COLORED MAXIMAL BRANCHING PROCESS*

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Abstract. We consider a certain version of the multi-type maximal branching process recently introduced by Lebedev. The main result of this paper is a limit theorem for empirical frequencies of the types. The result shows explicitly how the initial distribution of types is modified in the long run by a mechanism of selection among competing individuals in a maximal branching process.

Key words. multi-type maximal branching process, maximal branching process, asymptotic behavior of Markov chains, additive functionals of Markov chains.

AMS subject classifications. 60J80, 60F05, 60F15, 60G70

1. Introduction. The main goal of this paper is to study the mechanism of the selection of the ancestor in a certain version of the multi-type maximal branching process (following [18] we will use the abbreviation MTMBP for these processes). More precisely, we consider a version of the MTMBP where particles are colored at random and their offspring distribution depends on the color. Our main result (stated in Theorems 2.7 and 2.8 below) is a limit theorem for the distribution of the color of the direct ancestor of the n-th generation, as n goes to infinity.

Maximal branching processes (MBP) were introduced by Lamperti in [7]. An MBP is a Markov chain on the set of non-negative integers with a unique absorbing state at zero and transition kernel determined by the following recursive equation:

$$Z_{n+1} = \max_{1\le k\le Z_n} X_{n,k}, \qquad\qquad Z_n > 0,$$

where the random variables $X_{n,k}$ are i.i.d., non-negative, and integer-valued. The process can be thus described as an "extremal analogue" of the Galton-Watson branching processes, where the next generation is formed by the offspring of a most productive individual. The MBP is an elegant mathematical construction, and their theory turns out to be closely related to a general problem of the study of asymptotic behavior of (Markov) random processes in a half-line with asymptotically vanishing drift. The latter is sometimes referred to as Lamperti's problem (cf. [19, 20]) to acknowledge the contribution of Lamperti's pioneering work [9, 10, 11].

A generalization of the MBP from integer-valued population processes to their real-valued analogue is considered in a series of papers by Lebedev [12, 13, 14], see also a review of his results in [15]. An application of these processes to the queueing theory (for gated infinite-server queues, cf. [2]) is discussed in [13]. More recently, Lebedev studied in [16, 17, 18] an extension of the MBP to a multi-type setting. This paper intends to contribute to the understanding of the MTMBP by considering certain aspect of their asymptotic behavior in a setting where explicit computation is possible through a link to the one-dimensional theory of the MBP which has been developed by Lamperti in [7, 8].

The *colored maximal branching process* (CMBP for short) that we consider in this work is a vector-valued Markov chain $Z_n = \big(Z_n^{(1)}, Z_n^{(2)}, \ldots, Z_n^{(d)}\big)$ which describes

*Submitted: September 6, 2012

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evolution of a population of individuals of d different types (colors) in discrete time $n = 0, 1, \ldots$ We use the term *color* as an alternative to the *type*, to distinguish our model from a more general one of the MTMBP, introduced by Lebedev in [16, 17, 18]. The integer $d \geq 1$ is fixed and $Z_n^{(i)}$ represents the number of individuals of type i present in n-th generation. Transitions of the Markov chain Z_n consist of two stages: at the first stage the total size of the population $Y_n := \sum_{i=1}^d Z_n^{(i)}$ in the generation n is determined, and at the second stage colors are randomly assigned to the Y_n individuals who form the n-the generation. That is where our model differs from the one considered in [16, 17, 18], where, similarly to the ordinary multi-type Galton-Watson branching process, individuals are born being already of a certain type.

Let $\mathcal{D} := \{1, \ldots, d\}$ and let $\mathbb{Z}_+$ denote the set of nonnegative integers. The dynamics of Y_n is determined by the equation

$$Y_{n+1} = \max_{1 \leq i \leq d} \max_{1 \leq k \leq Z_n^{(i)}} X_{n,k}^{(i)},$$

where $X_{n,k}^{(i)}$ represents the number of children of the k-th individual of type i. We assume that the random variables $X_{n,k}^{(i)}$ take values in $\mathbb{Z}_+$, are independent and, moreover, $\{X_{n,k}^{(i)} : n \in \mathbb{Z}_+, k \in \mathbb{N}\}$ are identically distributed for each fixed $i \in \mathcal{D}$. We denote the common distribution function of $X_{n,k}^{(i)}$ by F_i.

Once a new generation is formed, a color from the set $\mathcal{D}$ is assigned to the individuals, independently of each other and of the previous history. Let $\chi_{n,k}$ denote the color of the k-th individual in the n-th generation. We assume that $\chi_{n,k}$, $n \in \mathbb{Z}_+, k \in \mathbb{N}$ are i.i.d $\mathcal{D}$-valued random variables, and denote

$$\mu_i := P(\chi_{n,k} = i), \qquad i \in \mathcal{D}, \tag{1.1}$$

and assume throughout that $\mu_i > 0$ for all $i \in \mathcal{D}$. The number of individuals of type i in the n-th generation thus can be written as

$$Z_n^{(i)} = \sum_{k=1}^{Y_n} \mathbf{1}_{\{\chi_{n,k} = i\}},$$

where $\mathbf{1}_A$ stands for the indicator of the event A in the underlying probability space.

For $n \in \mathbb{Z}_+$, let $\mathcal{X}_n := \{X_{n,k}^{(i)} : k \in \mathbb{N}, i \in \mathcal{D}\}$, $\mathcal{C}_n = \{\chi_{n,k} : k \in \mathbb{N}\}$, and let $\mathcal{F}_n = \sigma(\mathcal{X}_k, \mathcal{C}_k : k \leq n) = \sigma(\mathcal{X}_k, \mathcal{C}_k, Z_k : k \leq n)$ be the σ-algebra of the "events up to time n". To describe vectors Z_n when Y_n is given, we introduce the sets

$$\mathcal{M}_y = \big\{(k_1, \ldots, k_d) \in \mathbb{Z}_+^d : \sum_{i \in \mathcal{D}} k_i = y\big\}, \qquad y \in \mathbb{Z}_+.$$

Transition kernel of the Markov chain Z_n on $\mathbb{Z}_+^d$ is formally defined by the following equations:

$$P\big(Y_{n+1} \leq y \,|\, \mathcal{F}_n, Z_n^{(i)} = k_i, \, i \in \mathcal{D}\big) = \prod_{i=1}^d \big(F_i(y)\big)^{k_i}, \qquad y, k_i \in \mathbb{Z}_+,$$

and, for $(k_1, \ldots, k_d) \in \mathcal{M}_y$,

$$P\big(Z_{n+1}^{(i)} = k_i, \, i \in \mathcal{D} | \mathcal{F}_n, Y_{n+1} = y\big) = \frac{y!}{k_1! \cdots k_d!} \mu_1^{k_1} \cdots \mu_d^{k_d}.$$

Throughout the paper we will assume that $Y_0 = 1$ and the color of the first particle is chosen at random according to the distribution defined in (1.1). To support all the random variables defined above, the underlying probability space can be chosen as for the multi-type Galton-Watson branching processes. It can be formally constructed using the recipe given in [6, Chapter VI].

2. Statement of results. Following [7], we will say that the chain $(Z_n)_{n\in\mathbb{Z}_+}$ belongs to the class R if $P(Y \to \infty) = 0$, and to the class T otherwise. Here and henceforth $\{Y \to \infty\}$ serves as a shortcut for $\{\lim_{n\to\infty} Y_n = +\infty\}$. First, we will obtain criteria for the classification of the Markov chain Z_n. It turns out (cf. [7, 8]) that the asymptotic behavior of Z_n is best understood in terms of the dynamics of the random sequence

$$L_n := \log Y_n, \qquad n \in \mathbb{Z}_+.$$

Observe that both $(Y_n)_{n\geq 0}$ and $(L_n)_{n\geq 0}$ are Markov chains. In particular, transition kernel of the latter is determined by the following family of distribution functions:

$$\begin{aligned} H_L(\xi,\eta) &:= P\big(L_{n+1} \leq \eta \,|\, L_n = \xi\big) = P\big(Y_{n+1} \leq e^\eta \,|\, Y_n = e^\xi\big) \\ &= E\Big[\prod_{i=1}^d \big(F_i(e^\eta)\big)^{Z_n^{(i)}} \,\Big|\, Y_n = e^\xi, (Z_n^{(i)})_{i=1}^d\Big] = \sum_{k_1,\dots,k_d} \frac{(e^\xi)!}{k_1!\cdots k_d!} \prod_{i=1}^d \big(\mu_i F_i(e^\eta)\big)^{k_i} \\ &= \Big(\sum_{i=1}^d \mu_i F_i(e^\eta)\Big)^{e^\xi} = \Big(1 - E\big[G_\chi(e^\eta)\big]\Big)^{e^\xi}, \end{aligned} \tag{2.1}$$

where χ is a generic random color in $\mathcal{D}$ with the same distribution as $\chi_{n,k}$, and $G_\chi(x) := 1 - F_\chi(x)$.

Taking in account that $\sum_{i=1}^d \mu_i F_i(\,\cdot\,) = E\big[F_\chi(\,\cdot\,)\big]$ is a distribution function and that $Z_n^{(i)}$ is conditionally independent of $\mathcal{F}_n$ given Y_n, the corresponding Lamperti's results in [7, 8] yield the following recurrence-transience criteria for Z_n. Let γ denote Euler's constant, that is $\gamma = \lim_{n\to\infty}\big(\sum_{k=1}^n k^{-1} - \log n\big) \approx 0.57721$.

Lemma 2.1. *Let $(Z_n)_{n\geq 0}$ be a CMBP that satisfies Assumption 2.2. Then*

(i) If $\limsup_{x\to\infty} x\cdot E[G_\chi(x)] < e^{-\gamma}$, the chain Z_n is in the class R.

(ii) If $\liminf_{x\to\infty} x\cdot E[G_\chi(x)] > e^{-\gamma}$, the chain Z_n is in the class T.

(iii) If for some constant $\theta \in \mathbb{R}$,

$$E[G_\chi(x)] = \frac{e^{-\gamma}}{x} + \frac{\theta + o(1)}{x\log x}, \qquad x \to \infty, \tag{2.2}$$

and $\theta < \frac{\pi^2 e^{-\gamma}}{12}$, the chain Z_n is in the class R.

(iv) If (2.2) holds with $\theta > \frac{\pi^2 e^{-\gamma}}{12}$, the chain Z_n is in the class T.

In the rest of the paper we focus on the growing to infinity processes in a "critical" regime. More precisely, we will impose the following basic set of assumptions.

Assumption 2.2. *Let Z_n be a CMBP as described in Section 1. Suppose that*

(A1) $P(X_{n,1}^{(i)} = 0) = 0$ for all $i \in \mathcal{D}$.

(A2) For all $i \in \mathcal{D}$ there exists the limit $\alpha_i := \lim_{x\to\infty} x\big(1 - F_i(x)\big)$.

(A3) $\alpha_i \in (0,\infty)$ for all $i \in \mathcal{D}$. Furthermore, $\beta \in (e^{-\gamma},\infty)$, where

$$\beta := \sum_{i=1}^d \mu_i\alpha_i = E[\alpha_\chi]. \tag{2.3}$$

In particular, these assumptions imply that

$$P(Y \to \infty) = 1.$$

The following proposition is immediate from (2.1).

PROPOSITION 2.3. *Let Assumption 2.2 hold. Then, for any* $\lambda \in \mathbb{R}$,

$$\lim_{\xi \to \infty} H_L(\xi, \xi + \lambda) = J_\beta(\lambda),$$

where $J_\beta(\lambda) := \exp\bigl(-\beta e^{-\lambda}\bigr)$, *with* β *being defined in* (2.3).

Heuristically, the proposition suggests that the asymptotic behavior of the Markov chain L_n is that of a transient random walk with the increments distributed according to J_β. This intuition was made precise in the "comparison lemma" of [7], where a coupling of two processes is explicitly constructed.

We now turn to a law of large numbers for $\log Z_n^{(i)}$, which can be obtained using the comparison with a random walk. First, observe that the law of large numbers for triangular arrays implies that for all $i \in \mathcal{D}$,

$$\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \mathbf{1}_{\{\chi_{n,k} = i\}} = \mu_i, \qquad P - \text{a. s.},$$

where μ_i is introduced in (1.1). By passing in this limit result to the random subsequence of integers Y_n we obtain

PROPOSITION 2.4. *Let Assumption 2.2 hold. Then for any* $i \in \mathcal{D}$,

$$\lim_{n \to \infty} \frac{Z_n^{(i)}}{Y_n} = \mu_i, \qquad P - \text{a. s.}$$

For a single-type process, the analogue of the following result is stated in Section 3 of [8] (p. 52). The multi-type version follows from its single-type prototype by applying the latter to the MBP associated with the distribution function $E[F_\chi(\cdot)]$ and using Proposition 2.4.

LEMMA 2.5. *Let Assumption 2.2 hold. Then for any* $i \in \mathcal{D}$,

$$\lim_{n \to \infty} \frac{1}{n} \log Z_n^{(i)} = \gamma + \log \beta, \qquad P - \text{a. s.}$$

We remark that the random sequence $\log Z_n^{(i)}$, $n \in \mathbb{Z}_+$, satisfies a large deviation principle under P (see, for instance, Remark (ii) in [21, p. 594]).

We will next state a central limit theorem for $\log Z_n^{(i)}$ in the transient case. The "borderline" case $\beta = e^{-\gamma}$ and (2.2) holds with $\theta > \frac{\pi^2 e^{-\gamma}}{12}$ was studied by Lamperti in [8] (see Theorem 2 in [8]) using the theory he developed in [11]. A related result corresponding to the case $F_i(x) \sim 1 - \alpha_i x^{-\varepsilon}$ with $\varepsilon \in (0,1)$ is obtained in [20, Theorem 2.5]. In what follows we consider the case $F_i(x) \sim 1 - \alpha_i x^{-1}$ with $\beta > e^{-\gamma}$.

Let $D\bigl([0,1]; \mathbb{R}^d\bigr)$ denote the set of $\mathbb{R}^d$-vector valued càdlàg functions on $[0,1]$, endowed with the Skorokhod J_1-topology. Let

$$S_n(t) = \sqrt{\frac{6}{\pi^2 n}} \cdot \sum_{k=1}^{[nt]} (L_k - \gamma - \log \beta), \qquad t \in [0,1], \tag{2.4}$$

THEOREM 2.6. *Let Assumption 2.2 hold. Suppose in addition that* (2.2) *holds for some* $\theta \in \mathbb{R}$. *Then* S_n *converges weakly in* $D\big([0,1];\mathbb{R}^d\big)$, *as* $n \to \infty$, *to a standard* d*-dimensional Brownian motion.*

The proof of the theorem given in the Appendix uses a standard martingale technique and relies on certain moment estimates obtained in [8] (this is where the full extent of the extra condition (2.2) is exploited). We remark that Theorem 2.6 is "in spirit" of the results of [8], and even though it is not stated there it seems quite likely that the result was known to Lamperti.

The main results of this paper are stated in the next two theorems. Let τ_n denote the set of colors present among individuals at generation n with the maximum number of offspring. That is,

$$i \in \tau_n \Longleftrightarrow Y_{n+1} = X^{(i)}_{n,k} \text{ for some } k = 1, \ldots, Z^{(i)}_n.$$

First, we study the asymptotic distribution of the colors in τ_n. The proof of the following theorem is given in Section 3.

THEOREM 2.7. *Let Assumption 2.2 hold. Then for all* $i \in \mathcal{D}$,

$$\lim_{n\to\infty} P(i \in \tau_n) = \frac{\mu_i \alpha_i}{\beta}.$$

Our next result confirms the intuition that the limiting distributions found in Theorem 2.7 coincide with the asymptotic frequencies of the colors of the most productive individuals who serve as ancestors of the next generation.

THEOREM 2.8. *Let Assumption 2.2 hold. Then for all* $i \in \mathcal{D}$,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \mathbf{1}_{\{i \in \tau_k\}} = \frac{\mu_i \alpha_i}{\beta}, \qquad P-\text{a.s.}$$

The proof of the theorem is given below in Section 4.

Using the same decoupling techniques as in the proof of Theorem 2.8 one can establish the asymptotic of the "time of the rule of the longest-reigning dynasty" in the CMBP. More precisely, for $n \in \mathbb{Z}_+$ and $i \in \mathcal{D}$ let

$$r^{(i)}_n = \max\Big\{k \le n+1 : \prod_{s=n-k+1}^{n} \mathbf{1}_{\{i \in \tau_s\}} = 1\Big\}$$

and

$$R^{(i)}_n = \max_{0 \le k \le n} r^{(i)}_k.$$

We have:

THEOREM 2.9. *Let Assumption 2.2 hold. Then for all* $i \in \mathcal{D}$,

$$\lim_{n\to\infty} \frac{R^{(i)}_n}{\log n} = -\frac{1}{\log\big(1 - \frac{\mu_i \alpha_i}{\beta}\big)}, \qquad P-\text{a.s.}$$

Furthermore, for all $i \in \mathcal{D}$,

$$\limsup_{n\to\infty} \frac{r^{(i)}_n}{\log n} = -\frac{1}{\log\big(1 - \frac{\mu_i \alpha_i}{\beta}\big)} \quad \text{while} \quad \liminf_{n\to\infty} \frac{r^{(i)}_n}{\log n} = 0, \qquad P-\text{a.s.}$$

The above theorem states in fact that $R_n^{(i)}$ and $r_n^{(i)}$ exhibit the same almost sure asymptotic behavior as, respectively, the longest run and the current run of heads in a series of independent coin tossing trials where the probability of heads equals $\frac{\mu_i\alpha_i}{\beta}$. See, for instance, [5] and [3, pp. 54-55] for the corresponding results for the coin tossing. Although the proof of Theorem 2.9 is very similar to that of Theorem 2.8, for the sake of completeness and reader's convenience it is included in Section 5.

3. Proof of Theorem 2.7. To prove the theorem, it suffices to show that

$$\lim_{y\to\infty} P(i\in\tau_n|Y_n=y)=\frac{\mu_i\alpha_i}{\beta}. \tag{3.1}$$

Indeed, it follows from $P(Y\to\infty)=1$ and (3.1) that $\lim_{n\to\infty} P(i\in\tau_n|Y_n)=\frac{\mu_i\alpha_i}{\beta}$, $P-$a. s., and hence

$$\lim_{n\to\infty} P(i\in\tau_n)=\lim_{n\to\infty} E\big[P(i\in\tau_n|Y_n)\big]=\frac{\mu_i\alpha_i}{\beta},$$

by the bounded convergence theorem.

For any integer $y\geq 2$, similarly to (2.1), we have

$$\begin{aligned}
&P(i\in\tau_n|Y_n=y)\\
&\quad=\sum_{x=1}^{\infty}\sum_{k_1,\ldots,k_d}\frac{y!}{k_1!\cdots k_d!}\prod_{j\neq i}\big(\mu_jF_j(x)\big)^{k_j}\big\{\big(\mu_iF_i(x)\big)^{k_i}-\big(\mu_iF_i(x-1)\big)^{k_i}\big\}\\
&\quad=\sum_{x=1}^{\infty}\Big\{\Big(\sum_j\mu_jF_j(x)\Big)^y-\Big(\sum_{j\neq i}\mu_jF_j(x)+\mu_iF_i(x-1)\Big)^y\Big\}.
\end{aligned}\tag{3.2}$$

Fix any $\varepsilon>0$. Then (compare with Proposition 2.3),

$$\begin{aligned}
&\sum_{x=1}^{\lfloor\varepsilon y\rfloor}\Big\{\Big(\sum_j\mu_jF_j(x)\Big)^y-\Big(\sum_{j\neq i}\mu_jF_j(x)+\mu_iF_i(x-1)\Big)^y\Big\}\\
&\qquad\leq\sum_{x=1}^{\lfloor\varepsilon y\rfloor}\Big\{\Big(\sum_j\mu_jF_j(x)\Big)^y-\Big(\sum_j\mu_jF_j(x-1)\Big)^y\Big\}\\
&\qquad=\Big(\sum_j\mu_jF_j\big(\lfloor\varepsilon y\rfloor\big)\Big)^y\to e^{-\beta/\varepsilon},\qquad\text{as } y\to\infty.
\end{aligned}\tag{3.3}$$

By Taylor's expansion formula, for any $A>0$ and $b\in(0,A)$, we have

$$(A-b)^y=A^y-byA^{y-1}+\frac{b^2}{2}y(y-1)(A-c)^{y-2}$$

for some $c\in(0,b)$. In particular,

$$byA^{y-1}-\frac{b^2y^2}{2}A^{y-2}\leq A^y-(A-b)^y\leq byA^{y-1}. \tag{3.4}$$

Therefore, letting

$$A(x):=\sum_j\mu_jF_j(x)\qquad\text{and}\qquad b_i(x):=F_i(x)-F_i(x-1), \tag{3.5}$$

we obtain from (3.2) and (3.4) that

$$\begin{aligned}
P(i\in\tau_n|Y_n=y) &\geq \sum_{x=1+\lfloor\varepsilon y\rfloor}^{\infty}\Big\{\Big(\sum_j \mu_j F_j(x)\Big)^y-\Big(\sum_{j\neq i}\mu_j F_j(x)+\mu_i F_i(x-1)\Big)^y\Big\}\\
&= \sum_{x=1+\lfloor\varepsilon y\rfloor}^{\infty} y\mu_i\big(A(x)\big)^{y-1}\cdot b_i(x)-\sum_{x=1+\lfloor\varepsilon y\rfloor}^{\infty}\mu_i\big(A(x)\big)^{y-2}\cdot\frac{\big(b_i(x)y\big)^2}{2}\\
&:= I_1(y,\varepsilon)-I_2(y,\varepsilon). \qquad (3.6)
\end{aligned}$$

To evaluate $I_1(y,\varepsilon)$ and $I_2(y,\varepsilon)$ we will exploit the following implication of a general property of regularly varying sequences (see, for instance, [1, 4]):

$$\begin{aligned}
\lim_{x\to\infty}\frac{x^2b_i(x)}{\alpha_i} &= \lim_{x\to\infty}\frac{x\cdot\big\{\big(1-F_i(x-1)\big)-\big(1-F_i(x)\big)\big\}}{1-F_i(x)}\\
&= \lim_{x\to\infty}\frac{\lfloor x\rfloor\cdot\big\{\big(1-F_i(\lfloor x\rfloor-1)\big)-\big(1-F_i(\lfloor x\rfloor)\big)\big\}}{1-F_i(\lfloor x\rfloor)}\\
&= \lim_{n\to\infty}\frac{n\cdot\big\{\big(1-F_i(n-1)\big)-\big(1-F_i(n)\big)\big\}}{1-F_i(n)}=1. \qquad (3.7)
\end{aligned}$$

Since $\big(A(x)\big)^{y-1}\cdot b_i(x)$ is a step function, for the first term in (3.6) we have

$$\begin{aligned}
I_1(y,\varepsilon) &= \sum_{x=1+\lfloor\varepsilon y\rfloor}^{\infty} y\mu_i\big(A(x)\big)^{y-2}\cdot\frac{\big(b_i(x)y\big)^2}{2}\\
&\geq \int_{\varepsilon+1/y}^{\infty} y^2\mu_i\big(A(yt)\big)^{y-1}\cdot b_i(yt)dt. \qquad (3.8)
\end{aligned}$$

By (3.7), $y^2\mu_i\big(A(yt)\big)^{y-1}\cdot b_i(yt)$ converges, as $y\to\infty$, to the continuous distribution function $\alpha_i\mu_i e^{-\beta/t}t^{-2}$. Furthermore, the convergence is uniform on compact intervals. Therefore, first truncating the integral in the last display and then taking the limit as $y\to\infty$,

$$\liminf_{y\to\infty} I_1(y,\varepsilon)\geq\lim_{M\to\infty}\int_\varepsilon^M\alpha_i\mu_i e^{-\beta/t}t^{-2}dt=\int_\varepsilon^\infty\alpha_i\mu_i e^{-\beta/t}t^{-2}dt. \qquad (3.9)$$

Similarly,

$$\begin{aligned}
I_2(y,\varepsilon)\cdot y &= \sum_{x=1+\lfloor\varepsilon y\rfloor}^{\infty} y\mu_i\big(A(x)\big)^{y-2}\cdot\frac{\big(b_i(x)y\big)^2}{2}\\
&\leq \int_\varepsilon^\infty \mu_i\big(A(yt)\big)^{y-2}\cdot\frac{\big(b_i(yt)y^2\big)^2}{2}dt,
\end{aligned}$$

and hence, by Fatou's lemma,

$$\limsup_{y\to\infty}\{I_2(y,\varepsilon)\cdot y\}\leq\int_\varepsilon^\infty\mu_i\alpha_i^2e^{-\beta/t}t^{-4}dt=\int_0^{\varepsilon^{-1}}\mu_i\alpha_i^2e^{-\beta s}s^2ds. \qquad (3.10)$$

Since $\varepsilon > 0$ is an arbitrary positive real, it follows from (3.9), (3.10), and (3.6) that

$$\liminf_{y\to\infty} P(i \in \tau_n | Y_n = y) \geq \int_0^\infty \alpha_i \mu_i e^{-\beta/t} t^{-2} dt = \int_0^\infty \alpha_i \mu_i e^{-\beta s} ds = \frac{\mu_i \alpha_i}{\beta}. \tag{3.11}$$

On the other hand, using (3.2), (3.3), and the upper bound in (3.4), we obtain that

$$\limsup_{y\to\infty} P(i \in \tau_n | Y_n = y) \leq \limsup_{\varepsilon\to 0} \limsup_{y\to\infty} I_1(y,\varepsilon).$$

To conclude the proof of the theorem, observe that, similarly to (3.8) and (3.9),

$$I_1(y,\varepsilon) \leq \int_\varepsilon^\infty y^2 \mu_i \big(A(yt)\big)^{y-1} \cdot b_i(yt) dt \to \int_\varepsilon^\infty \alpha_i \mu_i e^{-\beta/t} t^{-2} dt, \qquad \text{as } y \to \infty.$$

Since $\lim_{\varepsilon\to 0} \int_\varepsilon^\infty \alpha_i \mu_i e^{-\beta/t} t^{-2} dt = \int_0^\infty \alpha_i \mu_i e^{-\beta/t} t^{-2} dt = \frac{\mu_i \alpha_i}{\beta}$, this implies

$$\limsup_{y\to\infty} P(i \in \tau_n | Y_n = y) \leq \frac{\mu_i \alpha_i}{\beta}.$$

In view of (3.11) this completes the proof of (3.1). □

4. Proof of Theorem 2.8. Let

$$\xi_n^{(i)} := \mathbf{1}_{\{i\in\tau_n\}}, \qquad n \in \mathbb{Z}_+,\ i \in \mathcal{D},$$

and $\xi_n := (\xi_n^{(1)}, \ldots, \xi_n^{(d)})$. Observe that $K_n^{(i)} := \sum_{k=1}^n \xi_k^{(i)}$ is an additive functional of the Markov chain Y_n. More precisely,

$$\begin{aligned} &P\big(\xi_n^{(i)} = s, Y_{n+1} = z \big| (Y_k, \xi_k)_{k<n}, Y_n = y\big) \\ &\quad = P(Y_{n+1} = z \big| Y_n = y) \cdot P(\xi_n^{(i)} = s \big| Y_n = y, Y_{n+1} = z), \end{aligned} \tag{4.1}$$

for any $y, z \in \mathbb{N}$ and $s \in \{0, 1\}$. This implies that the random sequence formed by the triples (Y_n, Y_{n+1}, ξ_n), $n \in \mathbb{Z}_+$ is a Markov chain (in fact, a hidden Markov model with ξ_n playing role of the "observable variables") and that one can generate a realization of this sequence step by step, at each step first generating the value of Y_{n+1} given Y_n and then the value of ξ_n given Y_n and Y_{n+1}. Heuristically, in view of Proposition 2.3 and Theorem 2.7, this suggests that the asymptotic behavior of the sequence $\xi_n^{(i)}$ along a typical trajectory of the chain Y_n is similar to that of the sequence of outcomes of i.i.d. coin tossing trials with the probability of heads equal to $\frac{\mu_i \alpha_i}{\beta}$. In what follows we will derive a formal version of this heuristic argument and deduce from it the result stated in Theorem 2.8.

Note that, similarly to (3.2),

$$\begin{aligned} &P(\xi_n^{(i)} = 1 \big| Y_n = y, Y_{n+1} = z) \\ &\quad = \frac{\Big(\sum_j \mu_j F_j(z)\Big)^y - \Big(\sum_{j\neq i} \mu_j F_j(z) + \mu_i F_i(z-1)\Big)^y}{\Big(\sum_j \mu_j F_j(z)\Big)^y - \Big(\sum_j \mu_j F_j(z-1)\Big)^y}. \end{aligned}$$

For $M>0$ let $\Omega_M:=\{(y,z)\in\mathbb{N}^2: z>y^{2/3} \text{ and } y,z>M\}$. Recall (3.5). By virtue of (3.4), we have for all $(y,z)\in\Omega_M$ with M sufficiently enough (namely, large enough to ensure that the denominator in(4.2) and the numerator in (4.3) below are positive),

$$\begin{aligned} &P(\xi_n^{(i)}=1\big|Y_n=y,Y_{n+1}=z)\\ &\quad\le \frac{y\mu_i b_i(z)\big(A(z)\big)^{y-1}}{y\Big(\sum_j \mu_j b_j(z)\Big)\big(A(z)\big)^{y-1}-\frac{y^2}{2}\Big(\sum_j \mu_j b_j(z)\Big)^2\big(A(z)\big)^{y-2}}\\ &\quad=\frac{\mu_i b_i(z)A(z)}{\Big(\sum_j \mu_j b_j(z)\Big)A(z)-\frac{y}{2}\Big(\sum_j \mu_j b_j(z)\Big)^2}, \end{aligned} \tag{4.2}$$

and

$$\begin{aligned} &P(\xi_n^{(i)}=1\big|Y_n=y,Y_{n+1}=z)\\ &\quad\ge \frac{y\mu_i b_i(z)\big(A(z)\big)^{y-1}-\frac{y^2}{2}\big(\mu_i b_i(z)\big)^2\big(A(z)\big)^{y-2}}{y\Big(\sum_j \mu_j b_j(z)\Big)\big(A(z)\big)^{y-1}}\\ &\quad=\frac{\mu_i b_i(z)A(z)-\frac{y}{2}\big(\mu_i b_i(z)\big)^2}{\Big(\sum_j \mu_j b_j(z)\Big)A(z)}. \end{aligned} \tag{4.3}$$

It follows from (4.2) and (4.3) that

$$\begin{aligned} &\lim_{M\to\infty}\sup_{(y,z)\in\Omega_M} P(\xi_n^{(i)}=1\big|Y_n=y,Y_{n+1}=z)\\ &\quad=\lim_{M\to\infty}\inf_{(y,z)\in\Omega_M} P(\xi_n^{(i)}=1\big|Y_n=y,Y_{n+1}=z)=\frac{\mu_i\alpha_i}{\beta}. \end{aligned} \tag{4.4}$$

By Lemma 2.1, $P(Y_n>M \text{ i. o})=0$ for any $M>0$. Furthermore, (2.1) yields

$$P(Y_{n+1}\le y^{2/3}|Y_n=y)=\big(1-E\big[G_\chi(y^{2/3})\big]\big)^y\le \exp\big(-yE\big[G_\chi(y^{2/3})\big]\big),$$

which, using Assumption 2.2 and Lemma 2.5, implies by a "conditional version" of the Borel-Cantelli lemma (see, for instance, [3, p. 240]) that

$$\lim_{M\to\infty}\mathbf{1}_{\{(Y_n,Y_{n+1})\in\Omega_M\}}=1, \qquad P-\text{a.s.} \tag{4.5}$$

This completes the proof of Theorem 2.8 by using (4.1) and the comparison with a sequence of independent coin tossing trials with the probability of heads equal to $\frac{\alpha_i\mu_i}{\beta}\pm\varepsilon$ with an arbitrary small $\varepsilon>0$, whenever $(Y_n,Y_{n+1})\in\Omega_M$. □

5. Proof of Theorem 2.9. We will continue to use notations introduced in the course of the proof of Theorem 2.8. Similarly to the proof of the latter, the proof of Theorem 2.9 rests on an application of (4.4) and (4.5).

Fix any $i\in\mathcal{D}$ and $\varepsilon>0$ such that $\varepsilon<\max\big\{1-\frac{\mu_i\alpha_i}{\beta},\frac{\mu_i\alpha_i}{\beta}\big\}$. By virtue of (4.4) one can choose $M=M(\varepsilon)>0$ be large that

$$\sup_{(y,z)\in\Omega_{M(\varepsilon)}} P(\xi_n^{(i)}=1\big|Y_n=y,Y_{n+1}=z)\le\frac{\mu_i\alpha_i}{\beta}+\varepsilon$$

and

$$\inf_{(y,z)\in\Omega_{M(\varepsilon)}} P(\xi_n^{(i)}=1\big|Y_n=y,Y_{n+1}=z)\ge\frac{\mu_i\alpha_i}{\beta}-\varepsilon.$$

In view of (4.5), there exists $N \in \mathbb{N}$ such that $(Y_n, Y_{n+1}) \in \Omega_{M(\varepsilon)}$ for all $n > N$. Therefore, using a standard coupling technique one can infer from (4.1) that the lim sup and the lim inf, as $n \to \infty$, of both $R_n^{(i)}/\log n$ and $r_n^{(i)}/\log n$ are dominated with probability one by the corresponding quantities in a series of i.i.d. coin tossing trials, with the probability of heads equal to $\frac{\mu_i \alpha_i}{\beta} \pm \varepsilon$. Since $\varepsilon > 0$ is arbitrary, this implies the results in Theorem 2.9 which merely states that the lim sup's and lim inf's coincide with their counterparts for the biased coin tossing when the probability of heads is equal to $\frac{\mu_i \alpha_i}{\beta}$ (cf. [5]). $\square$

Appendix A. Proof of Theorem 2.6. For $m \in \mathbb{N}$ and $x \in \mathbb{R}$ let

$$\nu_m(x) = E\big[(L_{n+1} - L_n)^m \big| L_n = x\big],$$

and set

$$M_n = L_n - \sum_{k=0}^{n-1} \nu_1(L_k), \qquad n \in \mathbb{N}.$$

Then $(M_n, \mathcal{F}_n)_{n \in \mathbb{N}}$ is a martingale, and the corresponding martingale difference sequence is

$$\Delta_n := M_n - M_{n-1} = L_n - L_{n-1} - \nu_1(L_{n-1}), \qquad n \in \mathbb{N}.$$

To apply a standard functional CLT for martingales to M_n, it suffices to verify the following two conditions (see, for instance, [3, p. 414]):

(i) $A_n(t) := \frac{1}{n} \sum_{k=1}^{\lfloor nt \rfloor} E\big[\Delta_k^2 \big| \mathcal{F}_{k-1}\big] \to \pi^2/(6t)$ in probability for all $t \in [0,1]$, and

(ii) $B_n(\varepsilon) := \frac{1}{n} \sum_{k=1}^{n} E\big[\Delta_k^2 \mathbf{1}_{\{|\Delta_k| > \varepsilon\sqrt{n}\}} \big| \mathcal{F}_{k-1}\big] \to 0$ in probability for any $\varepsilon > 0$.

The following asymptotic formulas (as $x \to \infty$) for the first two conditional moments are obtained in [8] (see Lemmas 1 and 2 in [8]):

$$\nu_1(x) = \gamma + \log\beta + \frac{\theta}{\beta x} + o(x^{-1}) \quad \text{and} \quad \nu_2(x) = (\gamma + \log\beta)^2 + \frac{\pi^2}{6} + o(1),$$

where the constant parameter θ is introduced in (2.2).

Since $E\big[\Delta_k^2 \big| \mathcal{F}_{k-1}\big] = \nu_2(L_{k-1}) - \big(\nu_1(L_{k-1})\big)^2$, these two formulas together with Lemma 2.1 imply that $E\big[\Delta_k^2 \big| \mathcal{F}_{k-1}\big] = \frac{\pi^2}{6} + o(1)$ as $k \to \infty$, and hence the first condition above holds.

To verify the second condition we will use the fact (see [8, Lemma 1]) that $\nu_4(x)$ is uniformly bounded in a neighborhood of infinity. More precisely, let $K > 0$ be a positive real such that $\nu_4(x) < K$ for all $x = \log k$, $k \in \mathbb{N}$. By using first the Cauchy-Sschwarz inequality and then Chebyshev's bound $P(X > a) \le a^{-4} E[X^4]$,

$$\begin{aligned} E\big[\Delta_k^2 \mathbf{1}_{\{|\Delta_k > \varepsilon\sqrt{n}\}} \big| \mathcal{F}_{k-1}\big] &\le \Big(E\big[\Delta_k^4 \big| \mathcal{F}_{k-1}\big] \cdot P\big(|\Delta_k| > \varepsilon\sqrt{n} \big| \mathcal{F}_{k-1}\big)\Big)^{1/2} \\ &\le \frac{1}{\varepsilon^2 n} E\big[\Delta_k^4 \big| \mathcal{F}_{k-1}\big]. \end{aligned}$$

By Minkowski's inequality, the right-most term in the last display is bounded with probability one by $\varepsilon^{-2} n^{-1} \big(\nu_4(L_{k-1}) + \big(\nu_1(L_{k-1})\big)^4\big)^{1/4}$, which implies that the second condition above for the martingale differences is satisfied for Δ_n.

To complete the proof, observe that the asymptotic formula for $\nu_1(x)$ yields

$$\frac{1}{\sqrt{n}}\big(L_n - (\gamma + \log\beta)n\big) = \frac{1}{\sqrt{n}}\big(M_n + O(\log n)\big),$$

where $O(x)$ is a function such that $O(x)/x$ is bounded away from both zero and infinity in a neighborhood of infinity. □

REFERENCES

[1] R. Bojanic and E. Seneta, *A unified theory of regularly varying sequences*, Math. Z. 134 (1973), 91–106.

[2] S. Browne, E. G. Coffman, Jr., E. N. Gilbert, and P. E. Wright, *The gated infinite-server queue: Uniform service times*, SIAM J. Appl. Math. 52 (1992), 1751–1762.

[3] R. Durrett, *Probability: Theory and Examples*, 2nd ed., Duxbury Press, Belmont, CA, 1996.

[4] J. Galambos and E. Seneta, *Regularly varying sequences* Proc. Amer. Math. Soc. 41 (1973), 110–116.

[5] L. Gordon, M. F. Schilling, and M. S. Waterman, *An extreme value theory for long head runs*, Probab. Th. Rel. Fields 72 (1986), 279–287.

[6] T. E. Harris, (1963), *The Theory of Branching Processes.* Springer, Berlin.

[7] J. Lamperti, *Maximal branching processes and "long-range percolation"*, J. Appl. Probab. 7 (1970), 89–98.

[8] J. Lamperti, *Remarks on maximal branching processes*, Theory Probab. Appl. 17 (1972), 44–53.

[9] J. Lamperti, *Criteria for the recurrence or transience of stochastic processes I*, J. Math. Anal. Appl. 1 (1960), 314–330.

[10] J. Lamperti, *Criteria for stochastic processes II: passage-time moments*, J. Math. Anal. Appl. 7 (1963), 127–145.

[11] J. Lamperti, *A new class of probability limit theorems*, J. Math. Mech. 11 (1962), 749–772.

[12] A. V. Lebedev, *A double exponential law for maximal branching processes*, Discrete Math. Appl. 12 (2002), 415–420.

[13] A. V. Lebedev, *Maximal branching processes with non-negative values*, Theory Probab. Appl. 50 (2006), 482–488.

[14] A. V. Lebedev, *Tail behavior of the stationary distributions of the maximal branching processes*, Theory Probab. Appl. 54 (2010), 699–702.

[15] A. V. Lebedev, *Maximal branching processes*, Modern Probl. of Math. and Mech. 4 (2009), 93–106 (In Russian). An electronic version of the survey manuscript is available at `http://www.math.msu.su/department/probab/svodny2.pdf`.

[16] A. V. Lebedev, *Limit theorems for stationary distributons of two-type maximal branching processes*, Modern Probl. of Math. and Mech. 7 (2011), 29–38 (In Russian). Available electronically at `http://www.math.msu.su/department/probab/.cheb190.pdf`.

[17] A. V. Lebedev, *Multitype maximal branching processes with extreme value copulas*, Modern Probl. of Math. and Mech. 7 (2011), 39–49 (In Russian). Available electronically at `http://www.math.msu.su/department/probab/cheb190.pdf`.

[18] A. V. Lebedev, *Maximal branching processes with several types of particles*, Moscow Univ. Math. Bull. 67 (2012), 97–101.

[19] M. V. Menshikov, M. Vachkovskaia, and A. R. Wade, *Asymptotic behaviour of randomly reflecting billiards in unbounded tubular domains*, J. Stat. Phys. 132 (2008), 1097–1133.

[20] M. V. Menshikov and A. R. Wade, *Rate of escape and central limit theorem for the supercritical Lamperti problem*, Stoch. Proc. Appl. 120 (2010), 2078–2099.

[21] P. Ney and E. Nummelin, *Markov additive processes II. Large deviations*, Ann. Probab. 15 (1987), 593–609.