theorem.

Theorem 2.4. *Let $F : \mathsf{Spec} \to \mathsf{Spec}$ be any homotopy functor. For all primes p, $n \geq 1$, and $d \geq 1$, the natural transformation in the Goodwillie tower of F*

$$p_d : P_dF \to P_{d-1}F$$

admits a natural homotopy section after applying finite $K(n)$-localization. Thus, there is a natural splitting

$$L^f_{K(n)}P_dF \simeq \prod_{i=0}^{d} L_{T(n)}D_iF.$$

Remark 2.5. Kuhn uses the notation $L_{T(n)}$ and the phrase '$T(n)$-localization' throughout his paper, but after much deliberation and consultation with my family pastor, it seems to me that 'finite $K(n)$-localization' $L^f_{K(n)}$ is much more appropriate for several reasons. First, $T(n)$ refers not to an actual spectrum, nor even a homotopy type, but merely a Bousfield class. Second, it's probably worth emphasizing the relationship between $K(n)$-localization and finite $K(n)$-localization, including the unlikely possibility that they may be the same functor. Finally, with such a limited alphabet and with so much mathematics yet to be done, we should only give meanings to letters with great care. Our culture of waste, excess, and consumption may teach us to use the letter T now and let the future handle itself, but it's this sort of thinking that is melting the polar ice caps. Who's to say that homotopy theorists of the 2030s may not need the letter T? With this in mind, I'll call this functor 'finite $K(n)$-localization' wherever possible, and urge others to do the same.

On the other hand, the functor arises so often in these notes that I'll be abbreviating it with the letter L where no ambiguity would be caused, making the above points moot.

3 Equivariant background and the Tate construction

3.1 Homotopy orbits and homotopy fixed points

The homogeneous layers of the Goodwillie tower are all of the form $D_rF(X) = (\partial_rF \wedge X^{\wedge r})_{\mathrm{h}\Sigma_r}$. So it's inevitable that we encounter equivariant homotopy theory in studying it, and indeed the key steps in Kuhn's proof involve an equivariant gadget called the Tate construction. We sketch this here. We work in the category $G\mathsf{Spec}$ of naïve G-spectra (which

[2] If $K(n)_*X = 0$, then any v_n-self map is null, and this telescope is contractible.

you can think of as just spectra with a G-action), for some finite group G. We can then define

$$X_{\mathrm{h}G} = (EG_+ \wedge X)/G, \qquad X^{\mathrm{h}G} = F(EG_+, X)^G.$$

Here EG is a contractible space with a free G-action, and EG_+ is E with a disjoint basepoint added. Both homotopy orbits and homotopy fixed points send equivariant weak equivalences to nonequivariant weak equivalences, and likewise with cofiber sequences. Homotopy orbits also commute with filtered homotopy colimits. Homotopy orbits preserve E-acyclic maps, and thus E_*-equivalences; dually, homotopy fixed points preserve E-local objects. As it turns out, we can also write the homotopy orbits functor as

$$X_{\mathrm{h}G} \simeq (EG_+ \wedge X)^G = (EG_+ \wedge F(EG_+, X))^G.$$

There's an equivariant cofiber sequence (the **isotropy separation sequence**)

$$EG_+ \to S \to \widetilde{EG},$$

the first map induced by the map $EG \to *$ of spaces, and this gives us a cofiber sequence

$$EG_+ \wedge F(EG_+, X) \to F(EG_+, X) \to \widetilde{EG} \wedge F(EG_+, X).$$

Taking G-fixed points gives a nonequivariant cofiber sequence

$$X_{\mathrm{h}G} \stackrel{N_G(X)}{\to} X^{\mathrm{h}G} \to t_G(X).$$

$N_G(X)$ is the **norm map**, and $t_G(X)$ the **Tate construction** of X.

Remark 3.1. If it helps, there's an algebraic analog of this. Given a G-module M, there's a map $N_G(M) : M_G \to M^G$ given by sending a class in M_G to the sum of its preimages in M, which is automatically G-invariant. If one has both a projective and an injective $\mathbb{Z}[G]$-resolution of M, one can use this norm map to join the two resolutions and obtain a $\mathbb{Z}$-graded cohomology theory, called **Tate cohomology**, which gives the homology of M in negative degrees below -1, its cohomology in positive degrees, and the kernel and cokernel of N_G in degrees 0 and -1.

Remark 3.2. Kuhn points out that the norm map is the unique (up to homotopy) natural transformation $(\cdot)_{\mathrm{h}G} \to (\cdot)^{\mathrm{h}G}$ such that $N_G(\Sigma^\infty G_+)$ is a weak equivalence. Indeed, call a functor $G\mathsf{Spec} \to \mathsf{Spec}$ **homological** if it preserves homotopy pushouts and filtered homotopy colimits. By a result of Klein, any functor has a universal left approximation by a homological functor, and one checks that the norm map is this approximation for the homotopy fixed points functor.

3.2 The transfer map

Before discussing the Tate construction in more detail, we mention an often-overlooked but useful equivariant construction. If $p : E \to B$ is a finite covering map of connected spaces with fiber of cardinality $n = [\pi_1(E) : \pi_1(B)]$, then *stably*, there is a natural **transfer map** $p^!$ going the opposite direction, such that for any homology theory h_*, the composition

$$h_*B \stackrel{p_*^!}{\to} h_*E \stackrel{p_*}{\to} h_*B$$

induces multiplication by n (some suspension isomorphisms have been suppressed). This appears to be old folklore, but a good source is [5].[3]

In particular, if X is a G-space and $H \leq G$, then we can model EH as EG with H-action given by restriction, and $X \wedge EG_+$ is a free G-space away from the basepoint. Thus there's a covering map

$$X_{\mathrm{h}H} = (X \wedge EG_+)_H \twoheadrightarrow (X \wedge EG_+)_G = X_{\mathrm{h}G}$$

with fiber $[G : H]$. The transfer thus gives a map of spectra $\Sigma^\infty X_{\mathrm{h}G} \to \Sigma^\infty X_{\mathrm{h}H}$, with the obvious composition inducing multiplication by $[G : H]$. If X is, for example, p-local for some prime p not dividing $[G : H]$, then this tells us that $X_{\mathrm{h}G}$ is a *summand* of $X_{\mathrm{h}H}$.

One can easily generalize this to X a G-spectrum. X is at any rate a homotopy colimit of G-spaces, and the above covering map is natural at the space level.

3.3 The Tate construction

In many cases, we'd like to prove that the Tate construction on some G-spectrum is E-acyclic, as this allows us to identify the E-homology of its homotopy orbits with the E-homology of its homotopy fixed points. One example, important later on, is the following.

Proposition 3.3. *Let K be a finite free G-CW-complex, and Y a G-spectrum. Then $t_G(F(K, Y)) \simeq *$.*

Proof. As homotopy orbits and homotopy fixed points preserve cofiber sequences, so does the Tate construction, and so we reduce to the case when $K = G$, so $F(K, Y) \simeq G_+ \wedge Y$.[4] The Tate spectrum is thus the G-fixed points of $\widetilde{EG} \wedge F(EG_+, G_+ \wedge Y) \simeq \widetilde{EG} \wedge G_+ \wedge F(EG_+, Y)$. But $\widetilde{EG} \wedge G_+$ is equivariantly contractible, by the equivariant Whitehead theorem: $\widetilde{EG}$ is nonequivariantly contractible, and $EG_+ \wedge G_+ \to G_+$ induces an equivalence on H-fixed points (the only one there is) for $0 \neq H \leq G$. □

The above observations on the transfer map give us another example of 'Tate vanishing.'

Proposition 3.4. *If Y is a p-local G-spectrum with p prime to $|G|$, then $t_G(Y) \simeq *$.*

Proof. The transfer $Y_{\mathrm{h}G} \to Y$ factors through the homotopy fixed points of Y via the norm map. Thus if f is the composition $Y^{\mathrm{h}G} \to Y \to Y_{\mathrm{h}G}$, we have $N(Y)f = fN(Y) = |G|$, which is invertible, so $N(Y)$ is an equivalence. □

The following proposition will be our main tool. C_p is the cyclic group of order p; if a group action is not specified, the trivial group action should be assumed.

Proposition 3.5. *Let R be a ring spectrum such that $t_{C_p}(R)$ is E-acyclic for each p. Then $t_G(M)$ is E-acyclic for any R-module M and any finite group G.*

Lemma 3.6. *If R is a ring spectrum with trivial G-action and M is an R-module, then $R^{\mathrm{h}G}$ and $t_G(R)$ are ring spectra, $M^{\mathrm{h}G}$ and $t_G(M)$ are modules over these ring spectra, and $R^{\mathrm{h}G} \to t_G(R)$ is a map of ring spectra (indeed, of R-algebras).*

[3] Incidentally, this paper proves that the transfer is realized as a map of spaces $\Sigma^k B \to \Sigma^k E$, where $k-1$ is the dimension of B and E.

[4] Yes, that's right. Finite G-sets are Spanier-Whitehead self-dual.

Proof. The homotopy fixed points are $F(BG_+, R)$, which clearly has a ring structure; put differently, $F(EG_+, R)$ is a G-equivariant ring spectrum. For the Tate spectrum, one must also observe that $\widetilde{EG}$ has a G-equivariant ring structure. Indeed, smashing it with the isotropy separation sequence gives a G-equivariant cofiber sequence

$$\widetilde{EG} \wedge EG_+ \to \widetilde{EG} \to \widetilde{EG} \wedge \widetilde{EG},$$

but by the G-Whitehead theorem, the left-hand term is trivial: $\widetilde{EG}$ is nonequivariantly contractible, while EG_+ and thus $\widetilde{EG} \wedge EG_+$ is H-free away from the basepoint for any nontrivial subgroup H of G. Thus the right-hand map is a weak equivalence; we take this to be the unit of $\widetilde{EG}$, and any homotopy inverse the multiplication. Now, since the smash product of (equivariant) ring spectra is an (equivariant) ring spectrum, we obtain our ring structure on $t_G(R)$. The same arguments apply for the second and third statements. ☐

Sketch of proof of Proposition 3.5. The map $G/H \mapsto R^{\mathrm{h}H}$ is a Mackey functor in spectra.[5] As such, it's a module over the Burnside ring Mackey functor $G/H \mapsto A(H)$, and this action extends to the completion $\widehat{A}$ of A at its augmentation ideal. It follows that $G/H \mapsto t_H(R)$ is also an $\widehat{A}$-module Mackey functor, and $G/H \mapsto E_*(t_H(R))$ is a $\widehat{A}$-module Mackey functor in graded abelian groups. By a theorem of May and McClure [9], $E_*(t_G(R)) = 0$ iff $E_*(t_{G_p}(R)) = 0$ for all p-Sylow subgroups G_p of G. We thus reduce to the case where G is a p-group.

But now G is solvable – indeed, it has a composition series where the quotients are all $\mathbb{Z}/p$. So we can do an induction on the normal subgroups of G. To be specific, let K be a normal subgroup and $Q = G/K$, and assume for the induction step that $t_K(R)$ and $t_Q(R)$ are both E-acyclic. We can factorize the G-norm map of R as

$$R_{\mathrm{h}G} \simeq (R_{\mathrm{h}K})_{\mathrm{h}Q} \xrightarrow{(N_K(R))_{\mathrm{h}Q}} (R^{\mathrm{h}K})_{\mathrm{h}Q} \xrightarrow{N_Q(R^{\mathrm{h}K})} (R^{\mathrm{h}K})^{\mathrm{h}Q} \simeq R^{\mathrm{h}G}.$$

To show this, one can go back to the definitions, or one can use Remark 3.2 and show that this map, natural in the spectrum R, is an equivalence on $\Sigma^\infty G_+$. Now, by assumption, $N_K(R)$ is an E_*-equivalence, and taking Q-homotopy orbits preserves this. Also by assumption, $N_Q(R)$ is an E_*-equivalence, so $t_Q(R)$ is E-acyclic. By the above lemma, $t_Q(R^{\mathrm{h}K})$ is a module over the ring spectrum $t_Q(R)$, and thus also E-acyclic. ☐

4 The Tate spectrum of the finitely $K(n)$-local sphere

The remainder of the proof is outlined as follows. Using Goodwillie calculus and its dual, one identifies the homogeneous layer D_dF as the homotopy orbits of the Σ_d-spectrum $\Delta_d(F)$ (which should be familiar), and the map $P_dF \to P_{d-1}F$ as a homotopy pullback of the natural map $\Delta_d(F)^{\mathrm{h}\Sigma_d} \to t_{\Sigma_d}(\Delta_d(F))$ (which is new). After finitely $K(n)$-localizing everything, and

[5] A Mackey functor is a pair of functors from the category of G-orbits G/H, a covariant functor giving you 'induction maps' and a contravariant functor giving you 'restriction maps,' that interact in a certain way. These pop up often in stable homotopy theory, often in the category of abelian groups or modules over a ring – for example, the homotopy and homology groups of a G-space naturally have this structure. If you're unfamiliar with this, ignore this proof.

composing with the map induced by $F \to LF$, finding the desired section reduces to showing that $t_{\Sigma_d}(\Delta_d(LF))$ is finitely $K(n)$-acyclic. But $\Delta_d(LF)$ is a module over the ring spectrum LS, and by Lemma 3.6, the Tate construction on this spectrum is a module over $t_{\Sigma_d}(LS)$. So it suffices to show that this is finitely $K(n)$-acyclic. In fact, one has

Proposition 4.1.

$$Lt_G LS \simeq *$$

for any finite group G acting trivially.

By Proposition 3.5, it suffices to prove this for $G = C_p$. As a final reduction, $t_{C_p}LS$ is a ring spectrum, so it suffices to show that the localization of its unit map $LS \to Lt_{C_p}LS$ is null.

And in a sense, we're going to do this by reversing the above steps. There's a restriction map $t_{\Sigma_p}LS \to t_{C_p}LS$, which is easily checked to be a ring homomorphism (and in particular, unital). So we reduce to showing that $LS \to Lt_{\Sigma_p}LS$ is null for each p. Finally, we identify this map using Goodwillie calculus, applied to the functor $\Sigma^\infty\Omega^\infty$.

The Tate spectrum of a C_p-spectrum can be described quite explicitly. C_p acts on $\mathbb{R}^p$ by permuting the coordinates (the regular representation); after quotienting by the 1-dimensional diagonal, we get a free representation which we'll call ρ. This representation is the restriction of a $(p-1)$-dimensional representation of Σ_p, also called ρ. Letting $S(\alpha)$ be the unit sphere in the real representation α, and S^α the one-point compactification of this representation, there are then cofiber sequences of spaces

$$S(k\rho)_+ \to S^0 \to S^{k\rho}$$

whose homotopy colimit is

$$S(\infty\rho)_+ \to S^0 \to S^{\infty\rho}.$$

Of course, this is a model for the isotropy separation sequence $(EC_p)_+ \to S^0 \to \widetilde{EC_p}$.

For any C_p-spectrum Y, we then have

$$Y^{\mathrm{h}C_p} = F(S(\infty\rho)_+, Y)^{\mathrm{h}C_p} \simeq \mathrm{holim}_k\, F(S(k\rho)_+, Y)^{\mathrm{h}C_p} \simeq \mathrm{holim}_k\, F(S(k\rho)_+, Y)_{\mathrm{h}C_p},$$

the last equivalence by Proposition 3.3. This implies that

$$t_{C_p}(Y) \simeq \mathrm{holim}_k\, \Sigma F(S^{k\rho}, Y)_{\mathrm{h}C_p}. \tag{1}$$

Computing $Lt_{C_p}LY$ is only slightly more complicated. The map $Y \to LY$ is an equivariant finite $K(n)$-equivalence, so $Y_{\mathrm{h}C_p} \to (LY)_{\mathrm{h}C_p}$ is a finite $K(n)$-equivalence, so $L(Y_{\mathrm{h}C_p}) \simeq L(LY)_{\mathrm{h}C_p}$. Meanwhile, $(LY)^{\mathrm{h}C_p}$ is already finitely $K(n)$-local, and repeating the arguments of (1) gives

$$L(LY)^{\mathrm{h}C_p} \simeq (LY)^{\mathrm{h}C_p} \simeq \mathrm{holim}_k\, F(S(k\rho)_+, LY)_{\mathrm{h}C_p} \simeq \mathrm{holim}_k\, LF(S(k\rho)_+, Y)_{\mathrm{h}C_p}.$$

We conclude that

$$Lt_{C_p}LY \simeq \mathrm{holim}_k\, \Sigma LF(S^{k\rho}, Y)_{\mathrm{h}C_p}. \tag{2}$$

By the same arguments as in Proposition 3.4, since LY is p-local and thus $(p-1)!$ is invertible, we see that the norm

$$F(S(k\rho)_+, LY)_{\mathrm{h}\Sigma_p} \to F(S(k\rho)_+, LY)^{\mathrm{h}\Sigma_p}$$

is also an equivalence, so that

$$Lt_{\Sigma_p} LY \simeq \operatorname{holim}_k \Sigma LF(S^{k\rho}, Y)_{\mathrm{h}\Sigma_p}. \tag{3}$$

Remark 4.2. The above arguments for C_p work for any localization functor L_E in place of L. Those for Σ_p work for any localization functor L_E such that $(p-1)!$ is invertible in E_*. In particular, they work for p-localization.

5 The Goodwillie tower of $\Sigma^\infty\Omega^\infty$

Observe that Ω^∞, being a right adjoint, preserves products, which in spectra are wedge sums. Thus $\Omega^\infty(X_1 \wedge \cdots \wedge X_d) = \Omega_1 X_1 \times \cdots \times \Omega_d X_d$. From the definition of the smash product, one then checks that

$$\Sigma^\infty\Omega^\infty(X_1 \wedge \cdots \wedge X_d) = \bigvee_{S \subseteq \mathbf{d}} \bigwedge_{i \in S} \Sigma^\infty\Omega^\infty X_i.$$

The maps in the cube are evidently the projections of this wedge onto the smaller wedges indexed by $T \subseteq S$ for some $S \subseteq \mathbf{d}$. Thus the dth cross effect is simply $\Sigma^\infty\Omega^\infty X_1 \wedge \cdots \wedge \Sigma^\infty\Omega^\infty X_d$.

I claim that the multilinearization of this is just $X_1 \wedge \cdots \wedge X_d$. I'll show this for $d = 1$, the generalization to higher values of d being obvious. There's a counit map

$$\Sigma^\infty\Omega^\infty X \to X.$$

If X is c-connected for $c > 0$, then so is $\Omega^\infty X$ (and it has the same homotopy groups). By the Freudenthal suspension theorem, $\pi_k \Sigma^\infty\Omega^\infty X = \pi_k^S \Omega^\infty X = \pi_k \Omega^\infty X$ for all $k \le 2c$. Thus the counit map is $2c$-connected. It follows that the multilinearization is as described.

Putting this together, we get

Proposition 5.1. $\Sigma^\infty\Omega^\infty : \mathsf{Spec} \to \mathsf{Spec}$ *is analytic, and its Goodwillie derivatives are*

$$D_d(X) = (X^{\wedge d})_{\mathrm{h}\Sigma_d}.$$

Remark 5.2. That's not a typo. These functors, called the **extended powers**, were called D_d before Goodwillie calculus was even invented!

We can write the Tate spectra from earlier in terms of the extended powers. Note that

$$D_p \Sigma^{-k} X \simeq (S^{-kp} \wedge X^{\wedge p})_{\mathrm{h}\Sigma_p} \simeq F(S^{kp}, X^{\wedge p})_{\mathrm{h}\Sigma_p} \simeq \Sigma^{-k} F(S^{k\rho}, X^{\wedge p})_{\mathrm{h}\Sigma_p}.$$

Thus, by (3), we get

$$Lt_{\Sigma_p} LS \simeq \operatorname{holim}_k L\Sigma^{k+1} D_p S^{-k}. \tag{4}$$

6 The Bousfield-Kuhn functor

6.1 Constructing the functor

We now reach part of the original point of this talk: a surprising interaction between the infinite loop space functor and our chromatic localization functors.

Theorem 6.1. *For any prime p and $n \geq 1$, there is a homotopy-commutative diagram, unique up to natural equivalence,*

$$\begin{array}{ccc} \mathsf{Spec} & \xrightarrow{L^f_{K(n)}} & L^f_{K(n)}\mathsf{Spec} \\ & {\scriptstyle \Omega^\infty}\searrow \quad \nearrow{\scriptstyle \Phi^f_n} & \\ & \mathsf{Spaces}. & \end{array}$$

By composing with $K(n)$-localization, one likewise gets a factorization $L_{K(n)} \simeq \Phi_n \Omega^\infty$ for another functor Φ_n, the ordinary Bousfield-Kuhn functor. Of course, the finite one is the one we'll be using, and implies the existence of the ordinary one. To save verbiage, though, I'll just construct the ordinary functor – all the spectra involved will be finite, so the finite one is no harder. The bizarreness of these things should be emphasized, and cuts two ways: on the one hand, it means that $K(n)$-localization can be constructed 'unstably'; on the other, it means that Ω^∞ has a section at the price of $K(n)$-localizing (for any $n \geq 1$!). This was done by Bousfield in [2] for $n = 1$, and Kuhn in [6] for larger n; a good exposition is in [8]. We'll construct the functor on the level of homotopy categories, following [6].

Definition 6.2. A $\mathcal{C}_n$**-resolution** of a spectrum X is a diagram of spectra *over* X

$$X_1 \to X_2 \to \cdots$$

such that each X_i is finite and $K(n-1)$-acyclic, and the natural map $\varinjlim K(m)_*(X_i) \to K(m)_*X$ is an isomorphism for all $m \geq n$.

Proposition 6.3. *For $n \geq 1$, every finite p-local spectrum has a $\mathcal{C}_n$-resolution.*

Proof. For $n = 1$, a one-term resolution exists, namely the fiber of $X \to X_{\mathbb{Q}}$. We proceed by induction on n. First, take X itself to be $K(n-1)$-acyclic and equipped with a choice of v_n-self map $v : \Sigma^d X \to X$. Let X_i be the cofiber of $v^i : \Sigma^{-1}X \to \Sigma^{-1-di}X$. Then each X_i is $K(n)$-acyclic. The maps of cofiber sequences

$$\begin{array}{ccccccc} \Sigma^{-1}X & \xrightarrow{v^i} & \Sigma^{-1-di}X & \longrightarrow & X_i & \longrightarrow & X \\ \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle v} & & \downarrow & & \downarrow{\scriptstyle 1} \\ \Sigma^{-1}X & \xrightarrow{v^{i+1}} & \Sigma^{-1-d(i+1)}X & \longrightarrow & X_{i+1} & \longrightarrow & X \end{array}$$

show that the X_i form a diagram of spectra over X. Finally, by the periodicity theorem, $K(m)_*v$ is nilpotent for each $m \geq n+1$, so $\operatorname{hocolim} X_i \to X$ induces a $K(m)_*$-equivalence for all $m \geq n+1$. Thus this is a $\mathcal{C}_{n+1}$-resolution.

Now let X be arbitrary, and let

$$X(1) \to X(2) \to \cdots \to X$$

be a $\mathcal{C}_n$-resolution. By the periodicity theorem, we can choose v_n-self maps $v_i : \Sigma^{d_i} X(i) \to X(i)$, with $d_i | d_{i+1}$, such that each square

$$\begin{array}{ccc} \Sigma^{d_{i+1}} X(i) & \longrightarrow & \Sigma^{d_{i+1}} X(i+1) \\ {\scriptstyle v_i^{d_{i+1}/d_i}} \downarrow & & \downarrow {\scriptstyle v_{i+1}} \\ X(i) & \longrightarrow & X(i+1) \end{array}$$

commutes. Letting $(X(i)_j)$ be the $\mathcal{C}_{n+1}$-resolution of $X(i)$ defined as above via the self map v_i, one easily observes that there are induced maps of $\mathcal{C}_{n+1}$-resolutions

$$\begin{array}{ccccc} X(1)_* & \longrightarrow & X(2)_* & \longrightarrow & \cdots \\ \downarrow & & \downarrow & & \\ X(1) & \longrightarrow & X(2) & \longrightarrow & \cdots \\ \downarrow & \swarrow & & & \\ X & & & & \end{array}$$

making everything commute. The diagonal of this diagram of spectra $X(i)_j$ is the sequence $Y_i = X(i)_i$, with the maps $Y_i \to Y_{i+1}$ given by the composites $X(i)_i \to X(i+1)_i \to X(i+1)_{i+1}$. These are $K(n-1)$-acyclic, map coherently to X, and are cofinal in the above diagram, so they have the same homotopy colimit. □

The thrust of the construction is as follows. Given any *space* Z with a self map $v : \Sigma^d Z \to Z$, and X a space, let $\Phi'_Z(X)$ be the spectrum[6] with mdth space $\mathrm{Maps}(Z, X)$, and with structure maps

$$\Phi'_Z(X)_{md} = \mathrm{Maps}(Z, X) \to \Omega^d \Phi'_Z(X)_{(m+1)d} = \Omega^d \mathrm{Maps}(Z, X) \cong \mathrm{Maps}(\Sigma^d Z, X)$$

induced by v. This is a functor $\Phi'_Z : \mathsf{Top} \to \mathsf{Spec}$, natural in Z. Also, $\Phi'_Z \cong \Phi'_{\Sigma^d Z}$, so by suspending sufficiently, we can take Z to be not just a space but any finite spectrum. Define $\Phi_Z = L_{K(n)} \Phi'_Z$.

Proposition 6.4. *If $K(n)_* v$ is an isomorphism, then for E a spectrum, there is a natural equivalence*

$$\Phi_Z(\Omega^\infty E) \simeq F(Z, L_{K(n)} E).$$

[6]This is following Adams' construction of spectra in the blue book, where he points out that one can define a spectrum (up to equivalence) by giving only a cofinal subset of its spaces and the obvious structure maps.

Proof. Without loss of generality, Z is a space. The mdth space of $\Phi'_Z(\Omega^\infty E)$ is $\mathrm{Maps}(Z', \Omega^\infty E) \simeq \Omega^\infty F(Z,E)$. Looking at the construction of this spectrum, one sees that in this case it's actually $\operatorname{hocolim} F(\Sigma^{md} Z, E) \simeq F(Z,E)[v^{-1}]$. Thus we want to show that $L_{K(n)}(F(Z,E)[v^{-1}]) \simeq F(Z, L_{K(n)}E)$. The latter spectrum is $K(n)$-local; since Z is finite, the maps

$$F(Z,E)[v^{-1}] \leftarrow F(Z,E) \to F(Z, L_{K(n)}E)$$

identify with

$$(DZ \wedge E)[v^{-1}] \leftarrow DZ \wedge E \to DZ \wedge L_{K(n)}E,$$

both of which are $K(n)_*$-equivalences. □

Proof of Theorem 6.1. Let (Z_i) be a $\mathcal{C}_n$-resolution of the p-local sphere, and for each Z_i, choose a v_n-self map $v_i : \Sigma^{d_i} Z_i \to Z_i$, such that the various v_i agree with each other in the sense of the above proof. Define $\Phi_n = \operatorname{holim} \Phi_{Z_i}$. Then

$$\Phi_n \Omega^\infty E = \operatorname{holim} \Phi_{Z_i} \Omega^\infty E \simeq F(\operatorname{hocolim} Z_i, L_{K(n)}E) \simeq L_{K(n)}E$$

since $\operatorname{hocolim} Z_i \to S$ is a $K(n)_*$-equivalence. □

6.2 Applying the functor

In our situation, the use of this is as follows. By elementary category theory, the natural transformation

$$\Omega^\infty \overset{\eta\Omega^\infty}{\to} \Omega^\infty\Sigma^\infty\Omega^\infty \overset{\Omega^\infty\epsilon}{\to} \Omega^\infty$$

is the identity; applying the Bousfield-Kuhn functor, we get a factorization of the identity natural transformation

$$L \to L\Sigma^\infty\Omega^\infty \to L.$$

But $\Sigma^\infty\Omega^\infty = \operatorname{holim}_d P_d$, and $P_1(X) = D_1(X) = X$, so the right-hand map factorizes, in particular, as $L\Sigma^\infty\Omega^\infty \to LP_p \to L$. There's thus a section of $LP_p \to L$, and also, for any k, of $L\Sigma^k P_p \Sigma^{-k} \to L$.

Applying a bunch of obvious exact functors to the sphere, there's a cofiber sequence

$$\operatorname{holim}_k L\Sigma^k P_p S^{-k} \to \operatorname{holim}_k L\Sigma^k P_{p-1} S^{-k} \to \operatorname{holim}_k L\Sigma^{k+1} D_p S^{-k}. \tag{5}$$

I claim that the middle term is just LS, and the first map is induced by the natural map $LP_pS \to LP_1S = LS$. With this understood, the above argument says that this map has a section; thus, the right-hand map, which is $LS \to Lt_{\Sigma_p}LS$, is null. This doesn't prove the Tate vanishing theorem, since this isn't necessarily the localization of the unit map of $t_{\Sigma_p}LS$. Nevertheless, I claim that the localized unit map factors through it. To complete the proof, we'll examine each of these claims in turn.

6.3 Extended powers of odd spheres

Lemma 6.5. *For k odd, p an odd prime, and $2 \leq d \leq p-1$, $D_d S^k$ is p-locally contractible.*

Proof. Since $d > 1$, $S^{kd} \wedge E\Sigma_{d+} \simeq S^{kd}$ is simply connected, so $\pi_1 D_d S^k = \Sigma_d$, which vanishes after p-localization. Thus we are led to consider the p-local homology of $D_d S^k$. But p-locally, the transfer map exhibits $D_d S^k$ as a stable summand of S^{kd}, so it suffices to show that $H_{kd}(D_d S^k) = 0$.

More precisely, there's a homotopy orbit spectral sequence (really just a Serre spectral sequence; s and t are used since p was taken)

$$H_s(\Sigma_d; H_t(S^{kd})) \Rightarrow H_{s+t}(D_d S^k).$$

The map of homotopy fibrations

$$\begin{array}{ccccc} S^{kd} & \longrightarrow & S^{kd} & \longrightarrow & * \\ \downarrow & & \downarrow & & \downarrow \\ S^{kd} & \longrightarrow & D_d S^k & \longrightarrow & B\Sigma_d \end{array}$$

induces a map of spectral sequences from the trivial spectral sequence

$$H_s(*; H_t(S^{kd})) \Rightarrow H_{s+t}(S^{kd})$$

to the spectral sequence for $D_d S^k$, and again the transfer map gives a p-local section of this. So finally we reduce to checking that $H_0(\Sigma_d; H_{kd}(S^{kd})) = 0$ p-locally. But since k is odd, $H_{kd} S^{kd}$ is the sign representation σ of Σ_d, which is to say the group $\mathbb{Z}$ with even permutations acting trivially and odd permutations acting by -1; one merely observes that swapping two odd-dimensional smash factors of $S^{kd} = S^k \wedge S^k \wedge \cdots \wedge S^k$ induces a sign of -1. Since $H_0(\Sigma_d; \sigma) \cong \mathbb{Z}/2$ and p is odd, we are done. □

Remark 6.6. The same argument shows that if k is even in the same situation, the map $S^{kd} \to D_d S^k$ is a p-local equivalence.

As a result, we find that $P_{p-1} S^k \to S^k$ is a p-local equivalence for k odd.[7] Taking the homotopy limit, we get that the map

$$\operatorname{holim}_k \Sigma^k P_{p-1} S^{-k} \to S$$

is a p-local equivalence. Thus the middle term in (5) is indeed LS (and we see why we had to take homotopy limits).

6.4 Factorizing the unit map

The map in (5) is of the form $\operatorname{holim}_k L\delta_k$, with $\delta_k : S \to \Sigma^{k+1} D_p S^{-k}$. By the remark after (3), the unit map we want can be written as $\operatorname{holim}_k L d_k$, where $d_k : S \to \Sigma^{k+1} D_p S^{-k}$ is defined p-locally. We conclude by showing that d_k factors through δ_k. In the following, all spectra are assumed p-local.

[7] This works for $p = 2$ too, since P_1 is the identity.

7 Calculus and cocalculus

We've at last proved the key Proposition 4.1. To properly apply it, a little bit of calculus is needed. As usual, let $cr_d F$ be the dth cross effect of the functor F, defined as the total fiber of the cube

$$[S \subseteq \mathbf{d}] \mapsto F\left(\bigvee_{i \in \mathbf{d}-S} X_i\right),$$

and let $(\Delta_d F)(X)$ be the Σ_d-spectrum $\mathcal{L}(cr_d F)(X, \dots, X)$, where $\mathcal{L}$ denotes multilinearization. Then $D_d F = (\Delta_d F)_{\mathrm{h}\Sigma_d}$.

In [10], McCarthy describes a dualization of Goodwillie calculus for functors $\mathsf{Spec} \to \mathsf{Spec}$. Briefly, we define $cr^d F(X_1, \dots, X_d)$ to be the total cofiber of the cube

$$[S \subseteq \mathbf{d}] \mapsto F\left(\prod_{i \in S} X_i\right),$$

and $(\Delta^d F)(X) = \mathcal{L}(cr^d F)(X, \dots, X)$. There are natural transformations $cr_d F \to cr^d F$ and $\Delta_d F \to \Delta^d F$, and since we're in a stable model category, these are weak equivalences.

Lemma 7.1. *If F is d-excisive, then $t_{\Sigma_d}(\Delta_d F)$ is $(d-1)$-excisive.*

Proof. It suffices to show that $cr_d(t_{\Sigma_d}(\Delta_d F)) \simeq *$. The Tate construction preserves wedges and fibers, so it commutes with cr_d. Since F is d-excisive, $cr_d F$ is already multilinear, and so $cr_d \Delta_d F \simeq F$ □

Proof of Theorem 2.4. There's a map of cofiber sequences

$$\begin{array}{ccccc}
(\Delta_d F)_{\mathrm{h}\Sigma_d} & \longrightarrow & (\Delta_d F)^{\mathrm{h}\Sigma_d} & \longrightarrow & t_{\Sigma_d}(\Delta_d F) \\
\sim\downarrow & & \downarrow & & \downarrow \\
D_d((\Delta_d F)^{\mathrm{h}\Sigma_d}) & \longrightarrow & P_d((\Delta_d F)^{\mathrm{h}\Sigma_d}) & \longrightarrow & P_{d-1}((\Delta_d F)^{\mathrm{h}\Sigma_d}).
\end{array}$$

Here the middle map comes from the Taylor tower for $(\Delta_d F)^{\mathrm{h}\Sigma_d}$, and the left-hand map from the fact that $(\Delta_d F)_{\mathrm{h}\Sigma_d}$ is d-excisive and homogeneous. By the above lemma, the right-hand map is an equivalence, so this is an equivalence of cofiber sequences (when F is d-excisive).

Cocalculus gives us a natural transformation

$$F(X) \to F(X^d)^{\mathrm{h}\Sigma_d} \to (\Delta^d F)(X)^{\mathrm{h}\Sigma_d} \simeq (\Delta_d F)(X)^{\mathrm{h}\Sigma_d}.$$

Applying this to a Goodwillie cofiber sequence for F allows us to extend the above diagram

$$\begin{array}{ccccc}
D_d F & \longrightarrow & P_d F & \longrightarrow & P_{d-1} F \\
\downarrow\sim & & \downarrow & & \downarrow \\
D_d((\Delta_d F)^{\mathrm{h}\Sigma_d}) & \longrightarrow & P_d((\Delta_d F)^{\mathrm{h}\Sigma_d}) & \longrightarrow & P_{d-1}((\Delta_d F)^{\mathrm{h}\Sigma_d}) \\
\uparrow\sim & & \uparrow\sim & & \uparrow\sim \\
(\Delta_d F)_{\mathrm{h}\Sigma_d} & \longrightarrow & (\Delta_d F)^{\mathrm{h}\Sigma_d} & \longrightarrow & t_{\Sigma_d}(\Delta_d F)
\end{array}$$

(still assuming F to be d-excisive). The top left-hand map is a weak equivalence by a similar argument as in the lemma, so we conclude that $P_dF \to P_{d-1}F$ is a homotopy pullback of $(\Delta_d F)^{\mathrm{h}\Sigma_d} \to t_{\Sigma_d}(\Delta_d F)$. Replacing an arbitrary F by P_dF (which is always d-excisive), we prove this for arbitrary F.

We apply this by passing along the natural transformation $F \to LF$ and localizing. This gives a commutative diagram, where the rows are cofiber sequences

$$\begin{array}{ccccc}
LD_dF & \longrightarrow & LP_dF & \longrightarrow & LP_{d-1}F \\
\sim\downarrow & & \downarrow & & \downarrow \\
LD_dLF & \longrightarrow & LP_dLF & \longrightarrow & LP_{d-1}LF \\
\sim\downarrow & & \downarrow & & \downarrow \\
L(\Delta_d(LF)_{\mathrm{h}\Sigma_d}) & \xrightarrow[\sim]{} & L(\Delta_d(LF)^{\mathrm{h}\Sigma_d}) & \longrightarrow & L(t_{\Sigma_d}(LF))
\end{array}$$

Now, D_d is defined by taking a cofiber, a multilinearization, and homotopy orbits, all things which preserve $T(n)$-equivalences. Thus $D_dF \to D_dLF$ is a $T(n)$-equivalence, so applying L gives a weak equivalence. We just proved that the next map in the chain is a weak equivalence. The final marked weak equivalence is so because the bottom right corner is a module over $Lt_{\Sigma_d}LS$, and thus contractible. Thus, finally, the map $LP_dF \to L(\Delta_d(LF)^{\mathrm{h}\Sigma_d}) \simeq LD_dF$ is a $T(n)$-local splitting to the Goodwillie tower. $\square$

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