Optimal control of a stochastic processing system driven by a fractional Brownian motion input

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Abstract

We consider a stochastic control model driven by a fractional Brownian motion. This model is a formal approximation to a queueing network with an ON-OFF input process. We study stochastic control problems associated with the long-run average cost, the infinite horizon discounted cost, and the finite horizon cost. In addition, we find a solution to a constrained minimization problem as an application of our solution to the long-run average cost problem. We also establish Abelian limit relationships among the value functions of the above control problems.

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1 Introduction

Self-similarity and long-range dependence of the underlying data are two important features observed in the statistical analysis of high-speed communication networks in heavy traffic, such as local area networks (LAN) (see for instance [10, 22, 26, 27, 31, 35, 36] and references therein). In theoretical models such traffic behavior has been successfully described by stochastic models associated with fractional Brownian motion (FBM) (see [14, 15, 20, 31, 33, 34]). It is well known that FBM exhibits both of these features when the associated Hurst parameter is above $\frac{1}{2}$. Therefore, understanding the behavior and control of these stochastic models are of significant interest. The non-Markovian nature of the fractional Brownian motion makes it quite difficult to study stochastic control problems for a state process driven by FBM. The techniques such as dynamic programming and analysis of the corresponding Hamilton-Jacobi-Bellmann equations which are the commonly used tools in

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the analysis of the stochastic control problems associated with the ordinary Brownian motion are not available for FBM-models.

In this paper, we study several basic stochastic control problems for a queueing model with an input described by a fractional Brownian motion process. Similar queueing models, but not in the context of control of the state process, were considered for instance in [19, 24, 37]. We are aware of only a few solvable stochastic control problems in FBM setting. Usually, the controlled state process is a solution of a linear (semi-linear in [7]) stochastic differential equation driven by FBM, and the control typically effects the drift term of the SDE. In particular, the linear-quadratic regulator control problem is addressed [16, 17] and a stochastic maximum principle is developed and applied to several stochastic control problems in [3]. We refer to [16] and to Chapter 9 of the recent book [4] for further examples of control problems in this setting. In contrast to the models considered in the above references, the model described here is motivated by queueing applications and involves processes with state-constraints. At the end of this section, we discuss an example of a queueing network which leads to our model. Our analysis relies on a coupling of the state process with its stationary version (see [19, 37]) which enables us to address control problems in a non-Markovian setting, and our techniques are different from those employed in [3, 7, 16, 17].

A real-valued stochastic process $W_H = \big(W_H(t)\big)_{t\geq 0}$ is called a fractional Brownian motion with Hurst parameter $H \in (0,1)$ if $W_H(0)=0$ and W_H is a continuous zero-mean Gaussian process with stationary increments and covariance function given by

$$\mathrm{Cov}\big(W_H(s), W_H(t)\big) = \frac{1}{2}[t^{2H} + s^{2H} - |t-s|^{2H}], \qquad s \geq 0,\, t \geq 0.$$

The fractional Brownian motion is a self-similar process with index H, that is for any $a>0$ the process $\frac{1}{a^H}\big(W_H(at)\big)_{t\geq 0}$ has the same distribution as $\big(W_H(t)\big)_{t\geq 0}$. If $H=\frac{1}{2}$ then W_H is an ordinary Brownian motion, and if $H \in [\frac{1}{2}, 1)$ then the increments of the process are positively correlated and the process exhibits long-range dependence, which means that

$$\sum_{n=1}^{\infty} \mathrm{Cov}\big(W_H(1), W_H(n+1) - W_H(n)\big) = \infty.$$

Notice that FBM is a recurrent process with stationary increments. Hence, $\lim_{t\to\infty} W_H(t)/t = 0$ a.s. and consequently $\lim_{t\to\infty}\big(W_H(t) - ut\big) = -\infty$ a.s. for any $u>0$. For additional properties and a more detailed description of this process we refer to [23, 24, 25, 28, 29].

We consider a single server stochastic processing network having deterministic service process with rate $\mu > 0$. For any time $t \geq 0$, the cumulative work input to the system over the time interval $[0,t]$ is given by $\lambda t + W_H(t)$, where λ is a fixed constant and W_H is a fractional Brownian motion with Hurst parameter $H \in [\frac{1}{2}, 1)$. We assume that the service rate μ satisfies $\mu > \lambda$ and that the parameter μ can be controlled. The workload present in the system at time $t \geq 0$ is given by $X_u^x(t)$ which is defined in (2.1) and (2.2) below. Here $x \geq 0$ is the initial workload and $u = \mu - \lambda > 0$ is the control variable. Assuming for simplicity that $x=0$, an equivalent representation for the process X_u^x is given by (see (2.12) below for the general case)

$$X_u^x(t) = \big(W_H(t) - ut\big) - \inf_{s\in[0,t]}\big(W_H(s) - us\big), \qquad t \geq 0. \tag{1.1}$$

For a given arrival process W_H, this is a common formulation for a simple stochastic network where the server works continuously unless there are no customers in the system. The first term above represents the difference between the cumulative number of job arrivals and completed services in the time interval $[0, t]$, and the last term ensures that the the queue-length is non-negative, and it is a non-decreasing process which increases only when the queue-length process is zero. For more examples of such formulations for queueing networks or stochastic processing networks, we refer to [11] and [34]. The above queueing model with W_H in (1.1) being a fractional Brownian motion was considered by Zeevi and Glynn in [37], and we are motivated by their work.

Our goal is to address several stochastic control problems related to the control of the workload process X_u^x described above. The organization of the paper is as follows. We will conclude this section with a motivating example of a queueing network which leads to our model. In Section 2, we introduce the model and describe three basic stochastic control problems associated with it, namely the long-run average cost problem, the infinite horizon discounted cost problem, and the finite horizon control problem. Here we also discuss some properties of the reflection map which will be used in our analysis.

In Section 3, we study the long-run average cost problem. Here we obtain an explicit deterministic representation of the cost functional for each control $u > 0$. This enables us to reduce the stochastic control problem to a deterministic minimization problem. We obtain an optimal control $u^* > 0$ and show its uniqueness under additional convexity assumptions for the associated cost functions. We show that the value function and the corresponding optimal control are independent of the initial data. It is well known that the above property is true for the classical long-run average cost problem associated with non-degenerate diffusion processes. Here we show that it remains valid for our model driven by FBM which is highly non-Markovian. The main results of this section are given in Theorems 3.6 and 3.7. Their generalizations are given in Theorems 3.8 and 3.9. Our proofs here rely on the use of a coupling of the underlying stochastic process with its stationary version introduced in [19]. In particular, we show that the coupling time has finite moments in Proposition 3.4. Because of the highly non-Markovian character of the fractional Brownian motion (it is well known that FBM cannot be represented as a function of a finite-dimensional Markov process), coupling arguments in general do not work for the models associated with FBM (we refer to [13] for an exception). In our case, the coupling is available due to the uniqueness results related to the reflection map described in Section 2.

We use our results in Section 3 to obtain an optimal strategy for a constrained optimization problem in Theorem 4.2 of Section 4. Similar stochastic control problems for systems driven by an ordinary Brownian motion were previously considered in [1, 32]. An interesting application of this model to wireless networks is discussed in [1]. Our constrained optimization problem (in the FBM setting) is a basic example of a general class of problems with an added bounded variation control process in the model. This class of problems has important applications to the control of queueing networks, but in FBM setting, it seems to be an unexplored area of research.

In Section 5, we address the infinite horizon discounted cost problem associated with a similar cost structure. An optimal control for this problem is given in Theorem 5.3.

In Section 6, we establish Abelian limit relationships among the value functions of the three stochastic control problems introduced in Section 2. The main result of this section is stated in Theorem 6.3. We show that the long term asymptotic for the finite horizon control problem and the asymptotic for the infinite horizon discounted control problem, as the discount factor approaches zero, share a common limit. This limit turns out to be the value of the long-run average cost control problem. Our proof also shows that the optimal control for the discounted cost problem converges to the optimal control for the long-run average cost control problem when the discount factor approaches zero. A similar result holds also for the optimal control of the finite horizon problem, as the time horizon tends to infinity. For a class of controlled diffusion processes analogous results were previously obtained in [32].

Motivating example. We conclude this section with a description of a queueing network related to the internet traffic data in which the weak limit of a suitably scaled queue-length process satisfies (2.1) and (2.2) (which are reduced to the above (1.1) when the initial workload x equals 0). For more details on this model we refer to [31] and Chapters 7 and 8 of [34].

We begin by defining a sequence of queueing networks with state dependent arrival and service rates, indexed by an integer $n \geq 1$ and a non-negative real-valued parameter $\tau \geq 0$. For each $n \geq 1$ and $\tau \geq 0$, the (n, τ)-th network has only one server and one buffer of infinite size, and the arrivals and departures from the system are given as follows. There are n input sources (e.g. n users connected to the server), and job requirements of each user is given by the so-called *ON-OFF process* $\{X_i^{n,\tau}, i \geq 1\}$ as defined in [31], namely each user stays connected to the server for a random ON-period of time with distribution function F_1, and stays off during a random OFF-period of time with distribution function F_2. The distribution F_i is assumed to have finite mean m_i but infinite variance, and in particular,

$$1 - F_i(x) \sim c_i x^{-\alpha_i},$$

where $1 < \alpha_i < 2$ for $i = 1, 2$. While connected to the server, each user demands service at unit rate (sends data-packets at a unit rate to the server for processing). The server is processing users requests at a constant rate, say $\mu_{n,\tau}$. Assume that the ON and OFF-periods are all independent (for each user as well as across users), the ON-OFF processes have stationary increments, and average rate of arrival of jobs (packets) from each source (customer) is given by $\lambda = m_1/(m_1 + m_2)$. The queue-length at time $t \geq 0$ is given by

$$X^{n,\tau}(t) = X^{n,\tau}(0) + \sum_{i=1}^{n} \int_0^t X_i^{n,\tau}(s)ds - \mu_{n,\tau}t + L^{n,\tau}(t),$$

where $L^{n,\tau}$ is a non-decreasing process that starts from 0, increases only when $X^{n,\tau}$ is zero, and ensures that $X^{n,\tau}$ is always non-negative. Physically, this implies that the server is non-idling, i.e it serves jobs continuously as long as the buffer is non-empty. The second term in the right-hand side of the above equation represents cumulative number of packets sent to the server by all the n customers in the interval $[0, t]$. We will assume that $X^{n,\tau}(0) = x_{n,\tau}$, where $x_{n,\tau}$ are fixed non-negative real numbers for each n and τ. In this setup, $\tau > 0$ represents the time scaling parameter, and it is well known (see [30] or Theorem 7.2.5 of [34]) that

$$\tau^{-H} n^{-\frac{1}{2}} \sum_{i=1}^{n} \int_0^{\tau \cdot} \big(X_i^{n,\tau}(s) - \lambda\big) ds \;\; \Rightarrow \;\; W_H(\cdot),$$

when $n \to \infty$ first and then $\tau \to \infty$. Here W_H is a fractional Brownian motion with Hurst parameter $H = (3 - \min\{\alpha_1, \alpha_2\})/2 \in \big(\frac{1}{2}, 1\big)$, and the convergence is the weak convergence in the space $D\big([0,\infty), \mathbb{R}\big)$ of right-continuous real functions with left limits equipped with the standard $J_{\alpha,1}$ topology (see [34] for details).

From the above convergence result it can be derived (see Theorem 8.7.1 in [34]) that if the service rates $\mu_{n,\tau}$ satisfy the following heavy traffic assumption

$$\tau^{-H} n^{-\frac{1}{2}}(\mu_{n,\tau} - n\lambda\tau) \;\to\; u,$$

as $(n,\tau) \to \infty$, then a suitably scaled queue-length satisfies equations (2.1) and (2.2). More precisely, if the above heavy traffic condition is satisfied, and $\tau^{-H} n^{-\frac{1}{2}} x_{n,\tau} \;\to\; x$, then the scaled queue-length $\tau^{-H} n^{-\frac{1}{2}} X^{n,\tau}(\tau\cdot)$ converges weakly to a limiting process $X^x_u(\cdot)$ that satisfies (2.1), (2.2) if we let $n \to \infty$ first and then $\tau \to \infty$.

Hence, we see that with a "super-imposed" ON-OFF input source and deterministic services times for the queueing processes, a suitably-scaled queue length in the limit satisfies our model. With a cost structure similar to either (2.5), (2.7), or (2.10) for the queueing network problem, one can consider the problem described in this paper as a formal fractional Brownian control problem (FBCP) of the corresponding control problem for the queueing network. We, however, do not attempt to solve the queueing control problem in this paper. A solution the limiting control problem provides useful insights into the queueing network control problem (see, for instance, [12]). For a broad class of queueing problems, it has been shown that the value function of the Brownian control problem (BCP) is a lower bound for the minimum cost in the queueing network control problem (see [6]). In many situations, the solution to the BCP can be utilized to obtain optimal strategies for the queueing network control problem (cf. [2], [5], [9] etc.). Here, we study just the Brownian control problem, which is an important problem in its own right. Our explicit solution to the FBCP can be considered as an "approximate solution" to the queueing network problem.

2 Basic setup

In this section we define the controlled state process (Section 2.1), describe three standard control problems associated with it (Section 2.2), and also discuss some basic properties of a reflection mapping which is involved in the definition of the state process (Section 2.3).

2.1 Model

Let $\big(W_H(t)\big)_{t\ge 0}$ be a fractional Brownian motion with Hurst parameter $H \ge 1/2$ and let $\sigma(\cdot)$ be a deterministic continuous function defined on $[0,\infty)$ and taking positive values. For a given initial value $x \ge 0$ and a control variable $u \ge 0$, the controlled state process X^x_u is defined by

$$X^x_u(t) = x - ut + \sigma(u) W_H(t) + L^x_u(t), \qquad t \ge 0, \tag{2.1}$$

where the process L^x_u is given by

$$L^x_u(t) = -\min\big\{0, \min_{s\in[0,t]}\big(x - us + \sigma(u) W_H(s)\big)\big\}, \qquad t \ge 0. \tag{2.2}$$

The control variable $u \geq 0$ remains fixed throughout the evolution of the state process X_u^x. It follows from (2.1) and (2.2) that $X_u^x(t) \geq 0$ for all $t \geq 0$. Notice that the process L_u^x has continuous paths, and it increases at times when $X_u^x(t) = 0$.

The process X_u^x represents the workload process of a single server controlled queue fed by a fractional Brownian motion, as described in the previous section (see also [37]). For a chosen control $u \geq 0$ that remains fixed for all $t \geq 0$, the controller is faced with a cost structure consisting of the following three additive components during a time interval $[t, t + dt]$:

1. a control cost $h(u)dt$,
2. a state dependent holding cost $C\big(X_u^x\big)dt$,
3. a penalty of $p\,dL_u^x(t)$, if the workload in the system is empty.

Here $p \geq 0$ is a constant, and h and C are non-negative continuous functions satisfying the following basic assumptions:

(i) The function h is defined on $[0, +\infty)$, and

$$h \text{ is non-decreasing and continuous, } h(0) \geq 0, \lim_{u\to+\infty} h(u) = +\infty. \tag{2.3}$$

(ii) The function C is also defined on $[0, +\infty)$, and it is a non-negative, non-decreasing continuous function which satisfies the following polynomial growth condition:

$$0 \leq C(x) \leq K(1 + x^\gamma) \tag{2.4}$$

for some positive constants $K > 0$ and $\gamma > 0$.

We will sometimes assume the convexity of h and C in order to obtain sharper results, such as the uniqueness of the optimal controls.

2.2 Three control problems

Here we formulate three cost minimization problems for our model. In the *long-run average cost minimization problem* (it is also called *ergodic control problem*), the controller minimizes the cost functional

$$\begin{aligned} I(u, x) &:= \limsup_{T\to\infty} \frac{1}{T} E\Big(\int_0^T \big[h(u) + C\big(X_u^x(t)\big)\big]dt + \int_0^T p\,dL_u^x(t)\Big) \\ &= h(u) + \limsup_{T\to\infty} \frac{1}{T} E\Big(\int_0^T C\big(X_u^x(t)\big)dt + p\,L_u^x(T)\Big), \end{aligned} \tag{2.5}$$

subject to the constraint $u > 0$ for a fixed initial value $x \geq 0$. Here $p > 0$ is a positive constant. Notice that since $L_u^x(t)$ is an increasing process, the integral with respect to $L_u^x(t)$ can be defined an ordinary Riemann-Stieltjes integral. The value function of this problem is given by

$$V_0(x) = \inf_{u>0} I(u, x). \tag{2.6}$$

In Section 3, we show that $I(u,x)$ and hence also the value function $V_0(x)$ are actually independent of x. In addition, we show the existence of a finite optimal control $u^* > 0$ and also prove that u^* is unique if the functions h and C are convex. We apply the results on the long run average cost problem to find an optimal strategy for a constrained optimization problem, in Section 4.

In Section 5, we solve the infinite horizon discounted cost minimization problem for the case when $\sigma(u) \equiv 1$ in (2.1). In this problem it is assumed that the controller wants to minimize the cost functional

$$J_\alpha(u,x) := E\Big(\int_0^\infty e^{-\alpha t}\big[h(u) + C\big(X_u^x(t)\big)\big]\,dt + p\int_0^\infty e^{-\alpha t} dL_u^x(t)\Big). \tag{2.7}$$

subject to $u > 0$ for a fixed initial value $x \geq 0$. Here the discount factor $\alpha > 0$ is a positive constant. The value function in this case is given by

$$V_\alpha(x) = \inf_{u>0} J_\alpha(u,x). \tag{2.8}$$

We study the asymptotic behavior of this model in Section 6. When α approaches zero, we prove that $\lim_{\alpha\to 0^+} \alpha J_\alpha(u,x) = I(u,x)$ for any control $u > 0$. Furthermore, we show that $\lim_{\alpha\to 0^+} \alpha V_\alpha(x) = V_0(x)$ and the optimal controls for the discounted cost problem converges to that of the long-run average cost problem as α tends to zero. In Section 6, we also consider the *finite horizon control problem* with the value function $V(x,T)$ defined by

$$V(x,T) := \inf_{u>0} I(u,x,T), \tag{2.9}$$

where

$$\begin{aligned} I(u,x,T) \;&:=\; E\Big(\int_0^T \big[h(u) + C\big(X_u^x(t)\big)\big]\,dt + pL_u^x(T)\Big) \\ &=\; h(u)T + p\,E\big(L_u^x(T)\big) + E\Big(\int_0^T C\big(X_u^x(t)\big)dt\Big), \end{aligned} \tag{2.10}$$

and $p \geq 0$ is a non-negative constant. We also prove that $\lim_{T\to\infty} \frac{V(x,T)}{T} = V_0(x)$. Furthermore, we show that the optimal controls for the finite horizon problem converges to that of the long-run average cost problem, as T tends to infinity.

2.3 The reflection map

The model equations (2.1) and (2.2) have an equivalent representation which is given below in (2.12) by using the reflection map. Therefore we briefly discuss some basic properties of the reflection map and of the representation (2.12).

Let $\mathcal{C}([0,\infty),\mathbb{R})$ be the space of continuous functions with domain $[0,\infty)$. The standard reflection mapping $\Gamma : \mathcal{C}([0,\infty),\mathbb{R}) \to \mathcal{C}([0,\infty),\mathbb{R})$ is defined by

$$\Gamma\big(f\big)(t) = f(t) + \sup_{s\in[0,t]} (-f(s))^+, \tag{2.11}$$

for $f \in \mathcal{C}([0,\infty),\mathbb{R})$. Here and henceforth we use the notation $a^+ := \max\{0,a\}$. This mapping is also known as the *Skorokhod map* or the *regulator map* in different contexts. For a detailed discussion we refer to [21, 34].

In our model (2.1)-(2.2), we can write X_u^x as follows:

$$X_u^x(t) = \Gamma\big(x - ue + \sigma(u)W_H\big)(t), \tag{2.12}$$

where $e(t) := t$ for $t \geq 0$, is the identity map.

Note that according to the definition, $\Gamma\big(f\big)(t) \geq 0$ for $t \geq 0$. We will also use the following two standard facts about Γ (see for instance [21, 34]). First, we have

$$\sup_{t\in[0,T]} |\Gamma\big(f\big)(t)| \leq 2 \sup_{t\in[0,T]} |f(t)|, \tag{2.13}$$

for $f \in \mathcal{C}([0,\infty),\mathbb{R})$. Secondly, let f and g be two functions in $\mathcal{C}([0,\infty),\mathbb{R})$ such that $f(0) = g(0)$ and $h(t) := f(t) - g(t)$ is a non-negative non-decreasing function in $\mathcal{C}([0,\infty),\mathbb{R})$. Then

$$\Gamma\big(f\big)(t) \geq \Gamma\big(g\big)(t), \qquad \text{for all } t \geq 0. \tag{2.14}$$

We shall also rely on the following convexity property of the reflected mapping. Let $\alpha \in (0,1)$ and f and g be two functions in $\mathcal{C}([0,\infty),\mathbb{R})$. Then, $\alpha f + (1-\alpha)g \in \mathcal{C}([0,\infty),\mathbb{R})$, and

$$\Gamma\big(\alpha f + (1-\alpha)g\big)(t) \leq \alpha\Gamma\big(f\big)(t) + (1-\alpha)\Gamma\big(g\big)(t) \tag{2.15}$$

for all $t \geq 0$. The proof is straightforward. Let $F(x) = x^- := \max\{0,-x\}$. Then F is a convex function and therefore $(\alpha f(s) + (1-\alpha)g(s))^- \leq \alpha f^-(s) + (1-\alpha)g^-(s)$. Consequently, $\sup_{s\in[0,t]}(\alpha f(s) + (1-\alpha)g(s))^- \leq \alpha \sup_{s\in[0,t]} f^-(s) + (1-\alpha)\sup_{s\in[0,t]} g^-(s)$. Since $\sup_{s\in[0,t]}(-f(s))^+ = \sup_{s\in[0,t]} f^-(s)$, the inequality (2.15) follows from the definition (2.11).

The reflection map also satisfies the following minimality property: if $\psi, \eta \in \mathcal{C}\big([0,\infty),\mathbb{R}\big)$ are such that ψ is non-negative, $\eta(0) = 0$, η is non-decreasing, and $\psi(t) = \varphi(t) + \eta(t)$ for $t \geq 0$, then

$$\psi(t) \geq \Gamma(\varphi)(t) \quad \text{and} \quad \eta(t) \geq \sup_{s\in[0,t]} \big(-\varphi(s)\big)^+, \qquad \text{for all } t \geq 0. \tag{2.16}$$

3 Long-run average cost minimization problem

In this section we address the control problem defined in (2.5)-(2.6). First we find a solution to the control problem for the particular case when $\sigma(u) \equiv 1$ in (2.1). This is accomplished in Sections 3.1–3.4. In Section 3.5, we show that the general case can be reduced to this simplified version.

3.1 Reduction of the cost structure

The controlled state space process X_u^x (corresponding to $\sigma(u) \equiv 1$) has the form

$$X_u^x(t) = x - ut + W_H(t) + L_u^x(t), \qquad t \geq 0. \tag{3.1}$$

The following lemma simplifies the expression for the cost functional (2.5) by computing $\lim\limits_{T\to\infty} \frac{1}{T} E\big(L_u^x(T)\big)$.

Lemma 3.1. *Let X_u^x be given by* (3.1)*. Then* $\lim_{T\to\infty} \frac{1}{T}E\big(L_u^x(T)\big) = u$.

Proof. Since $u > 0$, using (2.12), (2.13), and (2.14), we obtain:

$$0 \le X_u^x(t) \le X_0^x(t) = \Gamma\big(x + W_H\big)(t) \le 2\Big(|x| + \sup_{s\in[0,t]} |W_H(s)|\Big).$$

By the self-similarity of fractional Brownian motion process,

$$E\big(\sup_{s\in[0,T]} |W_H(s)|\big) \le K_1 T^H,$$

where $K_1 \in (0,\infty)$ is a constant independent of T (see for instance [25, p. 296]). Therefore,

$$0 \le E\big(X_u^x(T)\big) \le 2K_1\big(|x| + T^H\big). \tag{3.2}$$

Consequently, $\lim_{T\to\infty} \frac{1}{T}E\big(X_u^x(T)\big) = 0$. Since $W_H(T)$ is a mean-zero Gaussian process, it follows from (3.1) that $\frac{1}{T}E\big(L_u^x(T)\big) - u = \frac{1}{T}\Big(E\big(X_u^x(T)\big) - x\Big)$. Letting T tend to infinity completes the proof of the lemma. □

Remark 3.2. *Lemma 3.1 with literally the same proof as above remains valid if X_u^x satisfies* (2.1) *instead of* (3.1)*.*

With the above lemma in hand, we can represent the cost functional (2.5) and reformulate the long-run average cost minimization problem as follows. The controller minimizes

$$I(u,x) = \big(h(u) + pu\big) + \limsup_{T\to\infty} \frac{1}{T}E\Big(\int_0^T C\big(X_u^x(t)\big)dt\Big) \tag{3.3}$$

subject to $u > 0$ and X_u^x given in (3.1). Note that the above reduction shows that the original minimization problem (2.5) reduces to the case $p = 0$ with the function $h(u)$ replaced by $h(u) + pu$.

Our next step is to analyze the cost component $\limsup_{T\to\infty} \frac{1}{T}E\Big(\int_0^t C\big(X_u^x(t)\big)dt\Big)$. The following results are described in [19] and [8], and are collected in [37] in a convenient form for our application. We summarize them here using our notation.

(i) The random sequence $X_u^0(t)$ with $t \ge 0$ converges weakly, as t goes to infinity, to the random variable

$$Z_u := \max_{s\ge 0}\{W_H(s) - us\}. \tag{3.4}$$

(ii) There is a probability space supporting the processes X_u^0, L_u^0 (and hence X_u^x as well as L_u^x for any $x \ge 0$) and a stationary process $X_u^* = \{X_u^*(t) : t \ge 0\}$ such that

$$X_u^*(t) = W_H(t) - ut + \max\{X_u^*(0), L_u^0(t)\}, \qquad t \ge 0, \tag{3.5}$$

and

$$X_u^*(t) \stackrel{D}{=} Z_u, \qquad t \ge 0, \tag{3.6}$$

where "$\stackrel{D}{=}$" denotes the equality in distribution and Z_u is defined in (3.4).

(iii) The tail of the stationary distribution satisfies

$$\lim_{\mathfrak{z}\to\infty} \mathfrak{z}^{2H-2} \log P(Z_u \geq \mathfrak{z}) = -\theta^*(u), \tag{3.7}$$

where $\theta^*(u)$ is given by

$$\theta^*(u) = \frac{u^{2H}}{2H^{2H}(1-H)^{2(1-H)}} > 0. \tag{3.8}$$

In particular, all the moments of Z_u are finite.

Remark 3.3. *In (ii) above, the construction of X_u^* is rather simple: Consider a probability space with two independent FBM's $W_H(\cdot)$ and $\widetilde{W}_H(\cdot)$. By pasting $W_H(\cdot)$ and $\widetilde{W}_H(\cdot)$ at the point $t = 0$, first we construct the two-sided FBM $\{B_H(t) : -\infty < t < +\infty\}$, which has stationary Gaussian increments. Then we define*

$$X_u^*(t) = B_H(t) - ut + \sup_{s\in(-\infty,t]} \big(us - B_H(s)\big), \qquad t \in \mathbb{R}.$$

The random variables $X_u^*(t)$ are finite for all $t \in \mathbb{R}$ because $\lim_{|t|\to\infty} \frac{B_H(t)}{t} = 0$ a.s. Notice that for all $t \in \mathbb{R}$,

$$\begin{aligned} X_u^*(t) &= \sup_{s\in(-\infty,t]} \big(B_H(t) - B_H(s) - u(t-s)\big) \\ &\stackrel{D}{=} \sup_{s\in(-\infty,t]} \big(\widetilde{B}_H(t-s) - u(t-s)\big) = \sup_{r\in[0,\infty)} \big(\widetilde{B}_H(r) - ur\big), \end{aligned}$$

where $\widetilde{B}_H(\cdot)$ is a two-sided FBM. Hence (3.6) holds for all $t \in \mathbb{R}$. Furthermore, $X_u^*(0)$ is independent of $\{W_H(t) : t \geq 0\}$ and clearly (3.5) holds. For further discussion of (i)-(iii) we refer to [8, 18, 19, 24].

Throughout the rest of the paper, we use this probability space where all these processes are defined. Using (3.1) and (2.2), we can write for $t \geq 0$,

$$X_u^x(t) = W_H(t) - ut + \max\{x, L_u^0(t)\}, \tag{3.9}$$

where

$$L_u^0(t) = -\inf_{s\in[0,t]} \big(W_H(s) - us\big) = \sup_{s\in[0,t]} \big(us - W_H(s)\big). \tag{3.10}$$

The above representation (3.9) and (3.10) agrees with (3.5) if the process X_u^x is initialized with $X_u^*(0)$.

3.2 A coupling time

The following coupling argument is crucial to address the optimal control problems. In particular, it enables us to deal with the last term of $I(u,x)$ in (3.3).

Proposition 3.4. *Let $u>0$ and the initial point $x\geq 0$ be fixed. Consider the state process X_u^x in (3.1) and the stationary process X_u^* of (3.5) and (3.6). Then the following results hold:*

(i) There is a finite stopping time τ_0 such that $X_u^x(t)=X_u^*(t)$ for all $t\geq\tau_0$. Furthermore, $E(\tau_0^\beta)<\infty$ for all $\beta\geq 0$.

(ii) The cost functional $I(u,x)$ defined in (2.5) is finite and independent of x, that is $I(u,x)=I(u,0)<\infty$ for $x\geq 0$. Consequently, the value function $V_0(x)=\inf\limits_{u>0} I(u,x)$ is also finite and independent of x, that is $V_0(x)=V_0(0)<\infty$ for $x\geq 0$.

Proof. For $y>0$ introduce the stopping time

$$\lambda_y=\inf\{t>0: L_u^0(t)>x+y\}.$$

The stopping time λ_y is finite a.s. because $\lim\limits_{t\to+\infty} L_u^0(t)\geq \lim\limits_{t\to+\infty}\big(ut-W_H(t)\big)=+\infty$ a.s. Define the stopping time τ_0 by

$$\tau_0=\inf\{t>0: L_u^0(t)>x+X_u^*(0)\}. \tag{3.11}$$

Here X_u^* is the stationary process which satisfies (3.5) and (3.6). Hence, $\tau_0=\lambda_{X_u^*(0)}$ a.s. It follows that for $t\geq\tau_0$, we have $L_u^0(t)\geq L_u^0(\tau_0)=x+X_u^*(0)$ and $X_u^*(0)\geq 0$. Therefore, it follows from (3.9) and (3.10) that $X_u^x(t)=W_H(t)-ut+L_u^0(t)=X_u^*(t)$ for $t\geq\tau_0$.

Next, we prove that $E(\tau_0^\beta)<+\infty$ for each $\beta\geq 0$. Without loss of generality we can assume that $\beta\geq 1$. We then have:

$$E(\tau_0^\beta)\leq\sum_{m=0}^{\infty}E\big(\lambda_{m+1}^\beta\cdot\mathbf{1}_{[m\leq X_u^*(0)<m+1]}\big)\leq\sum_{m=0}^{\infty}\big[E(\lambda_{m+1}^{2\beta})P(X_u^*(0)\geq m)\big]^{1/2}, \tag{3.12}$$

where in the last step we have used the Cauchy-Schwartz inequality. Since $X_u^*(0)$ has the same distribution as $Z_u=\sup_{s\geq 0}\{W_H(s)-us\}$, it follows from (3.7) that for all m large enough,

$$P(X_u^*(0)\geq m)\leq e^{-\frac{1}{2}\theta^*(u)m^{2(1-H)}}, \tag{3.13}$$

where $\theta^*(u)$ is defined in (3.8). Next, we estimate $E(\lambda_m^{2\beta})$ for $m\geq 0$ and $\beta\geq 1$. For $m\in\mathbb{N}$, let $b_m=x+m$ and $T_m=\frac{2b_m}{u}$. We have

$$\begin{aligned} E(\lambda_m^{2\beta}) &= 2\beta\int_0^\infty t^{2\beta-1}P(\lambda_m>t)dt \;=\; 2\beta\int_0^\infty t^{2\beta-1}P(L_u^0(t)\leq x+m)dt \\ &\leq T_m^{2\beta}+2\beta\int_{T_m}^\infty t^{2\beta-1}P(L_u^0(t)\leq b_m)dt, \end{aligned} \tag{3.14}$$

where the second equality is due to the fact that $P(\lambda_m > t) = P(L_u^0(t) \le x+m)$ according to the definition of λ_m. Notice that

$$P\big(L_u^0(t) \le b_m\big) \;=\; P\Big(\sup_{s\in[0,t]}\{us - W_H(s)\} \le b_m\Big) \;\le\; P\big(W_H(t) \ge ut - b_m\big), \quad (3.15)$$

and recall that $Z := \frac{W_H(t)}{t^H}$ has a standard normal distribution. Therefore, by (3.15), for $t > T_m$ we have

$$\begin{aligned} P\big(L_u^0(t) \le b_m\big) &\le P\big(W_H(t) \ge ut - b_m\big) \;\le\; P\Big(W_H(t) \ge \frac{ut}{2}\Big) \\ &= P\Big(Z \ge \frac{ut^{1-H}}{2}\Big). \end{aligned} \quad (3.16)$$

It follows from (3.14) and (3.16) that

$$\begin{aligned} E(\lambda_m^{2\beta}) &\le T_m^{2\beta} + 2\beta \int_0^\infty t^{2\beta-1} P\Big(Z \ge \frac{ut^{1-H}}{2}\Big)dt \\ &= T_m^{2\beta} + 2\beta \int_0^\infty t^{2\beta-1} P\Big[\Big(\frac{2Z}{u}\Big)^{\frac{1}{1-H}} \ge t\Big]dt \\ &= T_m^{2\beta} + E\Big[\Big(\frac{2|Z|}{u}\Big)^{\frac{2\beta}{1-H}}\Big] = \frac{4^\beta}{u^{2\beta}}(x+m)^{2\beta} + E\Big[\Big(\frac{2|Z|}{u}\Big)^{\frac{2\beta}{1-H}}\Big] < \infty. \end{aligned} \quad (3.17)$$

The estimates (3.13) and (3.17) imply that the infinite series in the right-hand side of (3.12) converges. Thus $E(\tau_0^\beta) < \infty$ for all $\beta \ge 1$, and hence for all $\beta \ge 0$. This completes the proof of the first part of the proposition.

We turn now to the proof of part (ii). First, we will prove that

$$E\Big(\int_0^\infty \big|C\big(X_u^x(t)\big) - C\big(X_u^*(t)\big)\big|dt\Big) < \infty. \quad (3.18)$$

We will show later that part (ii) of the proposition is a rather direct consequence of this inequality.

Notice that

$$E\Big(\int_0^\infty \big|C\big(X_u^x(t)\big) - C\big(X_u^*(t)\big)\big|dt\Big) = E\Big(\int_0^{\tau_0} \big|C\big(X_u^x(t)\big) - C\big(X_u^*(t)\big)\big|dt\Big),$$

where τ_0 is given in (3.11) and X_u^* is given in (3.5) and (3.6). The definition of τ_0 implies $L_u^0(\tau_0) \le x + X_u^*(0)$. Therefore, it follows from (3.4) and (3.5) that for $t \in [0,\tau_0]$,

$$\max\{X_u^x(t), X_u^*(t)\} \le Z_u + x + X_u^*(0).$$

Since C is a non-decreasing function, this implies

$$\max\big\{C\big(X_u^x(t)\big), C\big(X_u^*(t)\big)\big\} \le C\big(Z_u + x + X_u^*(0)\big),$$

and consequently, using the Cauchy-Schwartz inequality,

$$\begin{aligned} E\Big(\int_0^{\tau_0} \big|C\big(X_u^x(t)\big) - C\big(X_u^*(t)\big)\big|dt\Big) &\le E\Big(\tau_0 C\big(Z_u + x + X_u^*(0)\big)\Big) \\ &\le \Big[E(\tau_0^2)E\Big(\big[C\big(Z_u + x + X_u^*(0)\big)\big]^2\Big)\Big]^{1/2}. \end{aligned}$$

Since $E(\tau_0^2) < \infty$ by part (i) of the lemma, (3.18) will follow once we show that

$$E\Big(\big[C\big(Z_u + x + X_u^*(0)\big)\big]^2\Big) < \infty. \tag{3.19}$$

Recall that $X_u^*(0)$ and Z_u have the same distribution. The tail asymptotic (3.7) implies that any moment of Z_u is finite. This fact together with (2.4) yield (3.19).

We will now deduce part (ii) of the proposition from (3.18). Toward this end, first observe that, since $X_u^*(t)$ is a stationary process,

$$\frac{1}{T}E\Big(\int_0^T C\big(X_u^*(t)\big)dt\Big) = \frac{1}{T}E\Big(\int_0^T C\big(X_u^*(0)\big)dt\Big) = E\big(C(Z_u)\big),$$

and recall that $E\big(C(Z_u)\big) < \infty$ by (3.7). Then notice that by (3.18),

$$\begin{aligned}\limsup_{T\to\infty}&\Big|\frac{1}{T}E\Big(\int_0^T C\big(X_u^*(t)\big)dt\Big) - \frac{1}{T}E\Big(\int_0^T C\big(X_u^x(t)\big)dt\Big)\Big| \\ &\le \limsup_{T\to\infty}\frac{1}{T}E\Big(\int_0^\infty \big|C\big(X_u^x(t)\big) - C\big(X_u^*(t)\big)\big|dt\Big) = 0.\end{aligned}$$

Therefore

$$\lim_{T\to\infty}\frac{1}{T}E\Big(\int_0^T C\big(X_u^x(t)\big)dt\Big) = E\big(C(Z_u)\big), \tag{3.20}$$

which implies part (ii) of the proposition in view of (3.3). Therefore, the proof of the proposition is complete. □

Remark. *The above proposition is in agreement with a result in Theorem 1 of [37] which shows that the limiting distributions of $M(t) = \max_{s\in[0,t]} X_u^x(s)$ and $M^*(t) = \max_{s\in[0,t]} X_u^*(s)$ coincide, as t tends to infinity.*

3.3 Properties of $E\big(C(Z_u)\big)$.

For $u > 0$, let $G(u) = E\big(C(Z_u)\big)$. We are interested in the behavior of $G(u)$ in view of the identity (3.20).

Lemma 3.5. *Let $G(u) = E\big(C(Z_u)\big)$ where Z_u is defined in* (3.4). *Then the following results hold:*

(i) $G(u)$ is a decreasing and continuous function of u on $[0,\infty)$.

(ii) If $C(x)$ is a convex function then so is $G(u)$.

(iii) $\lim_{u\to 0^+} G(u) = +\infty$.

Proof. First we observe that the polynomial bound (2.4) on the growth of C combined with (3.7), which describes the tail behavior of Z_u, imply that $G(u) = E\big(C(Z_u)\big)$ is finite for each $u \geq 0$. It is a decreasing function of u because C is non-decreasing while $Z_{u_1} \leq Z_{u_2}$ if $u_1 > u_2$.

To complete the proof of part (i), it remains to show that $G(u)$ is a continuous function. To prove this, first notice that, according to the definition (3.4), Z_u is a continuous function of the variable u a.s., as shown below. Indeed, if t_u is a random time such that $Z_u = W_H(t_u) - ut_u$ and $u \in (0, v)$, then

$$Z_u \geq Z_v \geq W_H(t_u) - vt_u = Z_u - t_u(v-u).$$

Hence $\lim_{v\to u^+} Z_v = Z_u$. A similar argument shows that $\lim_{v\to u^-} Z_v = Z_u$. Therefore, continuity of C implies that $C(Z_u)$ is a continuous function of u with probability one. Since $C(Z_u)$ is monotone in u, the dominated convergence theorem implies the continuity of G.

To prove part (ii), fix constants $r \in [0,1]$, $u_1 > 0$, $u_2 > 0$, and let $\bar{u}_r = ru_1 + (1-r)u_2$. Then $Z_{\bar{u}_r} = \sup_{t\geq 0}\{W_H(t) - \bar{u}_r t\} \leq rZ_{u_1} + (1-r)Z_{u_2}$. If C is a non-decreasing convex function, we have $E\big(C(Z_{\bar{u}_r})\big) \leq rE\big(C(Z_{u_1})\big) + (1-r)E\big(C(Z_{u_2})\big)$. Hence G is convex, and the proof of part (ii) is complete.

Turning to the part (iii), we first notice that $Z_0 = \sup_{t\geq 0} W_H(t) = +\infty$ with probability one. Let $(u_n)_{n\geq 0}$ be any sequence monotonically decreasing to zero. Then, Z_{u_n} is increasing and hence there exists a limit (finite or infinite) $\lim\limits_{n\to\infty} Z_{u_n} = L$ and $Z_{u_n} \leq L$ for all $n \geq 0$. Thus $W_H(t) - u_n t \leq L$ for all $n \geq 0$ and $t \geq 0$. By letting n go to infinity we obtain $\sup_{t\geq 0} W_H(t) \leq L$, and consequently $L = +\infty$ with probability one. Therefore $\lim\limits_{u\to 0^+} Z(u) = +\infty$ a.s. Since C is a non-decreasing function, the monotone convergence theorem implies that $\lim\limits_{u\to 0^+} E\big(C(Z_u)\big) = +\infty$. This completes the proof of the lemma. □

3.4 Existence of an optimal control

In the following two theorems we provide a representation of the cost functional $I(u,x)$ as well as the existence and uniqueness results for the optimal control $u^* > 0$.

Theorem 3.6. *Let $I(u,x)$ be the cost functional of the long-run average cost problem described in* (3.3)*. Then*

(i) $I(u,x)$ is independent of x and has the representation

$$I(u) := I(u,x) = h(u) + pu + G(u), \tag{3.21}$$

where $G(u)$ is given in Lemma 3.5. Furthermore, $I(u)$ is finite for each $u > 0$ and is continuous in $u > 0$.

(ii) $\lim\limits_{u\to 0^+} I(u) = +\infty$ *and* $\lim\limits_{u\to\infty} I(u) = +\infty$.

(iii) If $h(x)$ and $C(x)$ are convex functions, then $I(u)$ is also convex.

Proof. Part (i) follows from (3.3), Proposition 3.4, and Lemma 3.5.

The first part of claim (ii) follows from the fact that $I(u) \geq G(u)$ along with part (iii) of Lemma 3.5. To verify the second part, notice that $I(u) \geq h(u)$ for all $u > 0$, and $\lim_{u\to+\infty} h(u) = +\infty$. Consequently, $\lim_{u\to+\infty} I(u) = +\infty$.

Part (iii) follows from the representation (3.21) combined with the part (ii) of Lemma 3.5. This completes the proof of the theorem. □

Theorem 3.7.

(i) There is an optimal control $u^* > 0$ such that for all $x \geq 0$ we have

$$I(u^*) = \min_{u>0} I(u, x)$$

where I is given in (3.21). In particular, u^* is independent of x.

(ii) In the case $p > 0$, if h and C are convex functions, then u^* is unique.

(iii) In the case $p = 0$, if h is a strictly convex function and C is a convex function, then u^* is unique.

Proof. Since $I(u)$ is a continuous function, part (i) follows from parts (i) and (ii) of Theorem 3.6.

If h and C are convex functions, the representation (3.21) yields that I is a strictly convex function when $p > 0$. Therefore, u^* is unique in this case.

In the case $p = 0$, if h is a strictly convex function and C is a convex function, the results follows from the representation (3.21) in a similar way as in the case $p > 0$. □

3.5 Generalizations

In this section we generalize the results in Theorem 3.6 and 3.7 to the more general model introduced in (2.1). Notice that for a fixed control $u > 0$ in (2.1), the self-similarity of FBM yields that the process $\widehat{W}_H$ defined by

$$\widehat{W}_H(t) = \sigma(u) W_H\Big(\frac{t}{\sigma(u)^{\frac{1}{H}}}\Big), \ t \in \mathbb{R},$$

is a fractional Brownian motion. Let

$$Y_u^x(t) = X_u^x\Big(\frac{t}{\sigma(u)^{\frac{1}{H}}}\Big).$$

Then Y_u^x satisfies

$$Y_u^x(t) = x - \frac{ut}{\sigma(u)^{\frac{1}{H}}} + \widehat{W}_H(t) + \widehat{L}_u^x(t), \tag{3.22}$$

where

$$\widehat{L}_u^x(t) = L_u^x\Big(\frac{t}{\sigma(u)^{\frac{1}{H}}}\Big).$$

Using (2.2) and change of the variable $s = \frac{t}{\sigma(u)^{\frac{1}{H}}}$, we observe that

$$\widehat{L}_u^x(t) = L_u^x\Big(\frac{t}{\sigma(u)^{\frac{1}{H}}}\Big) = -\min\Big\{0, \min_{s\in[0,t]}\Big(x - \frac{su}{\sigma(u)^{\frac{1}{H}}} + \widehat{W}_H(s)\Big)\Big\}. \tag{3.23}$$

The equations (3.22) and (3.23) are analogous to (2.1) and (2.2).

We next consider the change in the cost structure due to the change of the variable $s = \frac{t}{\sigma(u)^{\frac{1}{H}}}$. We notice that

$$\frac{1}{T}\int_0^T C\big(X_u^x(t)\big)dt = \frac{1}{M(T)}\int_0^{M(T)} C\big(Y_u^x(t)\big)dt,$$

where $M(T) = \sigma(u)^{\frac{1}{H}}T$. Therefore, using the results in Theorem 3.6, we obtain

$$\lim_{T\to\infty}\frac{1}{T}E\Big(\int_0^T C\big(X_u^x(t)\big)dt\Big) = G\Big(\frac{u}{\sigma(u)^{\frac{1}{H}}}\Big). \tag{3.24}$$

We have the following result.

Theorem 3.8. *Consider the controlled state process X_u^x defined by (2.1) and (2.2) with the cost functional $I(x,u)$ given in (2.5). Define $f : [0,\infty) \to [0,\infty)$ by*

$$f(u) = \frac{u}{\sigma(u)^{\frac{1}{H}}}. \tag{3.25}$$

Then

$$I(u,x) = h(u) + p\,u + G\big(f(u)\big), \tag{3.26}$$

where the function G is given in Lemma 3.5. Furthermore, $I(u,x)$ is independent of x (we will henceforth denote the cost function by $I(u)$).

Proof. The same argument as in the proof of Lemma 3.1 yields $\lim_{T\to\infty}\frac{1}{T}E\big(L_u^x(T)\big) = u$. Combining this result with (3.24) we obtain the representation (3.26). □

Our next result is analogous to Theorems 3.6 and 3.7.

Theorem 3.9. *Assume that the function f in (3.25) is continuous and $\lim_{u\to 0^+} f(u) = 0$. Then, with $I(u) = I(u,x)$ as in (3.26), we have:*

(i) $\lim_{u\to 0^+} I(u) = \lim_{u\to +\infty} I(u) = +\infty$, and $I(u)$ is a finite continuous function on $[0,\infty)$. Furthermore, there is a constant $u^* > 0$ such that $I(u^*) = \min_{u>0} I(u)$.

(ii) If f is a concave increasing function then the statements similar to parts (i) and (ii) of Theorem 3.7 (regarding the uniqueness u^*) hold.

The proof of this theorem is a straightforward modification of the proofs of Theorems 3.6 and 3.7 and therefore is omitted.

Remark. *One can further generalize our model to cover the following situation. For given positive continuous functions $b(u)$ and $\sigma(u)$ let*

$$X_u^x(t) = x + \sigma(u)W_h(t) - b(u)t + L_u^x(t),$$

where for $u > 0$,

$$L_u^x(t) = -\min\left\{0, \min_{s\in[0,t]}\big(x - b(u)s + \sigma(u)W_H(s)\big)\right\}.$$

The optimization problem here is to minimize the cost functional $I(u,x)$ defined in (2.5).

Following the time change method described in Section 3.5, one can obtain an analogue of Theorem 3.9 regarding the derivation of the optimal control. In this situation, the function f defined in (3.25) *needs to be replaced by $f(u) = b(u)(\sigma(u))^{-\frac{1}{H}}$ with the assumptions that f is continuous and $\lim_{u\to 0^+} f(u) = 0$. We omit the details of the proof.*

4 A constrained minimization problem

In this section, we address a constrained minimization problem that can be solved by using our results in Section 3. Let $\{W_H(t) : t \geq 0\}$ be a FBM defined on a complete probability space $(\Omega, \mathcal{F}, P)$. Our model here is of the form

$$Y_u^x(t) = x - ut + \sigma(u)W_H(t) + K_u^x(t), \tag{4.1}$$

where σ is a non-negative continuous function, $K_u^x(\cdot)$ is a non-negative non-decreasing right-continuous with left limits (RCLL) process adapted to the natural filtration $(\mathcal{F}_t)_{t\geq 0}$, where $\mathcal{F}_t$ is the σ-algebra generated by $\{W_H(s) : 0 \leq s \leq t\}$ augmented with all the null sets in $\mathcal{F}$. Furthermore, $K_u^x(0) = 0$ and the process K_u^x is chosen by the controller in such a way that the state process Y_u^x is constrained to non-negative reals. In this situation, the controller is equipped with two controls: the choice of $u > 0$ and the choice of K_u^x process subject non-negativity of the Y_u^x process.

Throughout this section we keep the initial state $x \geq 0$ fixed and therefore we relabel Y_u^x process as Y_u^x. We will deduce using the results of the previous section that the value of this minimization problem as well as the optimal control are not affected by the initial data.

Let $m > 0$ be any fixed positive constant. The constrained minimization problem we would like to address here is the following:

Minimize

$$\limsup_{T\to\infty} \frac{1}{T} E\Big(\int_0^T \big[h(u) + C\big(Y_u^x(t)\big)\big]\,dt\Big) \tag{4.2}$$

Subject to:

$$\limsup_{T\to\infty} \frac{E\big(K_u^x(T)\big)}{T} \leq m, \tag{4.3}$$

over all feasible controls $u > 0$ and feasible processes $K_u^x(\cdot)$ which ensures that $Y_u^x(\cdot)$ is a non-negative process. A controlled optimization problem of this nature for diffusion processes was considered in [1], and for a more complete treatment in the case of diffusion processes we refer to [32].

Fix any integer $m > 0$ and define a class of state processes $\mathcal{U}_m$ as follows:

$$\mathcal{U}_m = \Big\{(Y_u^x, K_u^x) : \ Y_u^x(t) \geq 0 \text{ for } t \geq 0, \ (4.1) \text{ is satisfied}, \ \limsup_{T\to\infty} \frac{E\big(K_u^x(T)\big)}{T} \leq m\Big\}.$$

From our results in Section 3 it follows that for any $u \leq m$, the pair (X_u^x, L_u^x) in (2.1) and (2.2) belongs to $\mathcal{U}_m$, and hence $\mathcal{U}_m$ is non-empty. Therefore, the constrained minimization problem is to find

$$\inf_{(Y_u^x,K_u^x)\in\mathcal{U}_m} \limsup_{T\to\infty} \frac{1}{T} E\Big(\int_0^T \big[h(u) + C\big(Y_u^x(t)\big)\big]\, dt\Big).$$

In this section, we make the following additional assumptions:

(i) For functions h and C we assume:

$$h \text{ is strictly convex and satisfies (2.3)}, \ C \text{ is convex and satisfies (2.4)}. \tag{4.4}$$

(ii) Let $f(u) = \frac{u}{\sigma(u)^{\frac{1}{H}}}$. We assume:

$$f(u) > 0, \ \lim_{u\to 0^+} f(u) = 0, \text{ and } f \text{ is a convex increasing function.} \tag{4.5}$$

The following lemma enables us to reduce the set $\mathcal{U}_m$ to the collection of processes (X_u^x, L_u^x) described in the previous section, with $u \leq m$.

Lemma 4.1. *Let $u > 0$ and let (Y_u^x, K_u^x) be a pair of processes satisfying (4.1). Consider (X_u^x, L_u^x) which satisfies (2.1), (2.2), and is defined on the same probability space as (Y_u^x, K_u^x). Then*

(i) $L_u^x(t) \leq K_u^x(t)$ *and* $X_u^x(t) \leq Y_u^x(t)$ *for* $t \geq 0$.

(ii) $u = \lim_{T\to\infty} \frac{1}{T} E\big(L_u^x(T)\big) \leq \limsup_{T\to\infty} \frac{1}{T} E\big(K_u^x(T)\big)$.

(iii) $G\big(f(u)\big) = \lim_{T\to\infty} \frac{1}{T} E\big(\int_0^T C\big(X_u^x(t)\big)dt\big) \leq \limsup_{T\to\infty} \frac{1}{T} E\big(\int_0^T C\big(Y_u^x(t)\big)dt\big)$.

Proof. Since, $Y_u^x \geq 0$, $K_u^x(0) = 0$, and K_u^x is non-increasing process, the minimality property of the reflection map stated in (2.16) implies that $L_u^x(t) \leq K_u^x(t)$ and $X_u^x(t) \leq Y_u^x(t)$ for $t \geq 0$.

Next, observe that part (ii) of the lemma follows from the result in part (i) while the identity $u = \lim_{T\to\infty} \frac{1}{T} E\big(L_u^x(T)\big)$ is implied by Lemma 3.1 (see also the remark right after the proof of Lemma 3.1).

Finally, part (iii) of the lemma follows from (3.24), part (i), and from the fact that C is non-decreasing. The proof of the lemma is complete. □

Let

$$\mathcal{V}_m = \{(X_u^x, L_u^x) : \text{ (2.1),(2.2) are satisfied and in addition } u \le m\}.$$

From the above lemma it is clear that

$$\inf_{(K_u^x, Y_u^x)\in\mathcal{U}_m} \lim_{T\to\infty} \frac{1}{T} E\Big(\int_0^T \big[h(u) + C\big(Y_u^x(t)\big)\big]\,dt\Big)$$
$$= \inf_{(X_u^x, L_u^x)\in\mathcal{V}_m} \limsup_{T\to\infty} \frac{1}{T} E\Big(\int_0^T \big[h(u) + C\big(X_u^x(t)\big)\big]\,dt\Big). \tag{4.6}$$

Therefore, our minimization problem is reduced. Next, we can use the results in Section 3.5 and write for any $u > 0$,

$$\lim_{T\to\infty} \frac{1}{T} E\Big(\int_0^T \big[h(u) + C\big(X_u^x(t)\big)\big]\,dt\Big) = h(u) + G\big(f(u)\big). \tag{4.7}$$

Here G is given by Lemma 3.5, and f is described in (3.25) and (4.5). Consequently,

$$\inf_{(X_u^x, L_u^x)\in\mathcal{V}_m} \lim_{T\to\infty} \frac{1}{T} E\Big(\int_0^T \big[h(u) + C\big(X_u^x(t)\big)\big]\,dt\Big)$$
$$= \inf\{h(u) + G\big(f(u)\big) : 0 < u \le m\}. \tag{4.8}$$

We next consider the optimal control described in Theorem 3.9 corresponding to the case $p = 0$. In virtue of the assumptions (4.4) and (4.5), the optimal control is unique and we will label it by $u_0^* > 0$. We have the following result.

Theorem 4.2. *Let*

$$u^*(m) = \begin{cases} m & \text{if} \quad m < u_0^*, \\ u_0^* & \text{if} \quad m \ge u_0^* \end{cases}, \tag{4.9}$$

where u_0^* is the unique optimal control in Theorem 3.9 corresponding to $p = 0$.

Then the pair $(X_x^{u^*(m)}, L_x^{u^*(m)})$ is an optimal process for the constrained minimization problem (4.2) and (4.3). Furthermore, the optimal control $u^*(m)$ is a continuous increasing function of the parameter m.

Proof. Let $\Lambda(u) = h(u) + G\big(f(u)\big)$ where f and G are as in (4.7). Then, by the assumptions (4.4), (4.5) and Theorem 3.9, Λ is a strictly convex function which is finite everywhere on $(0,\infty)$. Furthermore, $\lim\limits_{u\to 0^+} \Lambda(u) = +\infty$ and $\lim\limits_{u\to+\infty} \Lambda(u) = +\infty$, and hence Λ has a unique minimum at u_0^*. Therefore, Λ is strictly increasing on (u_0^*, ∞). Clearly, with $u^*(m)$ defined in (4.9),

$$\Lambda\big(u^*(m)\big) = \inf_{u\le m} \Lambda(u),$$

and $u^*(m)$ is the unique number which has this property. By (4.6) and (4.8), we have

$$\Lambda\big(u^*(m)\big) = \inf_{(X_u^x, L_u^x)\in\mathcal{V}_m} \limsup_{T\to\infty} \frac{1}{T} E\Big(\int_0^T \big[h(u) + C\big(Y_u^x(t)\big)\big]\,dt\Big).$$

Consider the pair of processes $(X^x_{u^*(m)}, L^x_{u^*(m)})$ defined in (2.1) and (2.2). Then, in virtue of Lemma 3.1, we have $\lim_{T\to\infty} \frac{1}{T} E\big(L^x_{u^*(m)}(T)\big) = u^*(m) \le m$, and by Theorem 3.6,

$$\limsup_{T\to\infty} \frac{1}{T} E\Big(\int_0^T \big[h(u) + C\big(X^x_{u^*(m)}(t)\big)\big]\, dt\Big) = \Lambda\big(u^*(m)\big).$$

Hence $(X^x_{u^*(m)}, L^x_{u^*(m)})$ describes an optimal strategy. This completes the proof of the theorem. □

Remark. *Notice that the above optimal control $u^*(m)$ is independent of the initial point x.*

5 Infinite horizon discounted cost minimization problem

In this section we define an optimal control u^* for the infinite horizon discounted cost functional given in (2.7). Throughout this section we assume that $\sigma(u) \equiv 1$, the state process X^x_u satisfies (3.1), and that the functionals h and C are convex in addition to the assumptions stated in (2.3) and (2.4). In contrast with Section 3, our methods here do not readily extend to the case where the function $\sigma(u)$ is non constant.

The discounted cost functional $J_\alpha(x,u)$ is given by

$$\begin{aligned} J_\alpha(x,u) &= E\Big(\int_0^\infty e^{-\alpha t}\big[h(u) + C\big(X^x_u(t)\big)\big]\, dt + \int_0^\infty e^{-\alpha t} p\, dL^x_u(t)\Big) \\ &= \frac{h(u)}{\alpha} + E\Big(\int_0^\infty e^{-\alpha t}\big[C\big(X^x_u(t)\big) + \alpha p\, L^x_u(t)\big]\, dt\Big). \end{aligned} \tag{5.1}$$

Here $\alpha > 0$ is a constant discount factor and $p > 0$ is also a constant. To derive the last equality above we have used Fubini's theorem to obtain $\int_0^\infty e^{-\alpha t} dL^x_u(t) = \alpha \int_0^\infty e^{-\alpha t} L^x_u(t) dt$.

Let

$$J_{\alpha,1}(x,u) = E\Big(\int_0^\infty e^{-\alpha t} C\big(X^x_u(t)\big)\, dt\Big)$$

and

$$J_{\alpha,2}(x,u) = E\Big(\int_0^\infty e^{-\alpha t} L^x_u(t) dt\Big). \tag{5.2}$$

Next we use the convexity of the reflection mapping described in (2.15) to establish the convexity of the cost functional with respect to u.

Lemma 5.1. *Let $x \ge 0$ be fixed and let C be a convex function satisfying assumption (2.4). Then, $J_{\alpha,1}(x,u)$ and $J_{\alpha,2}(x,u)$ introduced above are finite for each $u \ge 0$ and are convex in u-variable.*

Proof. By (3.2), we have the bound $E\big(L_u^x(t)\big) \le ut + K_0(1+t^H)$, where $K_0 > 0$ is a constant independent of t. This implies, by using Fubini's theorem, that $J_{\alpha,2}(x,u)$ is finite.

Next, by using (3.18) we obtain

$$\Big|J_{\alpha,1}(x,u) - E\Big(\int_0^\infty e^{-\alpha t} C\big(X_u^*(t)\big)dt\Big)\Big| \le E\Big(\int_0^\infty \big|C\big(X_u^x(t)\big) - C\big(X_u^*(t)\big)\big|dt\Big) < \infty.$$

But, using the stationary of X_u^*, we have $E\Big(\int_0^\infty e^{-\alpha t} C\big(X_u^*(t)\big)dt\Big) = \alpha^{-1} E\big(C(Z_u)\big)$, where Z_u is given in (3.4). Notice that $E\big(C(Z_u)\big)$ is finite because C has polynomial growth and in virtue of (3.7). Consequently, $J_{\alpha,1}(x,u)$ is also finite.

To establish convexity of $J_{\alpha,1}(x,u)$, first recall that $X_u^x = \Gamma\big(x + W_H - ue\big)$, where $e(t) \equiv t$ for $t \ge 0$, and Γ is the reflection mapping described in Section 2.3. Now let $u_1 \ge 0$, $u_2 \ge 0$, and $r \in (0,1)$. Then, for $t \ge 0$,

$$\begin{aligned} x + W_H(t) - &\big(ru_1 + (1-r)u_2\big)e(t) \\ &= r\big(x + W_H(t) - u_1 e(t)\big) + (1-r)\big(x + W_H(t) - u_2 e(t)\big). \end{aligned}$$

Since the reflection map Γ satisfies the convexity property (2.15), we have:

$$X_{\bar u_r}^x(t) \le r X_{u_1}^x(t) + (1-r) X_{u_2}^x(t), \tag{5.3}$$

where $\bar u_r = ru_1 + (1-r)u_2$. Next, since C is a convex non-decreasing function, (5.3) implies that $C\big(X_{\bar u_r}^x(t)\big) \le rC\big(X_{u_1}^x(t)\big) + (1-r)C\big(X_{u_2}^x(t)\big)$ for $t \ge 0$. From this it follows that

$$J_{\alpha,1}(x,\bar u_r) \le rJ_{\alpha,1}(x,u_1) + (1-r)J_{\alpha,1}(x,u_2).$$

Hence $J_{\alpha,1}(x,u)$ is convex in u-variable.

Next, by (2.2) and (2.11), we have

$$\Gamma\big(x + W_H - ue\big)(t) - (x + W_H - ue)(t) = L_u^x(t), \qquad t \ge 0.$$

Then, since Γ is convex in u-variable by (2.15), the process L_u^x is also convex in u-variable. Consequently, with $\bar u_r = ru_1 + (1-r)u_2$, we obtain

$$L_{\bar u_r}^x(t) \le rL_{u_1}^x(t) + (1-r)L_{u_2}^x(t), \qquad t \ge 0.$$

Finally, it is evident that $J_{\alpha,2}(x,u)$ is convex in u-variable from the definition (5.2). $\square$

Corollary 5.2. *Under the conditions of Lemma 5.1, the discounted cost functional $J_\alpha(x,u)$ is finite for each $u \ge 0$ and is convex in u-variable.*

Proof. Notice that

$$J_\alpha(x,u) = \frac{h(u)}{\alpha} + J_{\alpha,1}(x,u) + \alpha p J_{\alpha,2}(x,u). \tag{5.4}$$

By our assumptions, h is a convex function and $p \ge 0$ and $\alpha > 0$ are constants. Therefore, the claim follows from Lemma 5.1. $\square$

The above lemma and the corollary lead to the following result.

Theorem 5.3. *Consider the X_u^x process satisfying* (3.1) *and the associated discounted cost functional $J_\alpha(x,u)$ described in* (5.1)*. Then, for each initial point $x \geq 0$, there is an optimal control $u^* \geq 0$ such that*

$$J_\alpha(x,u^*) = \inf_{u\geq 0} J_\alpha(x,u) \equiv V_\alpha(x),$$

where $V_\alpha(x)$ is the value function of the discounted cost problem defined in (2.8)*.*

Proof. Fix any $x \geq 0$ and $\alpha > 0$. By Corollary 5.2, $J_\alpha(x,u)$ is finite for each $u \geq 0$ and is convex in u-variable. By (5.4), we have that $J_\alpha(x,u) \geq \frac{h(u)}{\alpha}$ and hence, since $\lim\limits_{u\to\infty} h(u) = +\infty$, we obtain $\lim\limits_{u\to\infty} J_\alpha(x,u) = +\infty$. Since $J_\alpha(x,u)$ is convex in u-variable, we can conclude that there is a $u^* \geq 0$ (which may depend on x) such that $J_\alpha(x,u^*) = \inf_{u\geq 0} J_\alpha(x,u)$. This completes the proof of the theorem. $\square$

Remark. *Notice that in contrast with the long-run average cost minimization problem, we cannot rule out the possibility $u^* = 0$ here.*

Corollary 5.4. *For the special case $p = 0$, assume further that $h(x)$ is constant on an interval $[0,\delta]$ for some $\delta > 0$. Then, for every initial point $x \geq 0$, the optimal control u^* is strictly positive.*

Proof. It follows from (5.4) that $J_\alpha(x,u) = J_{\alpha,1}(x,u)$. The function C is increasing and $X_{u_1}^x(t) < X_{u_2}^x(t)$ for all $t \geq 0$ and $u_1 > u_2$. Therefore $J_{\alpha,1}(x,u_1) \leq J_{\alpha,1}(x,u_2)$ for $u_1 > u_2$. Consequently, $J_\alpha(x,0) \geq J_\alpha(x,u)$ for all $u > 0$. Hence, we can find an optimal control $u^* > 0$, and the proof of the corollary is complete. $\square$

6 Finite horizon problem and Abelian limits

In this section, we establish Abelian limit relationships among the value functions of three stochastic control problems introduced in Section 2.2. The main result is stated in Theorem 6.3 below. Throughout this section, for simplicity, we assume that $\sigma(u) \equiv 1$ in the model described in (2.1).

We begin with the existence of an optimal control for the finite horizon control problem introduced in Section 2.2.

Proposition 6.1. *Let $x \geq 0$, $T > 0$, and $I(u,x,T)$ be defined by* (2.10)*. Then*

(i) $I(u,x,T)$ is finite for each $u > 0$ and is continuous in $u > 0$.

(ii) $\lim\limits_{u\to\infty} I(u,x,T) = +\infty$.

(iii) If h and C are convex functions, then $I(u,x,T)$ is a convex function of the variable u.

Corollary 6.2. *For any fixed $x \geq 0$ and $T > 0$, we have:*

(i) There is an optimal control $u^*(x,T) \geq 0$ such that $I\big(u^*(x,T)\big) = \min_{u>0} I(u,x,T)$.

(ii) In the case $p > 0$, if h and C are convex functions, then $u^*(x,T)$ is unique.

(iii) In the case $p = 0$, if h is a strictly convex function and C is a convex function, then $u^*(x,T)$ is unique.

The proofs of the above proposition and the corollary are straightforward adaptations of the corresponding proofs given in Section 3, and are therefore omitted.

The following theorem is the main result of this section.

Theorem 6.3. *Let X_u^x satisfy (3.1) and let V_0, $V_\alpha(x)$, $V(x,T)$ be the value functions defined in (2.6), (2.8), and (2.9) respectively. Then the following Abelian limit relationships hold:*

$$\lim_{\alpha\to 0^+} \alpha V_\alpha(x) = \lim_{T\to\infty} \frac{V(x,T)}{T} = V_0.$$

We prove this result in Propositions 6.5 and 6.6 below. The following technical lemma gathers necessary tools to establish $\lim_{\alpha\to 0^+} \alpha V_\alpha(x) = V_0$.

Lemma 6.4. *Let $u > 0$ be given and X_u^x satisfy (3.1). Consider the cost functional $J_\alpha(x,u)$ as defined in (2.7). Then*

$$\lim_{\alpha\to 0^+} \alpha J_\alpha(x,u) = I(u),$$

where $I(u)$ is described in (3.21).

Proof. First consider $\lim_{\alpha\to 0^+} \alpha E\big(\int_0^\infty e^{-\alpha t} dL_u^x(t)\big)$, where L_u^x is as in (2.2). Similarly to (5.1), we have

$$\alpha E\Big(\int_0^\infty e^{-\alpha t} dL_u^x(t)\Big) = \alpha^2 E\Big(\int_0^\infty e^{-\alpha t} L_u^x(t)dt\Big) = \alpha^2 \int_0^\infty e^{-\alpha t} E\big(L_u^x(t)\big)dt, \quad (6.1)$$

where we used Fubini's theorem to obtain the last identity. By (3.1),

$$E\big(L_u^x(t)\big) = ut + E\big(X_u^x(t)\big) - x.$$

Therefore,

$$\begin{aligned}\alpha^2 \int_0^\infty e^{-\alpha t} E\big(L_u^x(t)\big)dt &= \alpha^2 u \int_0^\infty e^{-\alpha t} t dt + \alpha^2 \int_0^\infty e^{-\alpha t} E\big(X_u^x(t)\big)dt - \alpha x \\ &= u + \alpha^2 \int_0^\infty e^{-\alpha t} E\big(X_u^x(t)\big)dt - \alpha x. \quad (6.2)\end{aligned}$$

Further, by (3.2), $0 \le E\big(X_u^x(t)\big) \le K_0(1+t^H)$, where the constant $K_0 > 0$ is independent of t and u. Thus

$$\begin{aligned}0 &\le \alpha^2 \int_0^\infty e^{-\alpha t} E\big(X_u^x(t)\big)dt \le K_0\alpha^2 \int_0^\infty e^{-\alpha t}(1+t^H)dt \\ &\le K_0\big(\alpha + \mathrm{Gamma}(H)\alpha^{1-H}\big), \quad (6.3)\end{aligned}$$

where $\mathrm{Gamma}(H) := \int_0^\infty e^{-t}t^H dt = \alpha^{1+H}\int_0^\infty e^{-\alpha t}t^H dt$ is the Gamma function evaluated at H.

Since $H \in (0,1)$, it follows from (6.3) that $\lim_{\alpha\to 0^+} \alpha^2 \int_0^\infty e^{-\alpha t} E\big(X_u^x(t)\big)dt = 0$. Hence, using (6.1) and (6.2), we obtain

$$\lim_{\alpha\to 0^+} \alpha E\Big(\int_0^\infty e^{-\alpha t} dL_u^x(t)\Big) = u. \tag{6.4}$$

We next consider $\lim_{\alpha\to 0^+} \alpha E\big(\int_0^\infty e^{-\alpha t} C\big(X_u^x(t)\big)dt\big)$. It follows from (3.18) that

$$\begin{aligned}
&\Big|E\Big(\int_0^\infty e^{-\alpha t} C\big(X_u^x(t)\big)dt\Big) - E\Big(\int_0^\infty e^{-\alpha t} C\big(X_u^*(t)\big)\big|dt\Big)\Big| \\
&\qquad \le E\Big(\int_0^\infty \big|C\big(X_u^x(t)\big) - C\big(X_u^*(t)\big)\big|dt\Big) < \infty,
\end{aligned}$$

where X_u^* is the stationary process described in (3.5) and (3.6). Therefore,

$$\begin{aligned}
\lim_{\alpha\to 0^+} \alpha E\big(\int_0^\infty e^{-\alpha t} C\big(X_u^x(t)\big)dt\big) &= \lim_{\alpha\to 0^+} \alpha E\big(\int_0^\infty e^{-\alpha t} C\big(X_u^*(t)\big)dt\big) \\
&= \lim_{\alpha\to 0^+} \alpha \int_0^\infty e^{-\alpha t} E\big(C(Z_u)\big)dt = E\big(C(Z_u)\big) = G(u),
\end{aligned} \tag{6.5}$$

where $G(u)$ is defined in Section 3.3. It follows from (2.7), (6.4), and (6.5) that

$$\lim_{\alpha\to 0^+} \alpha J_\alpha(x,u) = h(u) + pu + G(u) = I(u),$$

where $I(u)$ is given in (3.21). This completes the proof of the lemma. $\square$

The next proposition contains the proof of the first part of Theorem 6.3.

Proposition 6.5. *Let X_u^x satisfy* (3.1) *and $V_\alpha(x)$ be the corresponding value function defined in* (2.8). *Then*

$$\lim_{\alpha\to 0^+} \alpha V_\alpha(x) = V_0 < \infty,$$

where V_0 is the value of the long-run average cost minimization problem given in (2.6).

Proof. Fix the initial point $x \ge 0$. For any $u > 0$, we have $\alpha V_\alpha(x) \le \alpha J_\alpha(x,u)$. Hence, by Lemma 6.4, $\limsup_{\alpha\to 0^+} \alpha V_\alpha(x) \le \lim_{\alpha\to 0^+} \alpha J_\alpha(x,u) = I(u)$. Therefore, minimizing the right-hand side over $u > 0$, we obtain

$$\limsup_{\alpha\to 0^+} \alpha V_\alpha(x) \le \inf_{u>0} I(u) = V_0.$$

It remains to show the validity of the reverse inequality, namely that

$$\liminf_{\alpha\to 0^+} \alpha V_\alpha(x) \ge \inf_{u>0} I(u) = V_0. \tag{6.6}$$

To this end, consider a decreasing to zero sequence $\alpha_n > 0$, $n \in \mathbb{N}$, such that

$$\liminf_{\alpha \to 0^+} \alpha V_\alpha(x) = \lim_{n\to\infty} \alpha_n V_{\alpha_n}(x). \tag{6.7}$$

Fix any $\varepsilon_0 > 0$ and let $u_n > 0$, $n \in \mathbb{N}$, be a sequence such that $V_{\alpha_n}(x) + \varepsilon_0 > J_{\alpha_n}(x, u_n)$. Then

$$\alpha_n V_{\alpha_n}(x) + \alpha_n \varepsilon_0 > \alpha_n J_{\alpha_n}(x, u_n) \geq h(u_n). \tag{6.8}$$

Letting $n \to \infty$ we obtain

$$\limsup_{n\to\infty} h(u_n) \leq \limsup_{n\to\infty} \alpha_n V_{\alpha_n}(x) \leq V_0. \tag{6.9}$$

Since $\lim\limits_{x\to\infty} h(x) = +\infty$, this implies that u_n is a bounded sequence. That is, there is $M > 0$ such that $u_n \in (0, M)$ for all $n \in \mathbb{N}$. Therefore, without loss of generality we can assume that u_n converges as $n \to \infty$ to some $u_\infty \in [0, M]$ (otherwise, we can consider a convergent subsequence of u_n).

Let $\delta \in (u_\infty, \infty)$. Then,

$$\alpha_n J_{\alpha_n}(x, u_n) \geq h(u_n) + \alpha_n^2 p \int_0^\infty e^{-\alpha_n t} E\big(L^x_{u_n}(t)\big) dt + \alpha_n \int_0^\infty e^{-\alpha_n t} E\big[C\big(X^x_\delta(t)\big)\big] dt$$

Since $E\big(L^x_{u_n}(t)\big) = E\big(X^x_{u_n}(t)\big) + u_n t - x \geq u_n t - x$, we obtain

$$\alpha_n^2 \int_0^\infty e^{-\alpha_n t} E\big(L^x_{u_n}(t)\big) dt \geq u_n - \alpha_n x,$$

and hence

$$\alpha_n J_{\alpha_n}(x, u_n) \geq h(u_n) + p\, u_n - p \alpha_n x + \alpha_n \int_0^\infty e^{-\alpha_n t} E\big[C\big(X^x_\delta(t)\big)\big] dt.$$

Therefore, letting n go to infinity and using (6.5), we obtain

$$\liminf_{n\to\infty} \alpha_n J_{\alpha_n}(x, u_n) \geq h(u_\infty) + p\, u_\infty + E\big[C\big(Z_\delta\big)\big]. \tag{6.10}$$

In particular, one can conclude that $u_\infty > 0$, because otherwise, letting δ tend to 0 and using Lemma 3.5, we would obtain that $\liminf_{n\to\infty} \alpha_n J_{\alpha_n}(x, u_n) = +\infty$ which contradicts the fact that $\limsup\limits_{n\to\infty} \alpha_n J_{\alpha_n}(x, u_n) \leq V_0 < \infty$ according to (6.8) and (6.9).

Therefore $u_\infty > 0$. Letting δ in (6.10) tend to u_∞ and using again the continuity of $G(u) = E\big[C\big(Z_u\big)\big]$ proved in Lemma 3.5, we obtain

$$\liminf_{n\to\infty} \alpha_n J_{\alpha_n}(x, u_n) \geq h(u_\infty) + p\, u_\infty + E\big[C\big(Z_{u_\infty}\big)\big] = I(u_\infty) \geq V_0. \tag{6.11}$$

The inequalities (6.8), (6.9), and (6.11) combined together yield

$$\lim_{n\to\infty} \alpha_n J_{\alpha_n}(x, u_n) = I(u_\infty) = V_0,$$

which completes the proof of the proposition in view of (6.7). Notice that (6.11) implies that u_∞ is an optimal control for the long-run average cost control problem. $\square$

The following proposition includes the second part of Theorem 6.3.

Proposition 6.6. *Under the conditions of Theorem 6.3, we have*

$$\lim_{T\to\infty} \frac{V(x,T)}{T} = V_0.$$

Proof. It follows from (2.5) and (2.10) that for any $x \geq 0$ and $u > 0$ we have

$$\limsup_{T\to\infty} \frac{I(u,x,T)}{T} = I(u),$$

where $I(u)$ is given in (3.21). Therefore $\limsup_{T\to\infty} \frac{V(x,T)}{T} \leq \limsup_{T\to\infty} \frac{I(u,x,T)}{T} = I(u)$, and minimizing the right-hand side over $u > 0$, we obtain

$$\limsup_{T\to\infty} \frac{V(x,T)}{T} \leq \inf_{u>0} I(u) = V_0. \tag{6.12}$$

It remains to show that

$$\liminf_{T\to\infty} \frac{V(x,T)}{T} \geq V_0. \tag{6.13}$$

The proof of (6.13) given below is quite similar to that of (6.6). Consider a sequence of positive real numbers $(T_n)_{n\in\mathbb{N}}$ such that $\lim_{n\to\infty} T_n = +\infty$ and

$$\liminf_{T\to\infty} \frac{V(x,T)}{T} = \lim_{n\to\infty} \frac{V(x,T_n)}{T_n}.$$

Fix any $\varepsilon_0 > 0$ and for each $n \in \mathbb{N}$ chose $u_n > 0$ such that $V(x,T_n) + \varepsilon_0 > I(u_n,x,T_n)$. Then, in view of (2.10) and (6.12), we have

$$\limsup_{n\to\infty} h(u_n) \leq \limsup_{n\to\infty} \frac{V(x,T_n)}{T_n} \leq V_0 < +\infty.$$

Since $\lim_{x\to\infty} h(x) = +\infty$, this implies that u_n is a bounded sequence. That is, there is $M > 0$ such that $u_n \in (0,M)$ for all $n \in \mathbb{N}$. Taking a further subsequence if necessary, we can assume without loss of generality that u_n converges to some $u_\infty \in [0,M]$, as $n \to \infty$. Furthermore, $u_\infty > 0$ because if $u_\infty = 0$, then by (3.20) we obtain

$$V_0 \geq \limsup_{n\to\infty} \frac{V(x,T_n)}{T_n} \geq \limsup_{n\to\infty} \frac{I(u_n,x,T_n)}{T_n} \geq E\big(C(Z_\delta)\big) \qquad \text{for any } \delta > 0.$$

This is impossible in view of part (iii) of Lemma 3.5. Therefore $u_\infty > 0$. Let v_1, v_2 be any numbers such that $0 < v_1 < u_\infty < v_2$. Then, by (2.10) and an argument similar to the derivation of (6.10), we have

$$V(x,T_n) + \varepsilon_0 > I(u_n,x,T_n) \geq h(v_1)T_n + E\big(L^x_{v_1}(T_n)\big) + \int_0^{T_n} E\big[C\big(X^x_{v_2}(t)\big)\big]\,dt,$$

for n large enough. Using Lemma 3.1 along with (3.20), we deduce that

$$\liminf_{n\to\infty} \frac{V(x,T_n)}{T_n} \geq h(v_1) + p\, v_1 + E\big(C(Z_{v_2})\big).$$

Since the functions $h(u)$ and $G(u) = E\big(C(Z_u)\big)$ are continuous, this implies

$$V_0 \geq \liminf_{n\to\infty} \frac{V(x,T_n)}{T_n} \geq h(u_\infty) + p\, u_\infty + E\big(C(Z_{u_\infty})\big) = I(u_\infty) \geq V_0.$$

The proof of (6.13), and hence the proof of the proposition is complete. In fact, the above inequality also shows that u_∞ is an optimal control for the long-run average cost control problem. □

Propositions 6.5 and 6.6 combined together yield Theorem 6.3.

Remark 6.7. *The proofs of Propositions 6.5 and 6.6 also imply the following results:*

1. *Let $(\alpha_n)_{n\geq 0}$ be a sequence of positive numbers converging to zero and let u_n be an ε-optimal control for $V_{\alpha_n}(x)$ in* (2.8)*. Then the sequence $(u_n)_{n\geq 0}$ is bounded and any limit point of $(u_n)_{n\geq 0}$ is an optimal control for V_0 defined in* (2.6)*.*
2. *Let $(T_n)_{n\geq 0}$ be a sequence of positive numbers such that $\lim_{n\to\infty} T_n = +\infty$. If u_n be an ε-optimal control for $V(x,T_n)$ in* (2.9)*, then the sequence $(u_n)_{n\geq 0}$ is bounded and any limit point of $(u_n)_{n\geq 0}$ is an optimal control for V_0 defined in* (2.6)*.*

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