Discrete-time Ornstein-Uhlenbeck process in a stationary dynamic environment

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Abstract

We study the stationary solution to the recursion $X_{n+1} = \gamma X_n + \xi_n$, where $\gamma \in (0,1)$ is a constant and ξ_n are Gaussian variables with *random* parameters. Specifically, we assume that $\xi_n = \mu_n + \sigma_n \varepsilon_n$, where $(\varepsilon_n)_{n\in\mathbb{Z}}$ is an i.i.d. sequence of standard normal variables and $(\mu_n, \sigma_n)_{n\in\mathbb{Z}}$ is a stationary and ergodic process independent of $(\varepsilon_n)_{n\in\mathbb{Z}}$, which serves as an exogenous dynamic environment for the model. We describe basic features of the stationary solution as a mixture of Gaussian random series, its asymptotic behavior when $\gamma \to 1$, and obtain limit theorems for its extreme values and partial sums.

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1 Introduction

This paper is devoted to the study of solutions to the following linear recursion (*stochastic difference equation*):

$$X_{n+1} = \gamma X_n + \xi_n, \tag{1}$$

where $\gamma \in (0,1)$ is a constant and $(\xi_n)_{n\in\mathbb{Z}}$ is a stationary and ergodic sequence of normal variables with *random* means and variances. More precisely, we suppose that

$$\xi_n = \mu_n + \sigma_n \varepsilon_n, \qquad n \in \mathbb{Z},$$

where $(\varepsilon_n)_{n\in\mathbb{Z}}$ is an i.i.d. sequence of standard (zero mean and variance one) Gaussian random variables and $(\mu_n, \sigma_n)_{n\in\mathbb{Z}}$ is an independent stationary and ergodic process. Denote

$$\omega_n = (\mu_n, \sigma_n) \in \mathbb{R}^2, \qquad n \in \mathbb{Z}, \tag{2}$$

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and $\omega = (\omega_n)_{n\in\mathbb{Z}}$. We refer to the sequence ω as a *random dynamic environment* or simply *dynamic environment*. We denote the probability law of the random environment ω by P and denote the corresponding expectation operator by E_P. Throughout the paper we impose the following conditions on the coefficients in (1).

Assumption 1.1. *Assume that:*

(A1) The sequence of pairs $(\omega_n)_{n\in\mathbb{Z}}$ *is stationary and ergodic.*

(A2) $E_P\big(\log^+|\mu_0| + \log^+|\sigma_0|\big) < +\infty$, *where* $x^+ := \max\{x, 0\}$ *for* $x \in \mathbb{R}$.

(A3) $\gamma \in (0, 1)$ *is a constant.*

The conditions stated in Assumption 1.1 ensure the existence of a limiting distribution for X_n and, consequently, the existence of a (unique) stationary solution to (1) (see Theorem 2.2 below). This solution is very well understood in the classical case when the innovations ξ_n form an i.i.d. sequence, in which case $(X_n)_{n\geq 0}$ defined by (1) is an *AR(1) (first-order autoregressive) process.* The AR(1) process often serves to model discrete-time dynamics of both the value as well as the volatility of financial assets and interest rates, see for instance, [66, 69].

The stochastic difference equation (1) has a remarkable variety of both theoretical as well as real-world applications; see, for instance, [22, 34, 57, 73] for a comprehensive survey of the literature. In particular, the introduction section in [34] includes a long list of applications in econometrics of the model (1) with an i.i.d. Gaussian noise term ξ_n. A sequence X_n that solves (1) can be thought as the AR(1) process with *stochastic variance.* The recognition that financial time-series, such as stock returns and exchange rates, exhibit changes in volatility over time goes back at least to [23, 52]. These changes are due for example to seasonal effects, response to the news and dynamics of the market. In this context, the random environment ω can be interpreted as an exogenous factor to the model determined by the current state of the underlying economy. For a comparative review of stochastic variance models we refer the reader to [30, 67, 69].

When ω_n is a function of the state of a Markov chain, the stochastic difference equation (1) is a formal analogue of the Langevin equation with regime switches, which was studied in [21]. The notion of *regime shifts* or *regime switches* traces back to the seminal paper [32], where it was proposed in order to explain the cyclical feature of certain macroeconomic variables. Discrete-time linear recursions with Markov-dependent coefficients have been considered, for instance, in [4, 5, 14, 18, 64, 68]. Certain non-Markovian sequences of coefficients ξ_n in (1) (in particular, general martingale differences and uniformly mixing sequence) were considered, for instance, in [9, 10, 29, 36, 37, 45, 46, 51, 76].

Equation (1) with i.i.d. but not necessarily Gaussian coefficients $(\xi_n)_{n\in\mathbb{Z}}$ has been considered, for example, in [1, 9, 26, 48, 49, 50, 53, 59, 75], see also references therein. The stationary solution to (1) is often referred to as a *discrete-time (generalized) Ornstein-Uhlenbeck process.* We adopt here a similar terminology, and call the above model, *discrete-time Ornstein-Uhlenbeck process in a stationary dynamic environment.* The case when γ is close to one is often of special interest in the context of stochastic volatility models; see [69, Section 3.5]. Such *nearly unstable* processes have been considered, for instance, in [7, 10, 16, 38, 40, 41, 45, 46, 48, 49, 54]. Remarkably, much of the work for nearly stable

$AR(1)$ processes focuses on the weak convergence of the process and its least-squares estimates, and is done in a general setting where the innovations ξ_n are martingale differences rather than i.i.d. variables. In particular, an interesting case of fractionally integrated innovations ξ_n is discussed in [7, 41] (for a survey on fractionally integrated models see, for instance, review articles [3, 61] and the monograph collection of papers [62]).

In this work we study the probabilistic structure of the (unique) stationary solution to (1) under the general Assumption 1.1. To enable some explicit computation, in a few illustrative examples in this paper we will consider the following setup.

Assumption 1.2. *Let $(y_n)_{n\in\mathbb{Z}}$ be an irreducible Markov chain defined on a finite state space $\mathcal{D} = \{1, \ldots, d\}$, $d \in \mathbb{N}$, and suppose that the sequence $(\xi_n)_{n\in\mathbb{Z}}$ is induced (modulated) by $(y_n)_{n\in\mathbb{Z}}$ as follows. Assume that for each $i \in \mathcal{D}$ there exists an i.i.d. sequence of pairs of reals $\omega_i = (\mu_{i,n}, \sigma_{i,n})_{n\in\mathbb{Z}}$ and that these sequences are independent of each other. Further, suppose that (A2) of Assumption 1.1 holds for each $i \in \mathcal{D}$, with (μ_0, σ_0) replaced by $(\mu_{i,0}, \sigma_{i,0})$. Finally, define $\mu_n = \mu_{y_n,n}$ and $\sigma_n = \sigma_{y_n,n}$.*

In a related context of linear regression models, it is remarked in [2] that "while the assumption of i.i.d. errors is convenient from the mathematical point of view, it is typically violated in regressions involving econometric variables". Testing a null hypothesis of a usual $AR(p)$ model versus a Markov switching framework is discussed, for instance, in [13, 35], using in particular classical examples of [32] and [31, 56] modeling, respectively, the postwar U. S. GNP growth rate and cartel market strategies. An application of a general "unit root versus strongly mixing innovations" statistical test to the model (1) is discussed in Section 3 of the classical reference [55].

We remark that though the Markovian setup of Assumption 1.2 is tractable analytically and thus is a natural starting point, "nothing in the approach ... precludes looking at more general probabilistic specifications" [33]. For instance, the Markov dynamics seems clearly inadequate for modeling socioeconomic factors involved in financial applications of regime-switching autoregressive models. In fact, while early regime-switching models assumed, in order to maintain the tractability of the theoretical framework, that the underlying Markov chain is stationary and the number of states is small (see, for instance, [21, 42, 43]), it has been proposed in more recent work to consider Markov models with a large number of highly connected states and to use a-prior Bayesian information (see, for instance, [8, 71]). Alternatively, one can replace the Markovian dynamics with that of *full shifts of finite type/chains of infinite order/chains with complete connections*, which are processes with long-range dependence (infinite, though a fading memory) preserving many key features of irreducible finite-state Markov chains [15, 24, 25, 27, 39, 60, 74]. Clearly, even the martingale difference setup considered in this paper (which is, for instance, more general than that of [34] and less general than that of [10]) does not cover the whole range of possible applications. According to [7], "in practice, econometric and financial time series often exhibit long-range dependent structure (see, e.g., [62, 63] and [17]) which cannot be encompassed by the martingale difference setting of [10]."

The rest of the paper is organized as follows. Section 2 is devoted to the study of the (Gaussian) sequence $(X_n)_{n\in\mathbb{N}}$ in a fixed environment. In Section 3 we study the asymptotic behavior of the limiting distribution of X_n, as $\gamma \to 1^-$. Section 4 contains a limit theorem for the extreme values $M_n = \max_{1\le k\le n} X_k$ and a related law of the iterated logarithm. In Section 5 we investigate the asymptotic behavior of the partial sums $S_n = \sum_{k=1}^n X_k$.

2 Stationary distribution of the sequence $(X_n)_{n\in\mathbb{N}}$

We denote the conditional law of $(\xi_n)_{n\in\mathbb{Z}}$, given an environment ω, by P_ω and the corresponding expectation by E_ω. To emphasize the existence of two levels of randomness in the model, the first one due to the random environment and the second one due to the randomness of $(\varepsilon_n)_{n\in\mathbb{N}}$, we will use the notations $\mathbb{P}$ and $\mathbb{E}$ for the unconditional distribution of $(\xi_n)_{n\in\mathbb{Z}}$ (and $(X_n)_{n\in\mathbb{Z}}$) and the corresponding expectation operator, respectively. We thus have

$$\mathbb{P}(\,\cdot\,) = \int P_\omega(\,\cdot\,)dP(\omega) = E_P\big[P_\omega(\,\cdot\,)\big]. \tag{3}$$

For any constants $\mu \in \mathbb{R}$ and $\sigma > 0$, we denote by Φ_{μ,σ^2} the distribution function of a normal random variable with mean μ and variance σ^2. That is, for $t \in \mathbb{R}$,

$$G_{\mu,\sigma^2}(t) := 1 - \Phi_{\mu,\sigma^2}(t) = \frac{1}{\sqrt{2\pi\sigma^2}} \int_t^\infty e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx. \tag{4}$$

It will be notationally convenient to extend the notion of "normal variables" to a class of distributions with random parameters μ and σ.

Definition 2.1. Let (μ,σ) be a random $\mathbb{R}^2$-valued vector with $\mathbb{P}(\sigma > 0) = 1$. We say that a random variable X has $\mathcal{N}(\mu,\sigma^2)$-distribution (in words, *normal* (μ,σ^2) *distribution*) and write $X \sim \mathcal{N}(\mu,\sigma^2)$ if

$$\mathbb{P}(X \le t) = \mathbb{E}\big[\Phi_{\mu,\sigma^2}(t)\big], \qquad t \in \mathbb{R}.$$

That is, conditional on the pair (μ,σ^2), the distribution of X is normal with mean μ and variance σ^2.

2.1 Limiting distribution of X_n

First we discuss the (marginal) distribution of an individual member of the sequence X_n. It follows from (1) that for $n \in \mathbb{N}$ we have

$$X_n = \gamma^n X_0 + \sum_{t=0}^{n-1} \gamma^{n-t-1}\xi_t. \tag{5}$$

The following general result can be deduced from (5) (see for instance [6]):

Proposition 2.2. *Assume that*

(i) $(\xi_n)_{n\in\mathbb{Z}}$ *is a stationary and ergodic sequence.*

(ii) $\mathbb{E}\big[\log^+|\xi_0|\big] < +\infty$, *where* $x^+ := \max\{x,0\}$ *for* $x \in \mathbb{R}$.

(iii) $\gamma \in (0,1)$ *is a constant.*

Then, for any initial value X_0*, the series* X_n *defined by* (1) *converges in distribution, as* $n \to \infty$*, to the random variable*

$$X = \sum_{t=0}^{\infty} \gamma^t \xi_{-t}, \tag{6}$$

which is the unique initial value making $(X_n)_{n\geq 0}$ *a stationary sequence. Furthermore, the series on the right-hand side of* (6) *converges absolutely with probability one.*

It follows from (5) and (6) that the stationary solution to (1) is given by

$$X_n = \sum_{k=0}^{\infty} \gamma^k \xi_{n-k} = \gamma^n \sum_{j=-\infty}^{n} \gamma^{-j} \xi_j, \qquad n \geq 0. \tag{7}$$

Our first result is a characterization of X (and hence of X_n in (7)) as a mixture of Gaussian random variables under Assumption 1.1. Let

$$\theta = \sum_{k=0}^{\infty} \gamma^k \mu_{-k} \qquad \text{and} \qquad \tau = \Big(\sum_{k=0}^{\infty} \gamma^{2k} \sigma_{-k}^2 \Big)^{1/2}. \tag{8}$$

Notice that by Proposition 2.2, the random variables θ and τ are well-defined functions of the environment. Recall Definition 2.1.

We have:

Theorem 2.3. *Let Assumption 1.1 hold and let* X *be defined by* (6)*. Then* $X \sim \mathcal{N}(\theta, \tau^2)$*.*

Proof. It follows from (5) that if $X_0 = 0$ then

$$X_n = \sum_{t=0}^{n-1} \gamma^{n-t-1} \xi_t, \qquad n \in \mathbb{N}. \tag{9}$$

In particular, X_n are Gaussian under the conditional law P_ω. By Proposition 2.2, it suffices to show that

$$\lim_{n\to\infty} \mathbb{E}\big[e^{itX_n} \big| X_0 = 0\big] = E_P\Big[e^{i\theta t - \frac{\tau^2 t^2}{2}}\Big], \qquad t \in \mathbb{R}.$$

Since the environment ω is a stationary sequence, (9) implies that $\mathbb{E}\big[e^{itX_n} \big| X_0 = 0\big] = \mathbb{E}\big[e^{itY_n}\big]$ where $Y_n \sim \mathcal{N}(\theta_n, \tau_n^2)$ with τ_n and θ_n given by

$$\theta_n = \sum_{k=0}^{n-1} \gamma^k \mu_{-k} \qquad \text{and} \qquad \tau_n = \Big(\sum_{k=0}^{n-1} \gamma^{2k} \sigma_{-k}^2 \Big)^{1/2}.$$

It follows from (8) and Proposition 2.2 that

$$\lim_{n\to\infty} \theta_n = \theta \qquad \text{and} \qquad \lim_{n\to\infty} \tau_n = \tau, \quad P - \text{a.s.}$$

Hence

$$\begin{aligned}\lim_{n\to\infty}\mathbb{E}\big[e^{itX_n}\big] &= \lim_{n\to\infty}\mathbb{E}\big[e^{itY_n}\big] = \lim_{n\to\infty}E_P\Big[E_\omega\big[e^{itY_n}\big]\Big] \\ &= \lim_{n\to\infty}E_P\Big[e^{i\theta_n t-\frac{\tau_n^2t^2}{2}}\Big] = E_P\Big[\lim_{n\to\infty}e^{i\theta_n t-\frac{\tau_n^2t^2}{2}}\Big] = E_P\Big[e^{i\theta t-\frac{\tau^2t^2}{2}}\Big].\end{aligned}$$

To justify interchanging of the limit and expectation operator in the last but one step, observe that $\big|e^{i\theta_n t-\tau_n^2t^2}\big| \le 1$, and therefore the bounded convergence theorem can be applied. $\square$

Corollary 2.4. *Let Assumption 1.1 hold. Then the distribution of X is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}$.*

Proof. For an event B in the underlying probability space, let $\mathbf{1}_B$ denote the indicator function of B. By Fubini's theorem, for any Borel set $A\subset\mathbb{R}$,

$$\begin{aligned}\mathbb{P}(X\in A)=\mathbb{E}[\mathbf{1}_{\{\mathcal{N}(\theta,\tau^2)\in A\}}] &= \int\bigg(\frac{1}{\sqrt{2\pi\tau^2}}\int_A e^{-\frac{(x-\theta)^2}{2\tau^2}}dx\bigg)\,dP(\omega) \\ &= \int_A\bigg(\int\frac{1}{\sqrt{2\pi\tau^2}}e^{-\frac{(x-\theta)^2}{2\tau^2}}dP(\omega)\bigg)\,dx,\end{aligned}$$

and, furthermore, the integral $\int\frac{1}{\sqrt{2\pi\tau^2}}e^{-\frac{(x-\theta)^2}{2\tau^2}}P(d\omega)$ exists for $m-$ a. e. $x\in\mathbb{R}$, where m denotes the Lebesgue measure of the Borel subsets of $\mathbb{R}$. $\square$

Corollary 2.5. *Let Assumption 1.1 hold. Then $\mathbb{E}\big[|X-\theta|^p\big]=\frac{2^{\frac{p}{2}}\Gamma\left(\frac{p+1}{2}\right)}{\sqrt{\pi}}\cdot E_P[\tau^p]$ for any constant $p>-1$.*

2.2 Distribution tails of X

The next theorem shows that under a mild extra assumption, the tails of the distribution of X have asymptotically a Gaussian structure. For a random variable Y denote

$$\|Y\|_p=\Big(\mathbb{E}\big[|Y|^p\big]\Big)^{1/p}\quad(p\ge 1)\quad\text{and}\quad\|Y\|_\infty=\inf\big\{y\in\mathbb{R}:\mathbb{P}\big(|Y|>y\big)=0\big\}.$$

Recall (see, for instance, [20, p. 466]) that $\|Y\|_\infty=\lim_{p\to\infty}\|Y\|_p$. Notice that (8) implies $\|\tau\|_\infty^2\le(1-\gamma^2)^{-1}\cdot\|\sigma_0\|_\infty^2$.

We have:

Theorem 2.6. *Let Assumption 1.1 hold and assume in addition that $P\big(|\mu_0|+\sigma_0<\lambda\big)=1$ for some constant $\lambda>0$. Then*

$$\lim_{t\to\infty}\frac{1}{t^2}\log\mathbb{P}(X>t)=\lim_{t\to\infty}\frac{1}{t^2}\log\mathbb{P}(X<-t)=-\frac{1-\gamma^2}{2\Lambda^2},$$

where $\Lambda:=\|\tau\|_\infty\in(0,\infty)$.

Proof. The proof relies on some well-known bounds for the tails of normal distributions. For the reader's convenience we will give a short derivation of these bounds here. We will only consider the upper tails $\mathbb{P}(X > t)$. The lower tails $\mathbb{P}(X < -t)$ can be treated in exactly the same manner, and therefore the proof for the lower tails is omitted.

Recall $G_{\mu,\sigma^2}(t)$ from (4). On one hand, we have:

$$\begin{aligned} G_{0,\sigma^2}(t) &= \frac{1}{\sqrt{2\pi\sigma^2}} \int_t^\infty e^{-\frac{x^2}{2\sigma^2}} dx \le \frac{1}{\sqrt{2\pi\sigma^2}} \int_t^\infty \frac{x}{t} e^{-\frac{x^2}{2\sigma^2}} dx = \frac{1}{2\sqrt{2\pi\sigma^2 t^2}} \int_{t^2}^\infty e^{-\frac{y}{2\sigma^2}} dy \\ &= \frac{\sigma^2}{\sqrt{2\pi\sigma^2 t^2}} \int_{t^2}^\infty \frac{1}{2\sigma^2} e^{-\frac{y}{2\sigma^2}} dy = \sqrt{\frac{\sigma^2}{2\pi t^2}} e^{-\frac{t^2}{2\sigma^2}}. \end{aligned} \tag{10}$$

On the other hand, denoting $t_\sigma = t/\sigma$ and using l'Hôspital's rule,

$$\lim_{t\to\infty} \frac{G_{0,\sigma^2}(t)}{\sqrt{\frac{\sigma^2}{2\pi t^2}} e^{-\frac{t^2}{2\sigma^2}}} = \lim_{t_\sigma\to\infty} \frac{\int_{t_\sigma}^\infty e^{-\frac{x^2}{2}} dx}{t_\sigma^{-1} e^{-\frac{t_\sigma^2}{2}}} = 1.$$

Recall that by our assumptions $\mu_0 > -\lambda$ and hence $\theta > -\lambda(1-\gamma)^{-1}$. Therefore, there exists $t_0 > 0$ such that if $t > \lambda t_0$, we have

$$\begin{aligned} G_{\theta,\tau^2}(t) &= G_{0,\tau^2}(t-\theta) \ge G_{0,\tau^2}\big(t + \lambda(1-\gamma)^{-1}\big) \\ &\ge \frac{1}{2}\sqrt{\frac{\tau^2}{2\pi t^2}} e^{-\frac{(t+\lambda(1-\gamma)^{-1})^2}{2\tau^2}} = \sqrt{\frac{\tau^2}{8\pi t^2}} e^{-\frac{(t+\lambda(1-\gamma)^{-1})^2}{2\tau^2}}. \end{aligned} \tag{11}$$

By Theorem 2.3, $\mathbb{P}(X > t) = E_P\big[P_\omega(X > t)\big] = E_P\big[G_{\theta,\tau^2}(t)\big]$. To get the upper bound, observe that $\mu_0 < \lambda$ and hence $\theta < \lambda(1-\gamma)^{-1}$. Thus

$$\begin{aligned} E_P\big[G_{\theta,\tau^2}(t)\big] &= E_P\big[G_{0,\tau^2}(t-\theta)\big] \le E_P\big[G_{0,\tau^2}\big(t - \lambda(1-\gamma)^{-1}\big)\big] \\ &\le E_P\Big[\sqrt{\frac{\tau^2}{2\pi t^2}} e^{-\frac{(t-\lambda(1-\gamma)^{-1})^2}{2\tau^2}}\Big] \le \sqrt{\frac{\tau^2}{2\pi t^2}} E_P\big[e^{-\frac{(t-\lambda(1-\gamma)^{-1})^2}{2\tau^2}}\big]. \end{aligned}$$

Therefore,

$$\begin{aligned} \lim_{t\to\infty} \frac{1}{t^2} \log \mathbb{P}(X > t) &\le \lim_{t\to\infty} \frac{1}{t^2} \log\Big(\big\|e^{-\frac{1}{2\tau^2}}\big\|_{t^2}\Big)^{t^2} = \log\Big(\big\|e^{-\frac{1}{2\tau^2}}\big\|_\infty\Big) \\ &= \log\big(e^{-\frac{1}{2\|\tau\|_\infty^2}}\big) = -\frac{1}{2\|\tau\|_\infty^2}. \end{aligned}$$

For the lower bound, we first observe that, in view of (11), we have for $t > \lambda t_0$,

$$E_P\big[G_{\theta,\tau^2}(t)\big] \ge E_P\Big[\sqrt{\frac{\tau^2}{8\pi t^2}} e^{-\frac{t^2}{2\tau^2}}\Big].$$

Now, let $\varepsilon > 0$ be any positive real number such that $P(\tau > \varepsilon) > 0$. Then

$$E_P\big[G_{\theta,\tau^2}(t)\big] \ge E_P\Big[\sqrt{\frac{\tau^2}{8\pi t^2}} e^{-\frac{t^2}{2\tau^2}} \cdot \mathbf{1}_{\{\tau>\varepsilon\}}\Big] \ge \sqrt{\frac{\varepsilon^2}{8\pi t^2}} E_P\Big[e^{-\frac{t^2}{2\tau^2}} \cdot \mathbf{1}_{\{\tau>\varepsilon\}}\Big],$$

which implies

$$\begin{aligned}
\lim_{t\to\infty}\frac{1}{t^2}\log\mathbb{P}(X>t) &\geq \lim_{t\to\infty}\frac{1}{t^2}\log\Big(\big\|e^{-\frac{1}{2\tau^2}}\cdot\mathbf{1}_{\{\tau>\varepsilon\}}\big\|_{t^2}\Big)^{t^2} = \log\Big(\big\|e^{-\frac{1}{2\tau^2}}\cdot\mathbf{1}_{\{\tau>\varepsilon\}}\big\|_{\infty}\Big) \\
&= \log\big(e^{-\frac{1}{2\|\tau\|_\infty^2}}\big) = -\frac{1}{2\|\tau\|_\infty^2}.
\end{aligned}$$

This completes the proof of the theorem. □

Next, we give a simple example of the situation when the distribution of τ can be explicitly computed, and the tails of X do not have the Gaussian asymptotic structure.

Example 2.7. *Let Assumption 1.2 hold and suppose that $P(\mu_0=0)=1$, $\mathcal{D}=\{1,2\}$, and the transition matrix of the Markov chain y_n is*

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Thus two states of the underlying Markov chain alternate in the deterministic manner, i.e. $P(y_{n+1}=3-y|y_n=y)=1$ for $y\in\mathcal{D}$. Further, assume that $\sigma_{1,n}^2$ and $\sigma_{2,n}^2$ have strictly asymmetric α-stable distributions with index $\alpha\in(0,1)$ and "Laplace transform" given by

$$E_P\big[e^{-\lambda\sigma_{i,n}^2}\big]=e^{-\theta_i\lambda^\alpha},\qquad \lambda>0,\ i=1,2,$$

for some positive constants θ_i, $\theta_1\neq\theta_2$. In notation of [65], these distributions belong to the class $S_\alpha(\theta,1,0)$ (see Section 1.1 and also Propositions 1.2.11 and 1.2.12 in [65]). The stationary distribution of the underlying Markov chain is uniform on $\mathcal{D}$, and therefore for the Laplace transform of the limiting variance τ^2 introduced in (8) *we have for any $\lambda>0$,*

$$\begin{aligned}
\mathbb{E}\big[e^{-\lambda\tau^2}\big] &= \frac{1}{2}\prod_{k=0}^{\infty}\mathbb{E}\Big[e^{-\lambda\gamma^{4k}\sigma_{1,0}^2}\Big]\cdot\prod_{k=0}^{\infty}\mathbb{E}\Big[e^{-\lambda\gamma^{4k+2}\sigma_{2,0}^2}\Big] \\
&\quad+\frac{1}{2}\prod_{k=0}^{\infty}\mathbb{E}\Big[e^{-\lambda\gamma^{4k}\sigma_{2,0}^2}\Big]\cdot\prod_{k=0}^{\infty}\mathbb{E}\Big[e^{-\lambda\gamma^{4k+2}\sigma_{1,0}^2}\Big] \\
&= \frac{1}{2}\prod_{k=0}^{\infty}e^{-\theta_1\lambda^\alpha\gamma^{4k\alpha}}\cdot e^{-\theta_2\lambda^\alpha\gamma^{(4k+2)\alpha}}+\frac{1}{2}\prod_{k=0}^{\infty}e^{-\theta_2\lambda^\alpha\gamma^{4k\alpha}}\cdot e^{-\theta_1\lambda^\alpha\gamma^{(4k+2)\alpha}} \\
&= \frac{1}{2}e^{-\frac{\lambda^\alpha(\theta_1+\theta_2\gamma^{2\alpha})}{1-\gamma^{4\alpha}}}+\frac{1}{2}e^{-\frac{\lambda^\alpha(\theta_2+\theta_1\gamma^{2\alpha})}{1-\gamma^{4\alpha}}}.
\end{aligned}$$

Therefore, Theorem 2.3 yields for $t\in\mathbb{R}$,

$$\mathbb{E}\big[e^{itX}\big] = \mathbb{E}\Big[e^{-\frac{\tau^2t^2}{2}}\Big]=\frac{1}{2}e^{-\frac{|t|^{2\alpha}(\theta_1+\theta_2\gamma^{2\alpha})}{2^\alpha(1-\gamma^{4\alpha})}}+\frac{1}{2}e^{-\frac{|t|^{2\alpha}(\theta_2+\theta_1\gamma^{2\alpha})}{2^\alpha(1-\gamma^{4\alpha})}}.$$

Thus X is a mixture of two symmetric (2α)-stable distributions. In particular, in contrast to the result obtained under the conditions of Theorem 2.6, X has power tails. Namely (see Property 1.2.15 on p. 16 of [65]) the following limits exist, are equivalent, and are both finite and strictly positive: $\lim_{t\to\infty}t^{2\alpha}\cdot\mathbb{P}(X>t)=\lim_{t\to\infty}t^{2\alpha}\cdot\mathbb{P}(X<-t)\in(0,\infty)$.

2.3 Covariance structure of $(X_n)_{n\in\mathbb{Z}}$ in a fixed environment

This section is devoted to a characterization in a given environment of the Gaussian structure of the stationary solution $(X_n)_{n\in\mathbb{Z}}$ to (1). Using (7), we obtain for any sequence of real constants $\mathbf{c} = (c_n)_{n\in\mathbb{Z}}$:

$$\sum_{k=0}^{n} c_k X_k \;=\; \sum_{k=0}^{n} c_k \sum_{j=-\infty}^{k} \gamma^{k-j}\xi_j = \sum_{j=-\infty}^{0} \xi_j \sum_{k=0}^{n} c_k\gamma^{k-j} + \sum_{j=1}^{n} \xi_j \sum_{k=j}^{n} c_k\gamma^{k-j},$$

where we used the absolute convergence of the series to interchange the summation signs. Therefore, under the measure P_ω, that is in a given environment ω,

$$\sum_{k=0}^{n} c_k X_k \sim \mathcal{N}(\chi_{\mathbf{c},n}, \eta^2_{\mathbf{c},n}),$$

where

$$\chi^2_{\mathbf{c},n} = \sum_{j=-\infty}^{0} \mu_j \sum_{k=0}^{n} c_k\gamma^{k-j} + \sum_{j=1}^{n} \mu_j \sum_{k=j}^{n} c_k\gamma^{k-j}.$$

and

$$\eta^2_{\mathbf{c},n} = \sum_{j=-\infty}^{0} \sigma_j^2\Big(\sum_{k=0}^{n} c_k\gamma^{k-j}\Big)^2 + \sum_{j=1}^{n} \sigma_j^2\Big(\sum_{k=j}^{n} c_k\gamma^{k-j}\Big)^2.$$

This shows that under P_ω, the process $(X_n)_{n\geq 0}$ is Gaussian. Note that Theorem 2.2 ensures the almost sure convergence of the infinite series in the formulas above.

The following corollary is immediate from Theorem 2.3.

Corollary 2.8. *Let Assumption 1.1 hold. Then, provided that the moments on the right-hand side exist, we have the following identities:*

(i) $\mathbb{E}[X] = \dfrac{E_P[\mu_0]}{1-\gamma}$.

(ii) $\mathrm{VAR}_{\mathbb{P}}(X) = \dfrac{E_P[\sigma_0^2]}{1-\gamma^2} + \mathrm{VAR}_P(\theta)$.

Proof. It follows from Theorem 2.3 that

$$m_X : \;=\; \mathbb{E}[X] = E_P\big[E_\omega\big[\mathcal{N}(\theta,\tau^2)\big]\big] = E_P[\theta] = \frac{E_P[\mu_0]}{1-\gamma}$$

and

$$\begin{aligned}\mathrm{VAR}_{\mathbb{P}}(X) \;&=\; E_P\big[E_\omega\big[X^2 - m_X^2\big]\big] = E_P[\tau^2+\theta^2] - m_X^2\\ &=\; E_P[\tau^2] + \mathrm{VAR}_P(\theta) = \frac{E_P[\sigma_0^2]}{1-\gamma^2} + \mathrm{VAR}_P(\theta),\end{aligned}$$

where we used the fact $m_X = E_P[\theta]$, and therefore $\mathrm{VAR}_P(\theta) = E_P[\theta^2] - m_X^2$. $\square$

In the case of a Markovian environment, $\mathrm{VAR}_P(\theta)$ can be expressed in terms of certain explicit transformations of the transition kernel of the underlying Markov chain. In the following lemma we compute $\mathrm{VAR}_P(\theta)$ under Assumption 1.2. To state the result we first need to introduce some notation. Denote by H the transition matrix of the underlying Markov chain, that is

$$H(i,j) = P(y_1 = j | y_0 = i), \qquad i,j \in \mathcal{D}.$$

Denote by $\pi = \big(\pi_1, \ldots, \pi_d\big)$ the stationary distribution of $(y_n)_{n\in\mathbb{Z}}$. Let $\bar{\mu}_i = E_P[\mu_{i,0}]$ and $a = \sum_{i=1}^d \pi_i \bar{\mu}_i = E_P[\mu_0]$. Let $\mathfrak{m}_2$ denote the d-dimensional vector whose i-th component is $\pi_i \bar{\mu}_i^2$ and introduce a $d \times d$ matrix K_γ by setting

$$K_\gamma(i,j) = \frac{\gamma}{\bar{\mu}_i} \cdot H(i,j) \cdot \bar{\mu}_j, \qquad i,j = 1, \ldots, d.$$

We have:

Lemma 2.9. *Let Assumption 1.2 hold. Then*

$$\mathrm{VAR}_P(\theta) = \frac{\mathrm{VAR}_P(\mu_0)}{1-\gamma^2} + \frac{2\gamma}{1-\gamma^2} \cdot \langle \mathfrak{m}_2, (I - K_\gamma)^{-1} \mathbf{1} \rangle - \frac{2\gamma a^2}{(1-\gamma^2)(1-\gamma)},$$

where $\langle \mathbf{x}, \mathbf{y} \rangle$ stands for the usual scalar product of two d-vectors $\mathbf{x}$ and $\mathbf{y}$.

Proof. For $n \in \mathbb{Z}$, let $\nu_n = \mu_{-n} - a$ and

$$\rho_n := E_P[\nu_i \nu_{n+i}] = \mathrm{COV}_P(\mu_{-i}, \mu_{-i-n}).$$

Then, according to (8),

$$\begin{aligned}
\mathrm{VAR}_P(\theta) &= E_P\Big[\Big(\sum_{n=0}^\infty \gamma^n \nu_n\Big)^2\Big] = E_P\Big[\sum_{n=0}^\infty \gamma^{2n} \nu_n^2\Big] + 2\sum_{n=0}^\infty \gamma^n \sum_{k=n+1}^\infty \gamma^k \rho_{k-n} \\
&= \frac{\mathrm{VAR}_P(\mu_0)}{1-\gamma^2} + 2\sum_{n=0}^\infty \gamma^n \sum_{m=1}^\infty \gamma^{n+m} \rho_m = \frac{\mathrm{VAR}_P(\mu_0)}{1-\gamma^2} + \frac{2}{1-\gamma^2} \cdot \sum_{m=1}^\infty \gamma^m \rho_m.
\end{aligned}$$

It remains to compute ρ_n for $n \geq 1$. We have

$$\begin{aligned}
\rho_n &= \sum_{i=1}^d \sum_{j=1}^d \pi_i H^{n-1}(i,j) E_P\Big[(\bar{\mu}_i - a)(\bar{\mu}_j - a)\Big] \\
&= \sum_{i=1}^d \sum_{j=1}^d \pi_i H^{n-1}(i,j) E_P\Big[(\bar{\mu}_i \bar{\mu}_j - a\bar{\mu}_i - a\bar{\mu}_j + a^2)\Big] \\
&= \sum_{i=1}^d \sum_{j=1}^d \pi_i H^{n-1}(i,j) E_P\big[\bar{\mu}_i \bar{\mu}_j - a^2\big] = E_P\Big[\sum_{i=1}^d \sum_{j=1}^d \pi_i \bar{\mu}_i H^{n-1}(i,j) \bar{\mu}_j\Big] - a^2.
\end{aligned}$$

Define the following *Doob transform* of matrix H :

$$K(i,j) = \frac{1}{\bar{\mu}_i} H(i,j) \bar{\mu}_j, \qquad i,j = 1, \ldots, d.$$

Then, a routine induction argument shows that for any $n \in \mathbb{N}$, $K^n(i,j) = \frac{1}{\bar{\mu}_i} H^n(i,j)\bar{\mu}_j$. Using this formula, we obtain

$$\rho_n = E_P\Big[\sum_{i=1}^d \sum_{j=1}^d \pi_i \bar{\mu}_i^2 K^{n-1}(i,j)\Big] - a^2 = \langle \mathfrak{m}_2, K^{n-1}\mathbf{1}\rangle - a^2,$$

and hence

$$\begin{aligned} \mathrm{VAR}_P(\theta) &= \frac{\mathrm{VAR}_P(\mu_0)}{1-\gamma^2} + \frac{2}{1-\gamma^2} \cdot \sum_{n=1}^\infty \gamma^n \big(\langle \mathfrak{m}_2, K^{n-1}\mathbf{1}\rangle - a^2\big) \\ &= \frac{\mathrm{VAR}_P(\mu_0)}{1-\gamma^2} + \frac{2\gamma}{1-\gamma^2} \cdot \langle \mathfrak{m}_2, (I-K_\gamma)^{-1}\mathbf{1}\rangle - \frac{2\gamma a^2}{(1-\gamma^2)(1-\gamma)}. \end{aligned}$$

The proof of the lemma is completed. □

Remark 2.10. *It is not hard to verify that with an appropriate modification of the definition of the Doob transform K_γ (as a positive integral kernel rather than a d-matrix), the statement of Lemma 2.9 remains true for a general, non-necessarily restricted to a finite-state, Markovian setup.*

In the remainder of this section we assume, for simplicity, that $P(\mu_n = 0) = 1$. The distribution of a mean-zero Gaussian sequence is entirely determined by its covariance structure. It follows from (5) that

$$X_{k+n} = \gamma^n X_k + \sum_{t=0}^{n-1} \gamma^t \xi_{n+k-t-1}, \qquad k \in \mathbb{Z},\, n \in \mathbb{N}.$$

Therefore, for any $k \in \mathbb{Z}$ and $n \in \mathbb{N}$, we have

$$\mathrm{COV}_\omega(X_k X_{k+n}) = E_\omega[X_k X_{k+n}] = \gamma^n E_\omega[X_k^2]. \tag{12}$$

In particular, random variables X_n and X_m are positively correlated for any $n, m \in \mathbb{Z}$.

3 Asymptotic behavior of X when $\gamma \to 1^-$.

To emphasize the dependence of the stationary solution to (1) on γ, throughout this section we use the notation X_γ for X and τ_γ^2 for the limiting variance τ^2, which is defined in (8). To illustrate the main result of this section, consider first the case when the coefficients ξ_n in (1) are independent and distributed according to $\mathcal{N}(0,\sigma^2)$ for some constant $\sigma > 0$. Then $X_\gamma \sim \frac{1}{\sqrt{1-\gamma^2}} \mathcal{N}(0,\sigma^2)$, and hence

$$\sqrt{1-\gamma} \cdot X_\gamma \overset{\mathbb{P}}{\Longrightarrow} \frac{1}{\sqrt{2}} \mathcal{N}(0,\sigma^2) \qquad \text{as } \gamma \to 1^-,$$

We next show that in a certain sense $(1-\gamma)^{-1/2}$ is always the proper scaling factor for the distribution of X_γ when $\gamma \to 1^-$.

Theorem 3.1. *Let Assumption 1.1 hold. Suppose, in addition, that* $P(\mu_0 = 0) = 1$, $E_P[\sigma_0^2] < \infty$, *and* $P(\sigma_0 > \delta) = 1$ *for some positive constant* $\delta > 0$. *Then*

$$\frac{\log |X_\gamma|}{\log(1-\gamma)} \xrightarrow{\mathbb{P}} -\frac{1}{2} \quad \text{as} \quad \gamma \to 1^-,$$

where $\xrightarrow{\mathbb{P}}$ *means convergence in probability under the law* $\mathbb{P}$.

Proof. We must prove that for any $\varepsilon > 0$,

$$\mathbb{P}\Big(\Big|\frac{\log |X_\gamma|}{\log(1-\gamma)} + \frac{1}{2}\Big| > \varepsilon\Big) \to_{\gamma\to 1^-} 0. \tag{13}$$

This is equivalent to the following two claims:

$$\mathbb{P}\Big(\frac{\log |X_\gamma|}{\log(1-\gamma)} > -\frac{1}{2} + \varepsilon\Big) \to_{\gamma\to 1^-} 0 \quad \text{and} \quad \mathbb{P}\Big(\frac{\log |X_\gamma|}{\log(1-\gamma)} < -\frac{1}{2} - \varepsilon\Big) \to_{\gamma\to 1^-} 0.$$

Since $\log(1-\gamma) < 0$, it suffices to show that

$$\mathbb{P}\Big(|X_\gamma| > (1-\gamma)^{-\frac{1}{2}-\varepsilon}\Big) \to_{\gamma\to 1^-} 0 \quad \text{and} \quad \mathbb{P}\Big(|X_\gamma| < (1-\gamma)^{-\frac{1}{2}+\varepsilon}\Big) \to_{\gamma\to 1^-} 0. \tag{14}$$

Toward this end, observe first that for any constant $\varepsilon > 0$,

$$\begin{aligned}\limsup_{\gamma\to 1^-} \mathbb{P}\Big(|X_\gamma| > (1-\gamma)^{-\frac{1}{2}-\varepsilon}\Big) &\le \limsup_{\gamma\to 1^-}\big\{(1-\gamma)^{\frac{1}{2}+\varepsilon}\cdot \mathbb{E}\big[|X_\gamma|\big]\big\}\\ &= \limsup_{\gamma\to 1^-}\big\{(1-\gamma)^{\frac{1}{2}+\varepsilon}\cdot E_P[2\tau_\gamma]\big\} = \limsup_{\gamma\to 1^-}\big\{(1-\gamma)^{\varepsilon}\cdot E_P[\sigma_0\sqrt{2}]\big\} = 0,\end{aligned}$$

On the other hand, using exponential Chebyshev's inequality

$$\mathbb{P}(Y<1) = \mathbb{P}(-Y > -1) \le \mathbb{E}\big[e^{-Y}\big]\cdot e \le 3\,\mathbb{E}\big[e^{-Y}\big],$$

we obtain

$$\begin{aligned}&\mathbb{P}\Big(|X_\gamma| < (1-\gamma)^{-\frac{1}{2}+\varepsilon}\Big) \le 3\,\mathbb{E}\big[\exp\big(-|X_\gamma|\cdot(1-\gamma)^{\frac{1}{2}-\varepsilon}\big)\big]\\ &\quad\le 6\,E_P\Big[\exp\Big(\frac{\tau_\gamma^2\cdot(1-\gamma)^{1-2\varepsilon}}{2}\Big)\cdot\frac{1}{\sqrt{2\pi\tau_\gamma^2}}\int_0^\infty \exp\Big\{-\Big(\frac{x}{\tau_\gamma\sqrt{2}} + \frac{\tau_\gamma(1-\gamma)^{\frac{1}{2}-\varepsilon}}{\sqrt{2}}\Big)^2\Big\}dx\Big]\\ &\quad\le 6\,E_P\Big[\exp\Big(\frac{\tau_\gamma^2\cdot(1-\gamma)^{1-2\varepsilon}}{2}\Big)\cdot\frac{1}{\sqrt{2\pi}}\int_{\tau_\gamma(1-\gamma)^{\frac{1}{2}-\varepsilon}}^\infty e^{-\frac{y^2}{2}}\,dy\Big].\end{aligned}$$

Therefore, in view of (10) and the conditions of the theorem, we have

$$\begin{aligned}\limsup_{\gamma\to 1^-}\mathbb{P}\Big(|X_\gamma| < (1-\gamma)^{-\frac{1}{2}+\varepsilon}\Big) &\le \limsup_{\gamma\to 1^-} 6\,E_P\Big[\frac{1}{\tau_\gamma(1-\gamma)^{\frac{1}{2}-\varepsilon}}\Big]\\ &\le \limsup_{\gamma\to 1^-}\big\{6\cdot\delta^{-2}(1-\gamma)^{\varepsilon}\big\} = 0.\end{aligned}$$

The proof of the theorem is thus completed. □

Approximation results that are much more accurate than those in Theorem 3.1 can be proved under additional assumptions on either the dependence structure of the environment or, assuming that the coefficient γ depends on n, the rate convergence of $\gamma = \gamma_n$ to one as $n \to \infty$. For instance, an application of [10, Lemma 2.1] to our model yields the following result (for i.i.d. errors ξ_n, a functional version of this result with the convergence of the scaled $AR(1)$ process to the Ornstein-Uhlenbeck was established in [16]).

Theorem 3.2. *(see [10, Lemma 2.1]) Let Assumption 1.1 hold. Suppose, in addition, that $P(\mu_0 = 0) = 1$ and $E_P[\sigma_0^2] < \infty$. Consider the recursion*

$$X_{k+1,n} = \gamma_n X_{k,n} + \xi_k, \qquad t = 0, 1 \ldots, n-1,$$

where $X_0 = 0$ and $\gamma_n = 1 - \alpha/n$ for some constant $\alpha > 0$ and $n \in \mathbb{N}$. Then,

$$\frac{X_{n,n}}{\sqrt{n} B_\alpha} \Rightarrow \mathcal{N}(0, 1),$$

where $B_\alpha := E_P[\sigma_0^2] \cdot \frac{1-e^{-2\alpha}}{2\alpha}$.

4 Extreme values of $(X_n)_{n \geq 0}$

The goal of this section is twofold. First, we prove a limit theorem for the running maxima $M_n = \max_{1 \leq k \leq n} X_k$ (Theorem 4.1 below). This result provides some information about the first passage times $T_a = \inf\{t > 0 : X_t > a\}$, through the identity of the events $\{T_a > n\}$ and $\{\max_{k \leq n} X_k < a\}$. Next, we obtain a law of the iterated logarithm type of result for the sequence $(X_n)_{n \in \mathbb{N}}$ (Theorem 4.3 below).

There is an extensive literature discussing the asymptotic behavior of maxima of Gaussian processes. The following general result suffices for our purposes (see [70] or Theorem A in [12]). For $n \in \mathbb{N}$, let

$$a_n = \sqrt{2 \log n} \qquad \text{and} \qquad b_n = a_n - \frac{\log a_n + \log \sqrt{2\pi}}{a_n}. \tag{15}$$

Theorem 4.1. *[70] Let $(X_n)_{n \in \mathbb{Z}}$ be a Gaussian sequence with $\mathbb{E}[X_n] = 0$ and $\mathbb{E}[X_n^2] = 1$. Let $\rho_{ij} = \mathbb{E}[X_i X_j]$ and $M_n = \max_{1 \leq k \leq n} X_k$. If*

(i) $\delta := \sup_{i<j} |\rho_{ij}| < 1$.

(ii) For some $\lambda > \frac{2(1+\delta)}{1-\delta}$,

$$\frac{1}{n^2} \sum_{1 \leq i < j \leq n} |\rho_{ij}| \cdot \log(j-i) \cdot \exp\big\{\lambda |\rho_{ij}| \cdot \log(j-i)\big\} \to 0, \qquad \text{as } n \to \infty, \tag{16}$$

then, for any $y \in \mathbb{R}$, $\mathbb{P}\big(M_n \leq b_n + a_n^{-1} y\big) \to \exp\big\{-e^{-y}\big\}$ as $n \to \infty$.

The theorem implies a sharp concentration of the running maximum around its long-term asymptotic average a_n. The limiting distribution in Theorem 4.1 is called the *standard Gumbel distribution* (cf. [44]).

Let

$$\lambda_k^2 := E_\omega[X_k^2] = \sum_{j=0}^{\infty} \gamma^{2j} \sigma_{k-j}^2, \qquad k \in \mathbb{Z}. \tag{17}$$

We next study the asymptotic distribution of the random variables

$$L_n = \max_{0 \le k \le n} \frac{X_k}{\lambda_k} \qquad \text{and} \qquad M_n = \max_{0 \le k \le n} X_k, n \in \mathbb{N}$$

under Assumption 1.1. We have:

Theorem 4.2. *Let Assumption 1.1 hold. Suppose in addition that $E_P[\mu_0] = 0$ and*

$$P\big(\sigma_0 \in (\delta, \delta^{-1})\big) = 1 \tag{18}$$

for some constant $\delta \in (0,1)$. Then

(a) For any constant $y \in \mathbb{R}$,

$$\lim_{n\to\infty} P_\omega\big(a_n(L_n - b_n) \le y\big) = \exp\{-e^{-y}\}, \qquad P-\text{a.s.}, \tag{19}$$

where a_n and b_n are defined in (15)*.*

(b) Further,

$$\frac{\log M_n}{\log\log n} \xrightarrow{P_\omega} \frac{1}{2}, \qquad P-\text{a.s.}$$

Proof.
(a) Let $U_k = \frac{X_k}{\lambda_k}$, $k \in \mathbb{Z}$. Then $E_\omega[U_k] = 0$ and $E_\omega[U_k^2] = 1$. Furthermore, (12) implies for any $k \in \mathbb{Z}$ and $n \in \mathbb{N}$,

$$\rho_{n,k+n} := \mathrm{COV}_\omega(U_k U_{k+n}) = E_\omega[U_k U_{k+n}] = \gamma^n \frac{\lambda_k}{\lambda_{k+n}}. \tag{20}$$

It suffices to verify that the conditions of Theorem 4.1 are satisfied for random variables U_n. Toward this end, observe that (17) implies

$$\lambda_{k+n}^2 = \gamma^{2n}\lambda_k^2 + \sum_{t=0}^{n-1} \gamma^{2t}\sigma_{k+n-t-1}^2,$$

and hence, by virtue of (17) and (18),

$$\frac{\lambda_{k+n}}{\gamma^n \lambda_k} = \sqrt{1 + \gamma^{-2n}\lambda_k^{-2} \sum_{t=0}^{n-1} \gamma^{2t}\sigma_{k+n-t-1}^2} >$$

(keeping only the last term in the sum, the one with $t = n-1$)

$$> \sqrt{1 + \gamma^{-2n}\lambda_k^{-2}\gamma^{2n-2}\sigma_k^2} > \sqrt{1 + \gamma^{-2}(1-\gamma^2)\delta^4}.$$

Thus

$$\mathfrak{r} := \sup_{k\in\mathbb{Z},n\in\mathbb{N}} \rho_{k,k+n} = \sup_{k\in\mathbb{Z},n\in\mathbb{N}} \Big\{\gamma^n \frac{\lambda_k}{\lambda_{k+n}}\Big\} < 1.$$

Furthermore, it follows from (20) and (17) that, under condition (18), we have for any constant $\mathfrak{s} > 0$:

$$\begin{aligned}
&\frac{1}{n^2} \sum_{1\le i<j\le n} |\rho_{ij}| \cdot \log(j-i) \cdot \exp\big\{\mathfrak{s}|\rho_{ij}| \cdot \log(j-i)\big\} \\
&\quad\le \frac{1}{n^2} \sum_{1\le i<j\le n} \frac{1}{\delta^4}\gamma^{(j-i)} \cdot \log(j-i) \cdot \exp\big\{\mathfrak{s}\delta^{-4} \cdot \log(j-i)\big\} \\
&\quad= \frac{1}{n^2\delta^4} \sum_{1\le i<j\le n} \gamma^{(j-i)} \log(j-i) \cdot (j-i)^{\mathfrak{s}\delta^{-4}} = \frac{1}{n^2\delta^4} \sum_{k=1}^{n-1} (n-k) \cdot \gamma^k \log k \cdot k^{\mathfrak{s}\delta^{-4}} \\
&\quad\le \frac{1}{n\delta^4} \sum_{k=1}^{\infty} \gamma^k \log k \cdot k^{\mathfrak{s}\delta^{-4}} \to 0, \qquad \text{as } n \to \infty.
\end{aligned}$$

Therefore, (19) holds for any $y \in \mathbb{R}$ by Theorem 4.1. The proof of part (a) of the theorem is complete.

(b) It follows from the conditions of the theorem that there exists $c_0 > 0$ such that for all $n \in \mathbb{Z}$,

$$c_0^{-1} < \frac{M_n}{L_n} < c_0, \qquad P-\text{a.s.}$$

Therefore, $P-$a.s., for any $\varepsilon > 0$, we have

$$P_\omega\Big(\frac{\log M_n}{\log\log n} > \frac{1}{2}+\varepsilon\Big) \quad=\quad P_\omega\big(M_n > (\log n)^{\frac{1}{2}+\varepsilon}\big) \le P_\omega\big(L_n > c_0^{-1}(\log n)^{\frac{1}{2}+\varepsilon}\big).$$

Part (a) of the theorem implies that, for any $y \in \mathbb{R}$,

$$\lim_{n\to\infty} P_\omega\big(L_n \le y a_n^{-1} + b_n\big) = \exp\{-e^{-y}\} \qquad P-\text{a.s.} \tag{21}$$

Since for any fixed $y > 0$ and $\varepsilon > 0$, eventually (for all n, large enough) we have

$$y a_n^{-1} + b_n < c_0^{-1}(\log n)^{\frac{1}{2}+\varepsilon},$$

it follows from (21) (because we can use arbitrarily large y while $\lim_{y\to\infty}\exp\{-e^{-y}\} = 1$) that

$$\lim_{n\to\infty} P_\omega\Big(\frac{\log M_n}{\log\log n} > \frac{1}{2}+\varepsilon\Big) = 0 \qquad P-\text{a.s.}$$

Similarly, since $P-$a.s., for any $\varepsilon > 0$,

$$P_\omega\Big(\frac{\log M_n}{\log\log n} < \frac{1}{2}-\varepsilon\Big) \quad=\quad P_\omega\big(M_n < (\log n)^{\frac{1}{2}-\varepsilon}\big) \le P_\omega\big(L_n < c_0(\log n)^{\frac{1}{2}-\varepsilon}\big),$$

while for any $y \in \mathbb{R}$, eventually,

$$c_0(\log n)^{1/2-\varepsilon} < ya_n^{-1} + b_n,$$

It follows from (21), using this time arbitrarily small (negative) values of y, that

$$\lim_{n\to\infty} P_\omega\Big(\frac{\log M_n}{\log\log n} < \frac{1}{2} - \varepsilon\Big) = 0 \qquad P-\text{a.s.}$$

The proof of the theorem is completed. $\square$

We next prove a "law of the iterated logarithm"-type asymptotic result for the sequence X_n. We have:

Theorem 4.3. *Let the conditions of Theorem 4.2 hold. Let $(X_n)_{n\geq 1}$ be the stationary solution to* (1) *defined by* (6)*. Then there exists a constant $c > 0$ such that*

$$\limsup_{n\to\infty} \frac{X_n}{\sqrt{2\log n}} = c, \qquad \mathbb{P}-\text{a.s.}$$

Proof. The claim follows from the bounds provided by a coupling of X_n with the following "extremal versions" of it. Let $(U_n)_{n\in\mathbb{Z}}$ and $(V_n)_{n\in\mathbb{Z}}$ be two stationary sequences that satisfy, respectively,

$$U_{n+1} = \gamma U_n + \delta^{-1}\varepsilon_n \qquad \text{and} \qquad V_{n+1} = \gamma V_n + \delta\varepsilon_n,$$

where δ is the constant introduced in the conditions of Theorem 4.2. Notice that, for all $n \in \mathbb{Z}$, we have $\text{VAR}_\mathbb{P}(U_n) = \delta^{-2}\cdot(1-\gamma^2)^{-1}$ and $\text{VAR}_\mathbb{P}(V_n) = \delta^2\cdot(1-\gamma^2)^{-1}$. Furthermore, it follows for instance from (12) that

$$\lim_{n\to\infty}\sup_{k\in\mathbb{Z}} \text{COV}_\mathbb{P}(U_k, U_{n+k}) = \lim_{n\to\infty}\sup_{k\in\mathbb{Z}} \text{COV}_\mathbb{P}(V_k, V_{n+k}) = 0.$$

Therefore, Theorem 2 in [47] implies that, with probability one,

$$\limsup_{n\to\infty}\frac{V_n}{\sqrt{2\log n}} = \limsup_{n\to\infty}\frac{|V_n|}{\sqrt{2\log n}} = \delta\cdot(1-\gamma^2)^{-1/2}$$

and

$$\limsup_{n\to\infty}\frac{U_n}{\sqrt{2\log n}} = \limsup_{n\to\infty}\frac{|U_n|}{\sqrt{2\log n}} = \delta^{-1}\cdot(1-\gamma^2)^{-1/2}.$$

For an event A, let A^c denote the complement of A and let the abbreviation "i. o." stand for *infinitely often*. Since the event $\{\limsup_{n\to\infty} W_n = a\}$ for a sequence of random variables W_n can be represented as the intersection of the following two events:

$$\bigcap_{\varepsilon>0}\big\{W_n > a-\varepsilon \text{ i. o.}\big\} \qquad \text{and} \qquad \bigcap_{\varepsilon>0}\big\{W_n > a+\varepsilon \text{ i. o.}\big\}^c,$$

then the following inequalities hold with probability one:

$$\limsup_{n\to\infty} \frac{|V_n|}{\sqrt{2\log n}} \leq \limsup_{n\to\infty} \frac{|X_n|}{\sqrt{2\log n}} \leq \limsup_{n\to\infty} \frac{|U_n|}{\sqrt{2\log n}}.$$

Thus, there exists a function $c(X)$ of $X = (X_n)_{n\in\mathbb{N}}$ such that, with probability one,

$$\delta \cdot (1-\gamma^2)^{-1/2} \leq \limsup_{n\to\infty} \frac{X_n}{\sqrt{2\log n}} = \limsup_{n\to\infty} \frac{|X_n|}{\sqrt{2\log n}} = c(X) \leq \delta^{-1} \cdot (1-\gamma^2)^{-1/2}.$$

The fact that $c(X)$ is actually a constant function follows from the ergodicity of the sequence X_n (which is implied by (6) along with the ergodicity of the sequence ξ_n) and the shift invariance of the limiting constant. By the latter we mean that

$$\limsup_{n\to\infty} \frac{X_n}{\sqrt{2\log n}} = \limsup_{n\to\infty} \frac{X_{n+1}}{\sqrt{2\log n}}.$$

The proof of the theorem is completed. □

5 Random walk $S_n = \sum_{k=1}^n X_k$

This section includes limit theorems describing the asymptotic properties of $S_n = \sum_{k=1}^n X_k$. Specifically, we prove a law of large numbers (Theorem 5.2), large deviation bounds associated with it (Theorem 5.3 and Corollary 5.4), and central limit theorems (Theorem 5.5 and Theorem 5.6) for the sequence S_n.

The random walk $S_n = \sum_{k=1}^n X_k$ associated with Equation (1) has been studied in [58] and [59]. The following decomposition of S_n, which is implied by (5), is useful:

$$\begin{aligned} S_n &= \sum_{k=1}^n \gamma^k X_0 + \sum_{k=1}^n \sum_{t=0}^{k-1} \gamma^{k-t-1}\xi_t = \sum_{k=1}^n \gamma^k X_0 + \sum_{t=0}^{n-1} \sum_{k=t+1}^{n} \gamma^{k-t-1}\xi_t = \\ &\qquad (\text{substitute } j = k-1) \ = \sum_{k=1}^n \gamma^k X_0 + \sum_{t=0}^{n-1}\Big(\sum_{j=t}^{\infty} \gamma^{j-t} - \sum_{j=n}^{\infty} \gamma^{j-t}\Big)\xi_t \\ &= \sum_{k=1}^n \gamma^k X_0 + (1-\gamma)^{-1} \sum_{t=0}^{n-1} \xi_t - (1-\gamma)^{-1} \sum_{t=0}^{n-1} \gamma^{n-t}\xi_t. \end{aligned} \tag{22}$$

Similar decompositions have been used, for instance, in [59] and [58]. Due to Assumption 1.1, the following inequalities hold with probability one (the right-most inequality in (24) is implied by Proposition 2.2):

$$\Big|\sum_{k=1}^n \gamma^k X_0\Big| \ \leq \ |X_0| \cdot \sum_{k=0}^{\infty} \gamma^k < \infty, \tag{23}$$

and

$$\Big|\sum_{t=0}^{n-1} \gamma^{n-t}\xi_t\Big| \stackrel{D}{=} \Big|\sum_{t=-n+1}^{0} \gamma^{1-t}\xi_t\Big| \leq \sum_{k=0}^{\infty} \gamma^{k+1} \cdot |\xi_{-k}| < \infty, \tag{24}$$

where $\stackrel{D}{=}$ means equivalence of distributions. This shows that only the second term in the right-most expression of (22) contributes to the asymptotic behavior of S_n. More precisely, we have the following lemma. Though the proof of the lemma is by standard arguments, we provide it below for the reader's convenience.

Lemma 5.1. *Let Assumption 1.1 hold. Then*

(a) For any sequence of reals $(a_n)_{n\in\mathbb{N}}$ increasing to infinity, we have

$$\frac{1}{a_n}\sum_{k=1}^{n}\gamma^k X_0 \to_{n\to\infty} 0, \qquad \mathbb{P}-\text{a.s.}$$

and

$$\frac{1}{a_n}\sum_{t=0}^{n-1}\gamma^{n-t}\xi_t \to_{n\to\infty} 0, \qquad \text{in probability.}$$

(b) If in addition $E_P\big[|\mu_0|\big]<\infty$ and $E_P\big[\sigma_0\big]<\infty$, then

$$\frac{1}{n}\sum_{t=0}^{n-1}\gamma^{n-t}\xi_t \to_{n\to\infty} 0, \qquad \mathbb{P}-\text{a.s.}$$

Proof.

(a) The first claim of part (a) is a direct consequence of (23). The second claim can be derived from (24) as follows. For any $\varepsilon>0$, we have in virtue of (24),

$$\begin{aligned}&\mathbb{P}\Big(\frac{1}{a_n}\Big|\sum_{t=0}^{n-1}\gamma^{n-t}\xi_t\Big|>\varepsilon\Big)=\\&\quad=\mathbb{P}\Big(\frac{1}{a_n}\Big|\sum_{t=-n+1}^{0}\gamma^{1-t}\xi_t\Big|>\varepsilon\Big)\le\mathbb{P}\Big(\frac{1}{a_n}\sum_{k=0}^{\infty}\gamma^{k+1}\cdot|\xi_{-k}|>\varepsilon\Big)\to_{n\to\infty}0,\end{aligned}$$

which implies the result.

(b) We must show that for any $\varepsilon>0$,

$$\mathbb{P}\Big(\frac{1}{n}\Big|\sum_{t=0}^{n-1}\gamma^{n-t}\xi_t\Big|>\varepsilon\ \text{i.o.}\Big)=0.$$

By the Borel-Cantelli lemma, it suffices to show that for any $\varepsilon>0$,

$$\sum_{n=1}^{\infty}\mathbb{P}\Big(\frac{1}{n}\Big|\sum_{t=0}^{n-1}\gamma^{n-t}\xi_t\Big|>\varepsilon\Big)<\infty. \tag{25}$$

Using (24), we obtain

$$\begin{aligned}\sum_{n=1}^{\infty}\mathbb{P}\Big(\frac{1}{n}\Big|\sum_{t=0}^{n-1}\gamma^{n-t}\xi_t\Big|>\varepsilon\Big)&\le\sum_{n=1}^{\infty}\mathbb{P}\Big(\frac{1}{\varepsilon}\sum_{k=0}^{\infty}\gamma^{k+1}\cdot|\xi_{-k}|>n\Big)\le\frac{1}{\varepsilon}\mathbb{E}\Big[\sum_{k=0}^{\infty}\gamma^{k+1}\cdot|\xi_{-k}|\Big]\\&=\frac{\gamma}{\varepsilon(1-\gamma)}\mathbb{E}\big[|\xi_0|\big].\end{aligned}$$

Since ξ_k are Gaussian random variables under P_ω, implies

$$\mathbb{E}\big[|\xi_0|\big] = E_P\Big[|\mu_0| + \sqrt{\frac{2\sigma_0^2}{\pi}}\Big]. \tag{26}$$

It hence follows from the conditions of the lemma that $\mathbb{E}\big[|\xi_k|\big] < \infty$. This establishes (25) and therefore completes the proof of part (b) of the lemma. □

In particular, one can obtain the following strong law of large numbers.

Theorem 5.2. *Let Assumption 1.1 hold and suppose in addition that* $E_P\big[|\mu_0|\big] < \infty$ *and* $E_P\big[\sigma_0\big] < \infty$. *Then,*

$$\lim_{n\to\infty} \frac{S_n}{n} = \mathbb{E}[X] = (1-\gamma)^{-1} E_P[\mu_0], \qquad \mathbb{P}-\text{a.s.} \tag{27}$$

Proof. Recall that under Assumption 1.1, $(\xi_n)_{n\in\mathbb{Z}}$ is a stationary and ergodic sequence. Furthermore, (26) implies that $\mathbb{E}\big[|\xi_0|\big] < \infty$. Therefore, by the Birkhoff ergodic theorem,

$$\lim_{n\to\infty} \frac{1}{n}\sum_{t=0}^{n-1} \xi_t = \mathbb{E}[\xi_0] = E_P[\mu_0], \qquad \mathbb{P}-\text{a.s.} \tag{28}$$

It follows now from (22) and Lemma 5.1 that

$$\lim_{n\to\infty} \frac{S_n}{n} = \lim_{n\to\infty} \frac{1}{1-\gamma}\frac{1}{n}\sum_{t=0}^{n-1} \xi_t = \frac{1}{1-\gamma} E_P[\mu_0], \qquad \mathbb{P}-\text{a.s.}$$

The proof of the theorem is completed. □

The above law of large numbers can be complemented by the following large deviation result. Recall that a sequence R_n of random variables is said to satisfy the large deviation principle (LDP) with a lower semi-continuous *rate function* $I : \mathbb{R} \to [0,\infty]$, if for any Borel set $E \subset \mathbb{R}$,

$$-\inf_{x\in E^\circ} I(x) \le \liminf_{n\to\infty} \frac{1}{n}\log \mathbb{P}(R_n \in E) \le \limsup_{n\to\infty} \frac{1}{n}\mathbb{P}(R_n \in E) \le -\inf_{x\in \overline{E}} I(x)$$

where $\overline{E}$ and E° denote, respectively, the closure and interior of E. The rate function is *good* if the level sets $\{x \in \mathbb{R} : I(x) \le c\}$ are compact for any $c \ge 0$.

We have:

Theorem 5.3. *Let the conditions of Theorem 4.2 hold. Assume in addition that the sequence* $R_n := \frac{1}{n}\sum_{k=1}^n \sigma_k^2$ *satisfies the LDP with a good rate function* $I(x)$. *Then* $\frac{S_n}{n}$ *satisfies the LDP with a good rate function* J, *such that* $J(x) \in (0,\infty)$ *for* $x \ne 0$.

Proof. Recall $G_{\mu,\sigma^2}(t)$ from (4). The l'Hôpital rule implies that, for $t_\sigma = t/\sigma$,

$$\lim_{t\to\infty} \frac{G_{0,\sigma^2}(t)}{\sqrt{\frac{\sigma^2}{2\pi t^2}}e^{-\frac{t^2}{2\sigma^2}}} \quad = \quad \lim_{t_\sigma\to\infty} \frac{\int_{t_\sigma}^\infty e^{-\frac{x^2}{2}}dx}{t_\sigma^{-1}e^{-\frac{t_\sigma^2}{2}}} = 1.$$

Therefore, there exists $t_0 > 0$ such that $t > t_0$ implies

$$\frac{1}{2}\sqrt{\frac{\sigma^2}{2\pi t^2}}e^{-\frac{t^2}{2\sigma^2}} \le G_{0,\sigma^2}(t) \le 2\sqrt{\frac{\sigma^2}{2\pi t^2}}e^{-\frac{t^2}{2\sigma^2}}.$$

It follows from (22) and (6) that $\mathbb{P}(S_n > nt) = E_P\big[P_\omega(S_n > nt)\big] = E_P\big[G_{0,\beta_n^2}(nt)\big]$, where

$$\beta_n^2 := \frac{\gamma^2(1-\gamma^n)^2\sum_{t=0}^\infty \sigma_{-t}^2\gamma^{2t}}{(1-\gamma)^2} + \frac{\sum_{t=0}^{n-1}\sigma_t^2\big(1-\gamma^{n-t}\big)^2}{(1-\gamma)^2}.$$

It then follows from (18) that for any $t > 0$,

$$\lim_{n\to\infty}\frac{1}{n}\log\mathbb{P}(S_n > nt) = \lim_{n\to\infty}\frac{1}{n}\log E_P\big(e^{-\frac{t^2n^2}{2\beta_n^2}}\big),$$

provided that the latter limit exists. We will next estimate the difference

$$\frac{1}{\beta_n^2} - \frac{(1-\gamma)^2}{\sum_{t=0}^{n-1}\sigma_t^2},$$

Using (18), we have:

$$\begin{aligned}
\Big|\frac{1}{\beta_n^2} - \frac{(1-\gamma)^2}{\sum_{t=0}^{n-1}\sigma_t^2}\Big| &\le \frac{\sum_{t=0}^{n-1}\sigma_t^2 - \sum_{t=0}^{n-1}\sigma_t^2\big(1-\gamma^{n-t}\big)^2 + \gamma^2(1-\gamma^n)^2\sum_{t=0}^\infty\sigma_{-t}^2\gamma^{2t}}{\Big(\sum_{t=0}^{n-1}\sigma_t^2(1-\gamma^{n-t})^2\Big)^2(1-\gamma)^{-2}} \\
&\le \frac{\delta^{-2}\Big(n - \sum_{k=1}^n(1-\gamma^k)^2 + \gamma^2(1-\gamma^2)^{-1}\Big)}{\Big(\sum_{t=0}^{n-1}\sigma_t^2\big(1-\gamma^{n-t}\big)^2\Big)^2(1-\gamma)^{-2}} \\
&\le \frac{\delta^{-6}\Big(2\sum_{k=1}^n\gamma^k + \gamma^2(1-\gamma^2)^{-1}\Big)}{\Big(\sum_{k=1}^n(1-\gamma^k)^2\Big)^2(1-\gamma)^{-2}} \le \frac{\delta^{-6}\Big(2\sum_{k=1}^n\gamma^k + \gamma^2(1-\gamma^2)^{-1}\Big)}{\Big(\sum_{k=1}^n(1-\gamma)^2\Big)^2(1-\gamma)^{-2}} \\
&\le n^{-2}\cdot\frac{\delta^{-6}\Big(2\sum_{k=1}^\infty\gamma^k + \gamma^2(1-\gamma^2)^{-1}\Big)}{(1-\gamma)^2}.
\end{aligned}$$

Thus

$$\lim_{n\to\infty}\frac{1}{n}\log\mathbb{P}(S_n > nt) = \lim_{n\to\infty}\frac{1}{n}\log E_P\Big(e^{-\frac{t^2n^2(1-\gamma)^2}{2\sum_{t=0}^{n-1}\sigma_t^2}}\Big),$$

provided that the limit in the right-hand side exists. By virtue of (18), we have

$$-\frac{t^2(1-\gamma)^2\delta^{-2}}{2} \le \frac{1}{n}\log E_P\Big(e^{-\frac{t^2n^2(1-\gamma)^2}{2\sum_{t=0}^{n-1}\sigma_t^2}}\Big) \le -\frac{t^2(1-\gamma)^2\delta^2}{2}.$$

Thus one can apply Varadhan's integral lemma (see [19, p. 137]) to $R_n := \frac{1}{n}\sum_{t=1}^n \sigma_t^2$ and the continuous function $\phi_t(x) = -\frac{t^2(1-\gamma)^2}{2x} : (0,\infty) \to \mathbb{R}$. It follows from Varadhan's lemma that

$$\lim_{n\to\infty} \frac{1}{n}\log \mathbb{P}(S_n > nt) = \lim_{n\to\infty}\frac{1}{n}\log E_P\big[e^{n\phi_t(R_n)}\big] = \sup_{x>0}\{\phi_t(x) - I(x)\}.$$

Furthermore, a symmetry argument shows that

$$\lim_{n\to\infty} \frac{1}{n}\log \mathbb{P}(S_n > nt) = \lim_{n\to\infty}\frac{1}{n}\log \mathbb{P}(S_n < -nt), \qquad t > 0.$$

Since $J(t) = -\sup_{x>0}\{\phi_t(x) - I(x)\} \in [0,\infty)$ is a strictly increasing function for $t \geq 0$, this implies that the LDP for S_n/n holds with rate function J (cf. [19, p. 31]).

It remains to show that J is a good rate function. Toward this end fix $c > 0$ and consider $\Psi(c) = \{t > 0 : J(t) > c\}$. Then $t \in \Psi(c)$ if and only if $t > 0$ and

$$\inf_{x>0}\Big\{\frac{t^2(1-\gamma)^2}{2x} + I(x)\Big\} > c.$$

It thus suffices to verify that $t_0 := \inf \Psi(c) \notin \Psi(c)$. Assume the contrary, that is suppose that for some $c_0 > c$

$$\inf_{x>0}\Big\{\frac{t_0^2(1-\gamma)^2}{2x} + I(x)\Big\} = c_0 > c. \tag{29}$$

Let $x_0 = \frac{t_0^2(1-\gamma)^2}{4c_0}$. We then can choose $t_1 < t_0$ such that

$$\inf_{x<x_0}\Big\{\frac{t_1^2(1-\gamma)^2}{2x} + I(x)\Big\} \geq \inf_{x<x_0}\Big\{\frac{t_1^2(1-\gamma)^2}{2x}\Big\} > c$$

and

$$\begin{aligned}&\inf_{x\geq x_0}\Big\{\frac{t_1^2(1-\gamma)^2}{2x} + I(x)\Big\}\\ &\qquad \geq \inf_{x\geq x_0}\Big\{\frac{t_0^2(1-\gamma)^2}{2x} + I(x)\Big\} - \sup_{x\geq x_0}\Big\{\frac{t_0^2(1-\gamma)^2}{2x} - \frac{t_1^2(1-\gamma)^2}{2x}\Big\} > c.\end{aligned}$$

Clearly, this contradicts (29) and hence shows that $t_0 \notin \Psi(c)$, as desired. the proof of the theorem is completed. □

The following is implied, for instance, by Theorem 3.1.2 in [19, p. 74].

Corollary 5.4. *Let Assumption 1.2 and the conditions of Theorem 4.2 hold. Then* $\frac{S_n}{n}$ *satisfies the LDP with a good rate function.*

It follows from (22) that if $E_P[\mu_0] = 0$ and $b_n^{-1}\sum_{k=1}^n \sigma_k^2$ converges in distribution to a random variable G for a suitable sequence $b_n \nearrow \infty$, then $S_n/\sqrt{b_n}$ converges in distribution to $\mathcal{N}(0, G)$. In a generic example, σ_n are in the domain of attraction of a symmetric stable law and the sequence $(\sigma_n)_{n\in\mathbb{Z}}$ satisfies certain mixing conditions. Limit theorems for S_n of this type can be found in [58]. We also refer the reader to [58] for a law of iterated logarithm for S_n. The special (Gaussian, once the environment is fixed) structure of the sequence ξ_n which is considered in this work, leads to the following result. It is different in essence from the limit theorems obtained in [58].

Theorem 5.5. *Let Assumption 1.1 hold and assume in addition that* $E_P[\mu_0] = 0$ *and* $E_P\big[\sigma_0^2\big] < \infty$*. Then,*

$$\frac{1}{\sqrt{n}} S_n \overset{\mathbb{P}}{\Longrightarrow} \frac{1}{1-\gamma}\mathcal{N}\big(0,\Sigma\big)$$

for $\Sigma := E_P[\sigma_0^2]$.

Proof. By the Birkhoff ergodic theorem,

$$\lim_{n\to\infty}\frac{1}{n}\sum_{t=0}^{n-1}\sigma_t^2 = E_P[\sigma_0^2], \qquad P-\text{a.s.}$$

Hence, letting $W_n = \sum_{t=0}^{n-1}\xi_t$, we obtain

$$\begin{aligned}\lim_{n\to\infty}\mathbb{E}\big[e^{it\frac{W_n}{\sqrt{n}}}\big] &= \lim_{n\to\infty} E_P\Big[E_\omega\big[e^{it\frac{W_n}{\sqrt{n}}}\big]\Big] = E_P\Big[\lim_{n\to\infty}E_\omega\big[e^{it\frac{W_n}{\sqrt{n}}}\big]\Big]\\ &= E_P\Big[\lim_{n\to\infty} e^{-t^2\frac{\sum_{t=0}^{n-1}\sigma_t^2}{2n}}\Big] = e^{-\frac{t^2\Sigma}{2}}.\end{aligned}$$

Therefore, $\frac{W_n}{\sqrt{n}} \overset{\mathbb{P}}{\Longrightarrow} \mathcal{N}\big(0,\Sigma\big)$. It follows now from (22) and part (a) of Lemma 5.1 that

$$\lim_{n\to\infty}\frac{S_n}{\sqrt{n}} = \lim_{n\to\infty}\frac{1}{1-\gamma}\frac{W_n}{\sqrt{n}} = \frac{1}{1-\gamma}\mathcal{N}\big(0,\Sigma\big),$$

where the limits in the above identities are understood in terms of convergence in distribution. The proof of the theorem is completed. □

The above theorem can be strengthened to a functional central limit result in the Skorokhod space $D[0,1]$ of *càdlàg* functions for the sequence of processes

$$J_n(t) = \frac{S_{[nt]}}{\sqrt{n\Sigma^2}}, \qquad t\in[0,1],$$

where $\Sigma^2 = \dfrac{E_P[\sigma_0^2]}{(1-\gamma)^2}$ as in the statement of Theorem 5.5, and $[x]$ denotes the integer part of $x\in\mathbb{R}$, that is $[x] = \max\{k\in\mathbb{Z} : k\le x\}$. We have:

Theorem 5.6. *Let the conditions of Theorem 4.2 hold. Then, for* P*-almost every environment* ω*, the sequence* J_n *converges in* $D[0,1]$ *under* P_ω *weakly to a standard Brownian motion. Consequently,* J_n *converges in* $D[0,1]$ *weakly to a standard Brownian motion also under* $\mathbb{P}$*.*

Proof. It is not hard to verify the convergence under P_ω of the finite-dimensional distributions of J_n to those of standard Brownian motion using characteristic functions and the Cramér-Wold device [20, p. 170]. The argument is based on an application of the law of large numbers to the sequence σ_n, and is nearly verbatim the same as in the proof of Theorem 5.5. On the other hand, the tightness under P_ω of the sequence of processes J_n in $D[0,1]$ is evident from the criterion stated in Example 1 in [28, p. 336]. Notice that the criterion can be applied to J_n in virtue of (18). Once the weak convergence of J_n to standard Brownian motion is proved under P_ω (for $P-$a.s. every environment ω), the same convergence under $\mathbb{P}$ follows from (3) and the bounded convergence theorem. □

References

[1] T. W. Anderson, *On asymptotic distributions of estimates of parameters of stochastic difference equations*, Ann. Math. Statist. **30** (1959), 676–687.

[2] A. Aue, L. Horváth, M. Hušková, and P. Kokoszka, *Change-point monitoring in linear models*, Econometrics Journal **9** (2006), 373–403.

[3] R. T. Baillie, *Long memory processes and fractional integration in econometrics*, Journal of Econometrics **73** (1996), 5–59.

[4] J. Benhabib, A. Bisin, and Z. Zhu, *The distribution of wealth and fiscal policy in economies with finitely lived agents*, Econometrica **79** (2011), 123–157.

[5] J. Benhabib and C. Dave, *Learning, large deviations and rare events*, 2012. Preprint is available at `http://www.nber.org/papers/w16816`.

[6] A. Brandt, *The stochastic equation* $Y_{n+1} = A_n Y_n + B_n$ *with stationary coefficients*, Adv. Appl. Probab. **18** (1986), 211–220.

[7] B. Buchmann and N. H. Chan, *Asymptotic theory of least squares estimators for nearly unstable processes under strong dependence*, Ann. Statist. **35** (2007), 2001–2017.

[8] L. Calvet and A. Fisher, *How to forecast long-run volatility: regime-switching and the estimation of multifractal processes*, Journal of Financial Econometrics **2** (2004), 49–83.

[9] N. H. Chan, *Inference for near-integrated time series with infinite variance*, J. Amer. Stat. Assoc. **85** (1990), 1069–1074.

[10] N. H. Chan and C. Z. Wei, *Asymptotic inference for nearly nonstationary AR(1) processes*, Ann. Stat. **15** (1987), 1050–1063.

[11] M. Chauvet and J. D. Hamilton, *Dating business cycle turning points*, in Nonlinear Analysis of Business Cycles, C. Milas, P. Rothman, and D. van Dijk eds., Elsevier, 2006.

[12] S. Chen and Z. Lin, *Almost sure max-limits for nonstationary Gaussian sequence*, Statist. Probab. Lett. **76** (2006), 1175–1184.

[13] J. S. Cho and H. White, *Testing for regime switching*, Econometrica **75** (2007), 1671–1720.

[14] J. F. Collamore, *Random recurrence equations and ruin in a Markov-dependent stochastic economic environment*, Ann. Appl. Probab. **19** (2009), 1404–1458.

[15] F. Comets, R. Fernández, and P. A. Ferrari, *Processes with long memory: regenerative construction and perfect simulation*, Ann. Appl. Probab. **12** (2002), 921–943.

[16] W. G. Cumberland and Z. M. Sykes, *Weak convergence of an autoregressive process used in modeling population growth*, J. Appl. Prob. **19** (1982), 450–455.

[17] P. Doukhan, G. Oppenheim, and M. S. Taqqu, eds., *Theory and Applications of Long-Range Dependence*, Birkhuser, Boston, 2003.

[18] B. de Saporta, *Tail of the stationary solution of the stochastic equation $Y_{n+1} = a_n Y_n + b_n$ with Markovian coefficients*, Stochastic Process. Appl. **115** (2005), 1954–1978.

[19] A. Dembo and O. Zeitouni, *Large Deviation Techniques and Applications*, 2nd ed., (Applications of Mathematics **38**), Springer, New York, 1998.

[20] R. Durrett, *Probability: Theory and Examples*, 2nd ed., Duxbury Press, Belmont, CA, 1996.

[21] P. Eloe, R. H. Liu, M. Yatsuki, G. Yin, and Q. Zhang, *Optimal selling rules in a regime-switching exponential Gaussian diffusion model*, SIAM J. Appl. Math. **69** (2008), 810–829.

[22] P. Embrechts and C. M. Goldie, *Perpetuities and random equations*, In *Asymptotic Statistics. Proc Fifth. Prague Symp.,* Eds. P. Mandl and M. Hušková, Physica, Heidelberg, 1993, pp. 75–86.

[23] E. F. Fama, *The behavior of stock market prices*, Journal of Business **38** (1965), 34–105.

[24] R. Fernández and G. Maillard, *Chains with complete connections and one-dimensional Gibbs measures*, Electron. Journ Probab. **9** (2004), 145–176.

[25] R. Fernández and G. Maillard, *Chains with complete connections: general theory, uniqueness, loss of memory and mixing properties*, J. Stat. Phys. **118** (2005), 555-588.

[26] M. Frisen and Cr. Sonesson, *Optimal surveillance based on exponentially weighted moving averages*, Sequential Analysis **25** (2006), 379–403.

[27] A. Galves and E. Löcherbach, *Stochastic chains with memory of variable length*, TICSP Series **38** (2008), 117–133.

[28] C. Genest, K. Ghoudib, and B. Rémillard, *A note on tightness*, Stat. Probab. Lett. **27** (1996), 331–339.

[29] A. P. Ghosh, D. Hay, V. Hirpara, R. Rastegar, A. Roitershtein, A. Schulteis, and J. Suh, *Random linear recursions with dependent coefficients*, Statist. Probab. Lett. **80** (2010), 1597–1605.

[30] E. Ghysels, A. Harvey, and E. Renault, *Stochastic volatility*, In *Statistical Methods in Finance (Handbook of Statistics **14**)*, G. S. Maddala, C. R. Rao, and H. D. Vinod eds., North-Holland, 1996, pp. 119–191.

[31] E. J. Green and R. H. Porter, *Noncooperative collusion under imperfect price information*, Econometrica **52** (1984), 87–100.

[32] J. D. Hamilton, *A new approach to the economic analysis of nonstationary time series and the business cycle*, Econometrica **57** (1989), 357–384.

[33] J. D. Hamilton, *Regime-switching models*, in New Palgrave Dictionary of Economics, 2nd edn., S. Durlauf and L. Blume eds., Palgrave McMillan Ltd., 2008.

[34] J. D. Hamilton and J. C. Wu, *Identification and estimation of Gaussian affine term structure models*, Journal of Econometrics **168** (2012), 315–331.

[35] B. E. Hansen, *The likelihood ratio test under non-standard conditions: Testing the Markov switching model of GNP*, Journal of Applied Econometrics **7** (1992), Supplement: Special Issue on Nonlinear Dynamics and Econometrics, S61–S82.

[36] D. Hay, R. Rastegar, and A. Roitershtein, *Multivariate linear recursions with Markov-dependent coefficients*, J. Multivariate Anal. **102** (2011), 521–527.

[37] U. Horst, *Stability of linear stochastic difference equations in strategically controlled random environments*, Adv. in Appl. Probab. **35** (2003), 961–981.

[38] K.-S. Hwang and T.-X. Pang, *Asymptotic inference for nearly nonstationary* $AR(1)$ *processes with possiblyinfinite variance*, Stat. Probab. Letters **79** (2009), 2374–2379.

[39] M. Iosifescu and S. Grigorescu, *Dependence with Complete Connections and its Applications*, Cambridge Tracts in Mathematics **96**, Cambridge University Press, 2009.

[40] P. Jeganathan, *On the asymptotic behavior of least square estimators in AR time series with roots near the unit circle*, Econometric Theory **7** (1991), 269–306.

[41] P. Jeganathan, *On asymptotic inference in cointegrated time series with fractionally integrated errors*, Econometric Theory **15** (1999), 583–621.

[42] C. J. Kim, *Dynamic linear models with Markov-switching*, Journal of Econometrics **60** (1994), 1–22.

[43] C. J. Kim and C. R. Nelson, *State-Space Models with Regime Switching*, MIT Press, 1999.

[44] C. Klüppelberg and A. Lindner, *Probability extreme value theory for moving average processes with light-tailed innovations*, Bernoulli **11** (2005), 381–410.

[45] J. Kormos, *Nearly nonstationary AR process with mixing innovation*, Annales Univ. Sci. Budapest., Sect. Comp. **22** (2004), 207–223.

[46] J. Kormos, G. Pap, T. van der Meer, and M. van Zuijlen, *Asymptotic inference of nearly nonstationary complex-valued* $AR(1)$ *processes*, In Exploring stochastic laws (Festschrift in honour of the 70th birthday of Academician Vladimir Semenovich Korolyuk), VSP, Utrecht, 1995, pp. 201–218.

[47] T. L. Lai, *Reproducing kernel Hilbert spaces and the law of the iterated logarithm for Gaussian processes*, Z. Wahrsch. verw. Gebiete **29** (1974), 7–19.

[48] H. Larralde, *A first passage time distribution for a discrete version of the Ornstein-Uhlenbeck process*, J. Phys. A: Math. Gen. **37** (2004), 3759–3767.

[49] H. Larralde, *Statistical properties of a discrete version of the Ornstein-Uhlenbeck process*, Phys. Rev. E **69** (2004), paper no. 027102.

[50] M. Lefebvre and J.-L. Guilbault, *First hitting place probabilities for a discrete version of the Ornstein-Uhlenbeck process*, International Journal of Mathematics and Mathematical Sciences, 2009, article no. 909835.

[51] R. Leipus and D. Surgailis, *Random coefficient autoregression, regime switching and long memory*, Adv. Appl. Probab. **35** (2003), 737–754.

[52] B. Mandelbrot, *Forecasts of future prices, unbiased markets, and martingale models*, Journal of Business **39** (1965), 241–255.

[53] A. A. Novikov, *On distributions of first passage times and optimal stopping of AR(1) sequences*, Theory Probab. Appl. **53** (2009), 419–429.

[54] M. Olvera-Cravioto, *On the distribution of the nearly unstable AR(1) process with heavy tails*, Adv. Appl. Probab. **42** (2010), 106–136.

[55] P. C. B. Phillips and P. Perron, *Testing for a unit root in time series regression*, Biometrika **75** (1988), 335–346.

[56] R. H. Porter, *A study of cartel stability: the Joint Executive Committee, 1880–1886*, Bell Journal of Economics **14** (1983), 301314.

[57] S. T. Rachev and G. Samorodnitsky, *Limit laws for a stochastic process and random recursion arising in probabilistic modeling*, Adv. Appl. Prob. **27** (1995), 185–202.

[58] R. Rastegar, A. Roitershtein, V. Roytershteyn, and J. Suh, *Discrete-time Langevin motion of a particle in a Gibbsian random potential*, 2012. Preprint is available at `hhtp:\\www.public.iastate.edu\~roiterst\preprints`

[59] E. Renshaw, *The discrete Uhlenbeck-Ornstein process*, J. Appl. Probab. **24** (1987), 908–917.

[60] J. Rissanen, *A universal data compression system*, IEEE Trans. Inform. Theory **29** (1983), 656–664.

[61] P. M. Robinson, *Time series with strong dependence*, In C. A. Sims ed., Advances in Econometrics: Sixth World Congress, vol I, New York: Cambridge University Press, 1994.

[62] P. M. Robinson, ed., *Time Series with Long Memory*, Oxford Univ. Press. 2003

[63] P. M. Robinson, *The distance between rival nonstationary fractional processes*, Journal of Econometrics **128** (2005), 283–300.

[64] A. Roitershtein, *One-dimensional linear recursions with Markov-dependent coefficients*, Ann. Appl. Probab. **17** (2007), 572–608.

[65] G. Samorodnitsky and M. S. Taqqu, *Stable Non-Gaussian Random Processes: Stochastic Models With Infinite Variance*, Chapman & Hall/CRC, 2000.

[66] L. Scott, *Option pricing when the variance changes randomly : theory, estimation and an application*, Journal of Financial and Quantitative Analysis **22** (1987), 419–438.

[67] N. Shephard, *Stochastic Volatility. Selected Readings (Advanced Texts in Econometrics)*, Oxford University Press, 2005.

[68] R. Stelzer, *Multivariate Markov-switching ARMA processes with regularly varying noise*, J. Multivariate Analysis **99** (2008), 1177–1190.

[69] S. J. Taylor, *Modeling stochastic volatility: A review and comparative study*, Mathematical Finance **4** (1994), 183–204.

[70] Xie Shengrong, *Asymptotic distributions of extremes in non-stationary Gaussian sequences*, Acta Mathematicae Applicatae Sinica (in Chinese) **7** (1984), 96–100.

[71] C. Sims and T. Zha, *Were there switches in U.S. monetary policy?*, American Economic Review **96** (2006), 54–81.

[72] T. Szabados and B. Székely, *An exponential functional of random walks*, J. Appl. Prob. **40** (2003), 413–426.

[73] W. Vervaat, *On a stochastic difference equations and a representation of non-negative infinitely divisible random variables*, Adv. Appl. Prob. **11** (1979), 750–783.

[74] R. Vilela Mendes, R. Lima, and T. Araújo, *A process-reconstruction analysis of market fluctuations*, Int. J. Theor. Appl. Finance **5** (2002), 797–821.

[75] A. Zeevi and P. W. Glynn, *Recurrence properties of autoregressive processes with super-heavy-tailed innovations*, J. Appl. Prob. **41** (2004), 639–653.

[76] Z.-W. Zhao, D.-H. Wang, and Y. Zhang, *Limit theory for random coefficient first-order autoregressive process under martingale difference error sequence*, J. Comput. Appl. Math. **235** (2011), 2515–2522.