# TRADING COOKIES IN A GAMBLER'S RUIN SCENARIO

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Abstract. We consider several variations of a two-person game between a "buyer" and a "seller", whose major component is a random walk of the buyer on an interval of integers. We assume a gambler's ruin scenario, where in contrast to the classical version the walker (buyer) has the option of consuming "cookies", which when used, increase the probability of moving in the desired direction for the next step. The cookies are supplied to the buyer by the the second player (seller). We determine the equilibrium price policy for the seller and the equilibrium "cookie store" location. An initial motivation for this question is provided by the popular model of "cookie" or "excited" random walks.

## 1. Introduction

Consider the following modification of the classical one-dimensional gambler's ruin problem [8, 9], where the walker has the option of consuming a "cookie" which, when used, changes transition probabilities for the next step in a desired way. The cookies are supplied to the walker (called "buyer" in what follows) by a "seller". The buyer starts at point $a \in \mathbb{N}$ located between 0 and $b \in \mathbb{N}$, $b \geq 2$, and performs a nearest-neighbor random walk on the integer lattice $\mathbb{Z}$. If the buyer gets to point $b$ before 0 she is rewarded with $r > 0$ dollars while if she gets to 0 before $b$ she wins 0 dollars. Meanwhile, the seller sets up a shop somewhere on integer sites within the interval $(0, b)$. The seller sells a certain amount of cookies at a fixed price, and each cookie gives the buyer an instant probability boost in the direction of $b$. The walker thus always moves one step to the right with a fixed probability $p \in (0, 1)$ from regular sites and with a larger probability $p + \varepsilon \in [p, 1]$ from the store locations, if she consumes a cookie there. The buyer seeks to maximize her expected utility function, and she can either accept the help of the "cookie service" for the offered price or reject it. Informally speaking, the goal of this paper is to determine the equilibrium price for a cookie as well as the optimal (from the perspective of the seller) placement for the store.

From the probability theory point of view, the problems that we investigate can be described collectively as an attempt to measure the gain of the walker from exploiting a reinforcing mechanism represented by "cookies"; see for instance (13) below. It is natural to study this type of problem within a game-theoretic framework, where exact features of the reinforcing mechanism are determined through the interaction between the walker and a supplier. This is in contrast to the usual *excited* or *cookie random walk* [2, 15] (see [13] for

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an up-to-date review and references), where the walker, as a price-taker in a large market, has no effect on determining the parameters of the cookie environment.

More specifically, we will study subgame perfect equilibria for several variants of a two-person Stackelberg game [10], i.e. a game where the seller takes an action first while the buyer observes the move of the seller and then acts. An action of the leader (seller) consists of setting the price for a cookie and choosing the store location, and a strategy of the follower (buyer) constitutes of specifying a set of seller's actions in response to which she would be willing to consume a cookie upon each visit to the store. The variations of the game that we study in the paper differ by the form of the payoff function that is assigned to the buyer. For instance, in the basic form, considered in Section 3 the buyer seeks to maximize her expected earnings, while in Section 5 the buyer is risk-averse and thus also takes the extent of the risk involved in her decisions into consideration. Throughout this paper that the buyer makes a simple a-priori commitment to either purchase a cookie each time when the opportunity is present or to "ignore" the store permanently, rather than devises a strategy contingent on the realization of the random walk path. It can be shown that this assumption is actually not restrictive for a risk-neutral buyer in the basic game considered in Sections 2 and 3 while, say a risk-averse buyer considered in Section 5 might benefit from employing a policy conditional on the number of cookies currently available at the store. Intuitively, the attractiveness of the investment in cookies decreases for a risk-averse buyer as the amount of available cookies is decreasing and hence the risk involved in the investment is increasing in the course of the game. We remark that the optimization problem which the seller faces is somewhat similar to that of a monopoly whose market is a spatially non-homogeneous Hotelling beach [1, 11] with demand curve varying randomly across the population.

The game can serve as a simplified model to explore the relationship between economic agents in a risky environment, for instance a firm in an innovative and competitive segment of a hi-tech industry and an experienced consulting company. The firm (buyer) seeks to reduce uncertainty and increase the expected profit by investing in the consulting service at a "bottleneck" point of its production line, while the consultant (seller) wants to optimize the configuration and the price of its service package.

We next define the underlying (buyer's) random walk. Fix any $p \in (0,1)$ and let $q = 1-p$. Fix the store placement $n \in \mathbb{N}$ and the cookie strength $\varepsilon \in [0,q]$. Let $X_k$ and $m_k$ the location of the walker and the number of cookies available at the store at time $k \in \mathbb{N}\cup\{0\}$, respectively. Formally, the pairs $(X_k, m_k)_{k\geq 0}$ form a Markov chain on $\mathbb{Z} \times \big(\mathbb{N} \cup \{0,\infty\}\big)$ with transition kernel given by

$$\begin{aligned}\mathbb{P}_n(X_{k+1} = j, m_{k+1} = m | X_k = i, m_k = l) = \\ = \begin{cases} p + \mathbf{1}_{\{i=n,l>0\}} \cdot \varepsilon & \text{if} \quad j = i+1, m = l-1 \\ q - \mathbf{1}_{\{i=n,l>0\}} \cdot \varepsilon & \text{if} \quad j = i-1, m = l-1 \\ 0 & \qquad\quad \text{otherwise.} \end{cases}\end{aligned}$$

Here we use the standard convention that $\infty - 1 = \infty$ and denote by $\mathbf{1}_A$ be the indicator function of the event $A$. That is, $\mathbf{1}_A$ is either 1 or 0 according to whether $A$ occurred or not.

The parameters $m_0, p$, and $\varepsilon$ (as well as the parameters $a, b$, and $r$ introduced later in this section) are considered as given exogenous variables. Let $\mathbb{P}_{a,n}$ denote the probability measure on the path of the random walk associated with the buyer starting with probability one at $X_0 = a$, while the cookie store is placed at $n$. Let $\mathbb{E}_{a,n}$ be the expectation operator associated with the probability measure $\mathbb{P}_{a,n}$. We will denote by $P_a$ and $E_a$, respectively, the distribution and the expectation associated with the corresponding usual random walk, i.e. the one with $\varepsilon = 0$.

Choose any $b \in \mathbb{N}$, $b > n$, and let

$$\mathcal{T} = \min\{T_0, T_b\} \qquad \text{with} \qquad T_j = \inf\{k : X_k = j\},\ j \in \mathbb{Z}. \tag{1}$$

Assume that $X_0 \in (0, b)$ with probability one and that 0 and $b$ are absorbing points for the buyer's random walk, that is $\mathbb{P}(X_{\mathcal{T}+k} = X_{\mathcal{T}} \text{ for } k \geq 0) = 1$. If the buyer visits $b$ before 0 she is rewarded with $r > 0$ dollars, otherwise she receives 0 dollars. The strategies of the seller are represented by the pairs $(c, n)$, where $c$ denotes the price for a cookie which remains fixed during the game (cf. Remark 3.1 below). The strategies of the buyer are represented by the mappings of the pairs $(c, n)$ into the set $\{\mathbb{P}_{a,n}, P_a\}$, where $\mathbb{P}_{a,n}$ means the decision to use the cookies whereas $P_a$ means the decision to ignore the cookie store and proceed as a usual random walk.

The usual cookie random walk model allows cookies to be located at each site of the integer lattice. Our assumption that all the cookies are placed in the same location makes the buyer's random walk into a nearly Markovian process, and thus ensures a more easily treatable model. In particular, the exit probabilities $\mathbb{P}_{a,n}(T_b < T_0)$ can be explicitly computed. Random walks defined by, in a sense small, local perturbations of $P_a$, have been considered by many authors. In the context of excited random walks see for instance [7, 12]. It turns out that even though our underlying random walk does not exhibit as interesting a deviation from the corresponding regular random walk as the excited random walks do (compare for instance Theorem 3.5 and Remark 7.4-(a) below), the perturbation by a single cookie store produces many interesting quantitative effects, and its influence is not negligible even when $b$ is taken to infinity. For instance, according to Theorem 3.5 either a supply of cookies $m_0$ or a reward $r$ of the same order as $b$ allow the seller to maintain expected revenue when $b$ goes to infinity, ceteris paribus. The structure of the equilibrium cost is quite curious, and is discussed in detail in Remark 3.1.

The rest of the paper is organized as follows. In Sections 2 and 3 we study the basic version of the game which is described above. In Section 4 we consider a walker with initial position uniformly distributed over the interval $[1, b-1]$. To further explore which factors are dominant in designing the equilibrium strategies of the players, we then consider buyers with utility functions different from the expected value of their earnings. In Section 5 we consider a risk-averse buyer whereas in Section 6 we study the game where the buyer is concerned not only with the expected reward but also with the expected time it takes to achieve the reward. For comparison, we then consider in Section 7 a variant of the game with the 1-excited random walk, that is when exactly one cookie is placed everywhere on $\mathbb{Z}$. Finally, some concluding remarks are given in Section 8.

## 2. Basic game: Preliminaries

In this section and the next section, we consider the following scenario. Fix any integer $b \geq 2$. There is one buyer, starting the random walk at a fixed integer point $X_0 = a$ between 0 and $b$. There is one seller, who is seeking to maximize her expected revenue by choosing store's location $n$ and the price of a cookie $c$. There is no production cost for the seller. The seller has $m$ cookies to sell to the buyer, either $m \in \mathbb{N}$ or $m = \infty$. The seller charges the same price for each cookie. The walker has an option to ignore (not to buy) cookies if the price is not attractive. If the buyer chooses to use the cookie she moves on $\mathbb{Z}$ according to

$\mathbb{P}_{a,n}$, otherwise her motion is according to $P_a$. The walker seeks to maximize her expected earnings.

Thus possible actions of the seller are represented by the collection of feasible pairs $(c, n)$, while possible strategies of the buyer are represented by the set of functions

$$B(c, n) : [0, \infty) \times \{1, \ldots, b - 1\} \to \{\mathbb{P}_{a,n}, P_a\}.$$

The walker chooses to consume or not to consume the cookies which are supplied by the seller at site $n$ for the marginal price $c$, according to whether $B(c, n) = \mathbb{P}_{a,n}$ or not. We next give a formal definition of the game. Let $\Omega_b := \{2, \ldots, b - 1\}$ be the set of feasible store's locations. Recall $\mathcal{T}$ from (1) and define

$$\eta_n = \sum_{i=0}^{\mathcal{T}} \mathbf{1}_{\{X_i = n\}} \qquad \text{and} \qquad \eta_{n,m} = \min\{\eta_n, m\}. \tag{2}$$

That is $\eta_{n,m}$ is the total number of "successful visits" to the store (i.e., visits when the cookies are still available) by the random walk before the absorption at either 0 or $b$.

**Definition 2.1** (Game $\Gamma_{m,a}$).

- *$\Gamma_{m,a}$ is a two-person Stackelberg game (the first player takes an action, the second player observes the action and then moves). The (random) payoffs of the players depends on the realization of the underlying random walk (action of Nature). In order to determine their strategies, the players consider the corresponding expected payoffs.*
- *The strategy set of the first player (seller) is $\mathcal{S} := [0, \infty) \times \Omega_b$. Each pair $(c, n) \in \mathcal{S}$ specifies the cookie's price $c > 0$ and the store's location $n \in \Omega_b$ chosen by the seller.*
- *The seller moves first and communicates her action to the second player (buyer). Then the second player (buyer) determines her strategy, and starts the random walk.*
- *Nature determines realization of the random walk.*
- *The strategies of the buyer are functions $B : [0, \infty) \times \Omega_b \to \{\mathbb{P}_{a,n}, P_a\}$. The buyer will either consume a cookie priced $c \in [0, \infty)$ upon each her visit to a store located at $n \in \Omega_b$ or will refrain from ever making a purchase, according to whether $B(c, n) = \mathbb{P}_{a,n}$ or $B(c, n) = P_a$, respectively.*
- *For given cookie price $c > 0$, store location $n \in (0, b)$, the response strategy $B$ of the buyer, and the realization of buyer's random walk, player's payoffs are defined as follows:*

$$u_{c,n,B} := r \cdot \mathbf{1}_{\{T_b < T_0\}} - c \cdot \eta_{n,m} \cdot \mathbf{1}_{\{B(c,n) = \mathbb{P}_{a,n}\}} \qquad \text{(buyer)}$$

*and*

$$v_{c,n,B} := c \cdot \eta_{n,m} \cdot \mathbf{1}_{\{B(c,n) = \mathbb{P}_{a,n}\}} \qquad \text{(seller)}.$$

Notice that the payoffs are random and depend on the realization of the underlying random walk. We next specify the game solution concept invoking expected utilities which is used throughout the paper. Denote by $\mathcal{B}$ the collection of all functions from $\mathcal{S}$ to $\{\mathbb{P}_{a,n}, P_a\}$. For any pair of strategies $S = (c, n) \in \mathcal{S}$ and $B \in \mathcal{B}$ denote by $U_{S,B}$ and $V_{S,B}$, respectively, the expected payoffs of the buyer and the seller who play according to the strategy profile $(S, B)$. That is,

$$U_{S,B} = \begin{cases} \mathbb{E}_{a,n}(u_{c,n,B}) & \text{if} \quad B(c, n) = \mathbb{P}_{a,n}, \\ E_a(u_{c,n,B}) & \text{if} \quad B(c, n) = P_a, \end{cases}$$

and

$$V_{S,B} = \begin{cases} \mathbb{E}_{a,n}(v_{c,n,B}) = c \cdot \mathbb{E}_{a,n}(\eta_{n,m}) & \text{if} \quad B(c,n) = \mathbb{P}_{a,n}, \\ E_a(v_{c,n,B}) = 0 & \text{if} \quad B(c,n) = P_a, \end{cases}$$

In the next sections we will consider several variants of the above game with different payoff functions for the seller. For the basic game $\Gamma_{m,a}$ we have

$$U_{S,B} = \begin{cases} r \cdot \mathbb{P}_{a,n}(T_b < T_0) - c \cdot \mathbb{E}_{a,n}(\eta_{n,m}) & \text{if} \quad B(c,n) = \mathbb{P}_{a,n}, \\ r \cdot P_a(T_b < T_0) & \text{if} \quad B(c,n) = P_a, \end{cases}$$

**Definition 2.2** ([10])**.** *A subgame perfect equilibrium of $\Gamma_{m,a}$ is defined as a profile of strategies $\big(S^*, B^*\big) \in \mathcal{S} \times \mathcal{B}$ such that*

$$U_{S^*,B^*} \geq U_{S,B^*} \qquad \forall\ S \in \mathcal{S}, \tag{3}$$

*and*

$$V_{S,B^*} \geq V_{S,B} \qquad \forall\ S \in \mathcal{S}, B \in \mathcal{B}. \tag{4}$$

More generally, (3) and (4) define a subgame perfect equilibrium for any Stackelberg (the leader moves first, the follower observes her action and then moves) two-person game with arbitrary payoffs $(U, V)$ and strategy sets $(\mathcal{S}, \mathcal{B})$. Throughout the paper we use "equilibrium" as a synonymous to the "subgame perfect equilibrium". The following remark is in order.

**Remark 2.3.** *The assumption that neither the seller can change the price during the course of the game, nor can the buyer reconsider her decision upon an arrival to the store might seem to be restrictive. However, it turns out that in fact this assumption does not put a real constraint on the strategies of the players. This is discussed in Remark 3.1 below, and is due to the fact that the equilibrium price for a cookie is actually independent of $m$, as long as $m > 0$.*

For given cookie price $c > 0$ and store location $n \in (0, b)$ let $U_c(a, n)$ denote the expected payoff of the buyer who uses the cookies. That is,

$$U_c(a, n) := \mathbb{E}_{a,n}(u_{c,n,\mathbb{P}_{a,n}}) = r \cdot \mathbb{P}_{a,n}(T_b < T_0) - c \cdot \mathbb{E}_{a,n}(\eta_{n,m}).$$

The corresponding expected revenue of the seller is denoted by $V_c(a, n)$. That is,

$$V_c(a, n) := \mathbb{E}_{a,n}(v_{c,n,\mathbb{P}_{a,n}}) = c \cdot \mathbb{E}_{a,n}(\eta_{n,m}). \tag{5}$$

Thus, for fixed $a$ and $n$, the seller will set the maximal possible price for each cookie. The maximal price $c^*(a, n)$ that the buyer would be willing to pay for a cookie is determined from the equation

$$U_{c^*(a,n)}(a, n) = r \cdot P_a(T_b < T_0), \tag{6}$$

where the right-hand side is the expected payoff of the buyer without cookie reinforcement. It will turn out that this equation has a unique solution for any feasible pair $(a, n)$. The optimal location of the store $n^* = n^*(a)$ is then given as the solution of the optimization problem

$$V_{c^*(a,n^*)}(a, n^*) = \max_{n \in \Omega_b} V_{c^*(a,n)}(a, n). \tag{7}$$

We will show below (see Lemma 3.2) that $n^*(a) = a$ is the unique solution to (7). The price is determined from equation (6), which can be alternatively written as

$$c^*(a,n) = \frac{r \cdot \mathbb{P}_{a,n}(T_b < T_0) - r \cdot P_a(T_b < T_0)}{\mathbb{E}_{a,n}(\eta_{n,m})} \tag{8}$$

The core result of this section is the following observation.

**Theorem 2.4.** *For a fixed store location $n$, the maximal price $c^*(a,n)$ that the buyer would be willing to pay for a cookie in a game $\Gamma_{m,a}$ is independent of the value of $a$.*

*Proof.* Given a store location $n \in (0,b)$, the maximal price $c^*(a,n)$ is determined from (6). If $n \geq a$, we have $U_{c^*(a,n)}(a,n) = P_a(T_n < T_0) \cdot U_{c^*(a,n)}(n,n)$. Thus, using the strong Markov property, identity (6) yields

$$P_a(T_n < T_0) \cdot U_{c^*(a,n)}(n,n) = rP_a(T_b < T_0) = rP_a(T_n < T_0)P_n(T_b < T_0),$$

which implies

$$U_{c^*(a,n)}(n,n) = rP_n(T_b < T_0). \tag{9}$$

If $n \leq a$, we have $U_{c^*(a,n)}(a,n) = P_a(T_n < T_b) \cdot U_{c^*(a,n)}(n,n) + rP_a(T_b < T_n)$. Hence, using again the strong Markov property, identity (6) yields

$$\begin{aligned} P_a(T_n < T_b) \cdot U_{c^*(a,n)}(n,n) + rP_a(T_b < T_n) &= rP_a(T_b < T_0) \\ &= rP_a(T_n < T_b)P_n(T_b < T_0) + rP_a(T_b < T_n), \end{aligned}$$

which also leads to (9) in the case $n \leq a$. This completes the proof of the theorem, since equation (9) for $c^*(a,n)$ is independent of the value of $a$. $\square$

Let $\rho = \frac{q}{p}$ and recall $T_n$ from (1). For any integer $n \in [0,a]$ we have [8, p. 274]:

$$P_a(T_n < T_b) = \begin{cases} \dfrac{\rho^b - \rho^a}{\rho^b - \rho^n} & \text{if } \; p \neq q \\ \dfrac{b-a}{b-n} & \text{if } \; p = q. \end{cases}$$

We conclude this section with the computation of $\mathbb{E}_{a,n}(\eta_{n,m})$. Let $J_n$ (respectively, $K_n$) denote the probability of returning (not returning) to $n$ after consuming a cookie at $n$:

$$\begin{aligned} K_n \;&=\; 1 - J_n = R_n + L_n, \\ &\quad\text{where } R_n := (p+\varepsilon)P_{n+1}(T_b < T_n) \text{ and } L_n := (q-\varepsilon)P_{n-1}(T_0 < T_n). \end{aligned} \tag{10}$$

We have

$$\mathbb{E}_{a,n} = \mathbb{P}_{a,n}(T_n < \mathcal{T}) \cdot \mathbb{E}_{n,n}, \quad \mathbb{E}_{n,n} = (1-J_n)\left[\sum_{i=1}^{m-1} iJ_n^{i-1}\right] + mJ_n^{m-1} = \frac{1-J_n^m}{1-J_n}. \tag{11}$$

Throughout the paper, we use the convention that if $m = \infty$ then $J^m = mJ^m = 0$ for any constant $J \in (0,1)$ in our calculations.

## 3. Basic Game: Main Results

Our main results in this section are collected in Theorem 3.3 which includes explicit results for the values of the equilibrium price and store location in $\Gamma_{m,a}$. Theorem 3.4 extends the results to the infinite interval $(-\infty, 0]$ when $\rho < 1$.

In Theorem 3.5, for the case $p = q$, we find a natural scaling of the parameters $r$ and $m$ when $b$ goes to infinity. In particular, this theorem shows that an increase in cookie supply proportional to the change in the value of $b$ allows the seller to maintain her revenue. In other words, the effect of a single store with an adequate cookie supply on the underlying random walk cannot be neglected, even asymptotically.

Finally, Theorem 3.6 establishes monotonicity of the seller's equilibrium revenue as a function of the parameter $\varepsilon$. The latter result is interesting because even though the higher quality (i.e., higher strength of the cookie, $\varepsilon$) means higher price for a cookie, it also means that the buyer is expected to finish the game sooner and hence implies the drop in the expected amount of cookies sold.

We will frequently make use of the "decomposition according to the first step" arguments for the underlying Markov chain $(X_k, m_k)_{k\geq 0}$, in particular exploiting the following equality:

$$P_k(\mathcal{T} = T_x) = pP_{k+1}(\mathcal{T} = T_x) + qP_{k-1}(\mathcal{T} = T_x) \tag{12}$$

with $x \in \{0, b\}$ and $n = 1, \ldots, b-1$. The recurrence relationship (12) can be equivalently stated as the martingale-type identity $E(Z_{k+1}|X_k) = Z_k$ for $Z_k = P_{X_k}(\mathcal{T} = T_x)$. We will denote $c^*\big(a, n^*(a)\big)$ (which will turn out to be $c^*(a, a)$, see Lemma 3.2 below) by $c^*(a)$ and refer to this value as the equilibrium price of the cookie in $\Gamma_{m,a}$. Thus, according to (8),

$$c^*(a) = \frac{r \cdot \mathbb{P}_{a,n^*(a)}(T_b < T_0) - r \cdot P_a(T_b < T_0)}{\mathbb{E}_{a,n^*(a)}(\eta_{n^*(a),m})} \tag{13}$$

We next compute the equilibrium price $c^*(a)$. We will first calculate $U_c(n, n)$ for general $c > 0$ and $n \in (0, b)$. To simplify notation we will abbreviate $U_c(n, n)$ to $U_c(n)$ and $V_c(n, n)$ to $V_c(n)$. Recall (10). We have:

$$\begin{aligned} U_c(n) \;=\; & \sum_{k=1}^{m-1} \big[(r - kc)R_n J_n^{k-1} - kcL_n J_n^{k-1}\big] \\ & + J_n^{m-1}\Big[r\big[(p+\varepsilon)P_{n+1}(T_b < T_0) + (q-\varepsilon)P_{n-1}(T_b < T_0)\big] - mc\Big]. \end{aligned} \tag{14}$$

It follows from (12), that

$$\begin{aligned} U_c(n) = & \frac{R_n r(1 - J_n^{m-1})}{K_n} - c\left[\frac{(m-1)J_n^m - mJ_n^{m-1} + 1}{K_n} + mJ_n^{m-1}\right] \\ & + J_n^{m-1} r\big[P_n(T_b < T_0) + \varepsilon\big(P_{n+1}(T_b < T_0) - P_{n-1}(T_b < T_0)\big)\big]. \end{aligned}$$

Therefore, using (6) and the following identity (recall that $K_n = 1 - J_n$):

$$\frac{(m-1)J_n^m - mJ_n^{m-1} + 1}{K_n} + mJ_n^{m-1} = \frac{1 - J_n^m}{K_n},$$

we obtain

$$\frac{c^*(n)(1 - J_n^m)}{K_n} = \frac{R_n r(1 - J_n^{m-1})}{K_n}$$

$$+J_n^{m-1} r\big[P_n(T_b < T_0) + \varepsilon\big(P_{n+1}(T_b < T_0) - P_{n-1}(T_b < T_0)\big)\big] - rP_n(T_b < T_0).$$

Thus $c^*(n)$ can be expressed as

$$c^*(n) = \frac{c_1(n) + c_2(n)}{1 - J_n^m},$$

where

$$\begin{aligned}
c_1(n) &= r\big[R_n - K_n \cdot P_n(T_b < T_0)\big] \\
c_2(n) &= rJ_n^{m-1}\Big[K_n\big[P_n(T_b < T_0) + \varepsilon\big(P_{n+1}(T_b < T_0) - P_{n-1}(T_b < T_0)\big)\big] - R_n\Big] \\
&= J_n^{m-1}\Big[r\varepsilon K_n\big[P_{n+1}(T_b < T_0) - P_{n-1}(T_b < T_0)\big] - c_1(n)\Big].
\end{aligned}$$

We have:

$$\begin{aligned}
c_1(n) &= r(p+\varepsilon)P_{n+1}(T_b < T_n) \\
&\qquad - r\big[(p+\varepsilon)P_{n+1}(T_b < T_n) + (q-\varepsilon)P_{n-1}(T_0 < T_n)\big] \cdot P_n(T_b < T_0) \\
&= r(p+\varepsilon)P_{n+1}(T_b < T_n) \cdot P_n(T_0 < T_b) - r(q-\varepsilon)P_{n-1}(T_0 < T_n) \cdot P_n(T_b < T_0) \\
&= r\big[pP_{n+1}(T_b < T_n) \cdot P_n(T_0 < T_b) - qP_{n-1}(T_0 < T_n) \cdot P_n(T_b < T_0)\big] \\
&\qquad + \varepsilon r\big[P_{n+1}(T_b < T_n) \cdot P_n(T_0 < T_b) + P_{n-1}(T_0 < T_n) \cdot P_n(T_b < T_0)\big] \\
&:= c_{1,1}(n) + c_{1,2}(n),
\end{aligned}$$

where the last equality serves as the definition of $c_{1,1}(n)$ and $c_{1,2}(n)$. Using the Markov property and (12), we obtain

$$\begin{aligned}
c_{1,1}(n) &= r\big[p\big(1 - P_{n+1}(T_n < T_b)\big) \cdot P_n(T_0 < T_b) - q\big(1 - P_{n-1}(T_n < T_0)\big) \cdot P_n(T_b < T_0)\big] \\
&= r\big[p\big(P_n(T_0 < T_b) - P_{n+1}(T_0 < T_b)\big) - q\big(P_n(T_b < T_0) - P_{n-1}(T_b < T_0)\big)\big] \\
&= r\big[p\big(P_n(T_0 < T_b) - P_{n+1}(T_0 < T_b)\big) - q\big(P_{n-1}(T_0 < T_b) - P_n(T_0 < T_b)\big)\big] = 0,
\end{aligned}$$

and

$$\begin{aligned}
c_{1,2}(n) &= \varepsilon r\big[\big(1 - P_{n+1}(T_n < T_b)\big) \cdot P_n(T_0 < T_b) + \big(1 - P_{n-1}(T_n < T_0)\big) \cdot P_n(T_b < T_0)\big] \\
&= \varepsilon r\big[1 - P_{n+1}(T_n < T_b)P_n(T_0 < T_b) - P_{n-1}(T_b < T_0)\big] \\
&= \varepsilon r\big[P_{n-1}(T_0 < T_b) - P_{n+1}(T_0 < T_b)\big] = \varepsilon r\big[P_{n+1}(T_b < T_0) - P_{n-1}(T_b < T_0)\big].
\end{aligned}$$

Further,

$$\begin{aligned}
c_2(n) &= J_n^{m-1}\Big[r\varepsilon K_n\big[P_{n+1}(T_b < T_0) - P_{n-1}(T_b < T_0)\big] - c_1(n)\Big] \\
&= -J_n^{m-1} r\varepsilon(1 - K_n) \cdot \big[P_{n+1}(T_b < T_0) - P_{n-1}(T_b < T_0)\big] \\
&= -J_n^m r\varepsilon \cdot \big[P_{n+1}(T_b < T_0) - P_{n-1}(T_b < T_0)\big].
\end{aligned}$$

Thus

$$c^*(n) = \frac{c_1(n) + c_2(n)}{1 - J_n^m} = r\varepsilon\big[P_{n+1}(T_b < T_0) - P_{n-1}(T_b < T_0)\big], \tag{15}$$

which yields

$$c^*(n)=\begin{cases}\dfrac{r\varepsilon\rho^n(\rho^{-1}-\rho)}{1-\rho^b} & \text{if } \ p\neq q\\ \dfrac{2\varepsilon r}{b} & \text{if } \ p=q.\end{cases} \tag{16}$$

**Remark 3.1.** *Remarkably, $c^*(n)$ is independent of $m$. Furthermore, (15) implies that, given the store location $n$, the equilibrium price $c^*(n)$ is the unique positive constant $c$ which makes $M_k = r\cdot P_{X_k}(T_b<T_0)-c\cdot\sum_{i=0}^{\min\{k,m\}}\mathbf{1}_{\{X_i=n\}}$ into a martingale under $\mathbb{P}_{a,n}$ with respect to the natural filtration $\mathcal{F}_k=\sigma\big((X_i,\mathbf{y}_i):i\leq k\big)$ of the Markov chain formed by the pairs $(X_k,\mathbf{y}_k)$. Notice that $P_{X_k}(T_b<T_0)$, $k\geq 0$, is a martingale with respect to its natural filtration under $P_a$, but not under $\mathbb{P}_{a,n}$.*

*The independence of $c^*(n)$ of $m$ is an implication of the Markov property and our assumption that the buyer is risk neutral, and thus is concerned only with the expected value of her earnings. Using the Markov property, equation (8) can be rewritten as*

$$c^*(n)=\frac{r\cdot\mathbb{P}_{n,n}(T_b<T_0)-r\cdot P_n(T_b<T_0)}{\mathbb{E}_{n,n}(\eta_{n,m})}.$$

*The difference $\mathbb{P}_{n,n}(T_b<T_0)-r\cdot P_n(T_b<T_0)$ can be decomposed into the sum of the expected gain from using 1 cookie until either returning to the store or finishing the game. Notice that between two successive visits to the store, the buyer's motion is described by the measure $P_a$. Given the possibility to reconsider her decision to use cookies upon the next return to the store, the buyer would evaluate her expected earnings again according to (6). Therefore, using the Markov property, the buyer's gain from using one cookie is, up to the multiplicative factor $r$,*

$$\begin{aligned}(p+\varepsilon)P_{n+1}(T_b<T_n)+\big[(p+\varepsilon)P_{n+1}(T_n<T_b)+(q-\varepsilon)P_{n-1}(T_n<T_0)\big]P_n(T_b<T_0)\\ -pP_{n+1}(T_b<T_n)+\big[pP_{n+1}(T_n<T_b)+qP_{n-1}(T_n<T_0)\big]P_n(T_b<T_0)\\ =\varepsilon P_{n+1}(T_b<T_n)+\varepsilon\big[P_{n+1}(T_n<T_b)-P_{n-1}(T_n<T_0)\big]P_n(T_b<T_0)\\ =\varepsilon\big[P_{n+1}(T_b<T_0)-\varepsilon P_{n-1}(T_b<T_0)\big],\end{aligned}$$

*in agreement with (15).*

*As we already mentioned in Remark 2.3, the fact that $c^*(n)$ is independent of $m$ implies that the buyer would not change her decision regarding the use of cookies during the course of the game. This implies that the equilibrium price policy for the seller is to maintain a fixed cookie price throughout the game even if the buyer were allowed to change it according to the number of the cookies left in stock. The fact that the price $c^*(a)$ is a multiple of the boost $\varepsilon$ is not surprising, though it is not trivial a priori and interesting.*

We are now in a position to find the seller's expected revenue with the store located at $n$. For an arbitrary $c>0$, write, using (11):

$$V_c(n)=c\cdot\mathbb{E}_{n,n}(\eta_{n,m})=\frac{c(1-J_n^m)}{1-J_n}. \tag{17}$$

Recall the convention $J_n^m=mJ_n^m=0$ for $m=\infty$. We have:

**Lemma 3.2.** *For a fixed starting point of the buyer $a$, the unique subgame perfect location of the cookie store is at $n^*(a)=a$.*

*Proof.* The strong Markov property and Theorem 2.4 imply that

$$V_{c^*(a,n)}(a,n) = P_a(T_n < \mathcal{T}) \cdot V_{c^*(n)}(n), \tag{18}$$

where $\mathcal{T}$ is defined in (1). For real $x \in (0,b)$ define

$$J(x) = \begin{cases} (p+\varepsilon)\left(1 - \dfrac{1}{b-x}\right) + (q-\varepsilon)\left(1 - \dfrac{1}{x}\right) & \text{if } \ p = q \\ (p+\varepsilon)\left(\dfrac{\rho^b - \rho^{x+1}}{\rho^b - \rho^x}\right) + (q-\varepsilon)\left(\dfrac{\rho^{x-1} - 1}{\rho^x - 1}\right) & \text{if } \ p \neq q. \end{cases}$$

For real numbers $x \in (0,b)$ define

$$f_{\rho,m}(x) = \begin{cases} \dfrac{1}{x} \cdot \dfrac{1 - J^m(x)}{1 - J(x)} & \text{if } \ x \geq a \text{ and } \rho = 1 \\ \dfrac{1}{b-x} \cdot \dfrac{1 - J^m(x)}{1 - J(x)} & \text{if } \ x \leq a \text{ and } \rho = 1 \\ \dfrac{\rho^x}{\rho^x - 1} \cdot \dfrac{1 - J^m(x)}{1 - J(x)} & \text{if } \ x \geq a \text{ and } \rho \neq 1 \\ \dfrac{\rho^x}{\rho^b - \rho^x} \cdot \dfrac{1 - J^m(x)}{1 - J(x)} & \text{if } \ x \leq a \text{ and } \rho \neq 1. \end{cases}$$

Then $J_n = J(n)$, where $J_n$ is given by (10). It follows from (16), (17), and (18) that $f_{\rho,m}(x)$ differs form $V_{c^*(x)}(a,x)$ only by a positive constant multiplicative factor on both the intervals $[1,a]$ and $[a,n]$. Considering the sign of the derivative $f'_{\rho,m}(x)$ and using the fact that

$$\left(\frac{1 - J^m(x)}{1 - J(x)}\right)' = J'(x) \sum_{k=0}^{m} kJ^{k-1}(x),$$

it is easy to verify that if the lemma is true for $m = \infty$, it is true for any $m \in \mathbb{N}$. It is then routine to check, using the first derivative test, that $f_{\rho,\infty}(x)$ (and hence $V_{c^*(x)}(a,x)$) attains its maximum when $x = a$ for any $\rho > 0$. The proof of the lemma is completed. □

Note that in the extreme case $\varepsilon = 1 - p$, any location $n \leq a$ will have the same effect from perspective of the buyer. Thus in that case, the seller is only concerned with optimizing the chances of the buyer to ever visit the store. We summarize our results for the subgame perfect equilibrium strategy $\big(c^*(a), n^*(a)\big)$ of the seller and her corresponding revenue in the following statement.

**Theorem 3.3.** *Consider a game* $\Gamma_{m,a}$. *Then*

(a) *For a fixed starting point of the buyer* $a$, *the unique subgame perfect equilibrium location of the store is at* $n^* = a$.

(b) *The subgame perfect equilibrium cost* $c^*(a)$ *is given by* (16) *with* $n = a$, *and thus*

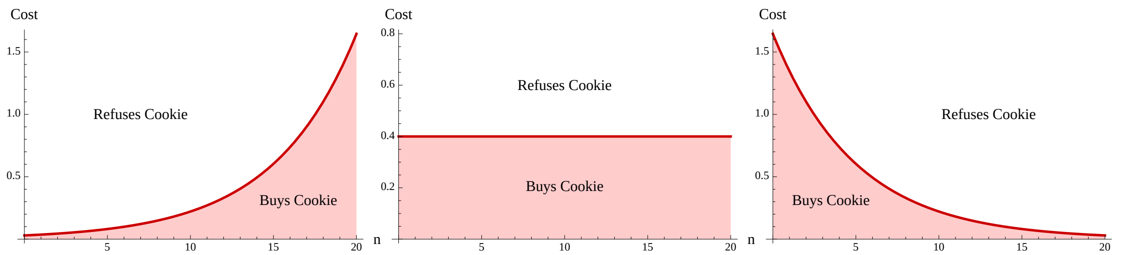
$$c^*(a) = \begin{cases} \dfrac{r\varepsilon\rho^a(\rho^{-1} - \rho)}{1 - \rho^b} & \text{if } \ p \neq q \\ \dfrac{2\varepsilon r}{b} & \text{if } \ p = q. \end{cases}$$

*In particular,* $c^*(a)$ *is independent of the value of* $m$.

(c) *The expected revenue of the seller at equilibrium is given by*

$$V^*(a) := V_{c^*(a)}(a) = \frac{c^*(a)(1 - J_a^m)}{1 - J_a}.$$

The following picture provides a graphical representation of the equilibrium strategy of the buyer in the first quadrant of the plane $(c, n)$, where each point corresponds to an available strategy of the seller. In all three cases illustrated below, $b = r = 10$ and $\varepsilon = 0.2$.




(A) $\rho > 1$ (B) $\rho = 1$ (C) $\rho < 1$

FIGURE 1. Sketch of the graph of $c^*(n)$ and buyer's equilibrium policy

Assuming $q > p$, one can consider a version of the game on the interval $(-\infty, 0]$ with a reward $r > 0$ given to the walker when (and if) she arrives at 0. The equilibrium strategies for the game on $(-\infty, 0)$ can be formally obtained from the corresponding results for a finite interval by replacing $a$ with $a + b$ and taking limit as $b \to \infty$. We state this as follows. Let

$$\begin{aligned} J_a^{inf} &= (p+\varepsilon)P_{a+1}(T_a < T_0) + (q-\varepsilon)P_{a-1}(T_a < T_{-\infty}) \\ &= (p+\varepsilon)\frac{\rho^{a+1}-1}{\rho^a - 1} + (q-\varepsilon)\rho^{-1} = 1 + \varepsilon(\rho - \rho^{-1}) - (p+\varepsilon)\frac{\rho - 1}{1-\rho^a}. \end{aligned}$$

We have:

**Theorem 3.4.** *Consider the variant of $\Gamma_{m,a}$ where the buyer's random walk is taking place on the infinite interval $(-\infty, 0]$, the site 0 is the unique absorbing point for the random walk, $\rho > 1$, the buyer is rewarded with \$$r > 0$ when (and if) she reaches 0, and the buyer's starting point is a fixed constant $a \in (-\infty, 0)$.*

*Then*

(a) *The equilibrium location of the store is at $n^*_{inf} = a$.*

(b) *For a given price for a cookie $c$, provided that the buyer will use the cookies, the expected revenue of the seller is given by $V_c^{inf}(a) = c \cdot \frac{1-(J_a^{inf})^m}{1-J_a^{inf}}$.*

(c) *The equilibrium cost $c^*_{inf}(a)$ is given by $c^*_{inf}(a) = r\varepsilon\rho^a(\rho - \rho^{-1})$. In particular, $c^*_{inf}(a)$ is independent of the value of $m$.*

The explicit formulas provided by Theorem 3.3 allow one also to study how the main characteristics of the buyer-seller game depend on the parameters $b$ and $\varepsilon$. In the next theorem we find natural scalings for the parameters $m$ and $r$ when $p = q$ and $b$ goes to infinity. The scaling factors turn out to be of order $b$, indicating that the effect of the cookie store on the simple random walk is considerably large. For $x \in \mathbb{R}$, let $[x]$ denote the integer part of $x$, that is $[x] = \max\{n \in \mathbb{N} : n \leq x\}$.

**Theorem 3.5.**

(a) *For any $a \in \mathbb{N}$ and $m \in \mathbb{N} \cup \{\infty\}$, if $\lim_{b\to\infty} b^{-1} \cdot r(b) = \alpha$ for some constant $\alpha \in (0, \infty)$, we have $\lim_{b\to\infty} c^*(a) = 2\varepsilon\alpha$.*

(b) *For any* $x > 0$, *if* $r > 0$ *and* $\lim_{b\to\infty} b^{-1} \cdot m(b) = \beta$ *for some constant* $\beta \in (0, \infty)$, *we have* $\lim_{b\to\infty} V^*\big([bx]\big) = 2\varepsilon r \cdot \frac{1-e^{-\beta K_0}}{K_0}$, *where* $K_0 = \frac{1}{2}\big(\frac{1+2\varepsilon}{1-x} + \frac{1-2\varepsilon}{x}\big)$.

We next investigate the equilibrium revenue of the seller $V^*(a)$ as a function of the parameter $\varepsilon$. On one hand, the seller provides cookies creating a positive reinforcement to the random walk to terminate at $b$. On the other hand, in order to increase consumption of cookies, she is interested in keeping the walker in the game as long as possible. The following result shows that in the trade-off between the equilibrium price $c^*(a)$ (increasing function of $\varepsilon$) and the expected number of visits to the store (decreasing function of $\varepsilon$), the former is the dominant factor for establishing the equilibrium policy of the seller.

**Theorem 3.6.** $V^*(a)$ *is an increasing function of the parameter* $\varepsilon$.

*Proof.* Observe that for any $\rho > 0$, $c^*(a)$ has the form $c^*(a) = C\varepsilon$ where $C > 0$ does not depend on $\varepsilon$. Therefore, by Theorem 3.3

$$\frac{\partial V^*(a)}{\partial \varepsilon} = \frac{C(1-J_a^m)}{1-J_a} + \frac{\partial J_a}{\partial \varepsilon} \cdot \frac{C\varepsilon(1-mJ_a^{m-1}+(m-1)J_a^m)}{(1-J_a)^2}. \tag{19}$$

According to (10),

$$\frac{\partial J_a}{\partial \varepsilon} = P_{a+1}(T_a < T_b) - P_{a-1}(T_a < T_0) > \frac{J_a - 1}{\varepsilon}.$$

Furthermore,

$$\frac{1-mJ_a^{m-1}+(m-1)J_a^m}{(1-J_a)^2} = \frac{\partial}{\partial J_a}\left(\frac{1-J_a^m}{1-J_a}\right) = \sum_{k=1}^{m} kJ_a^{k-1} > 0.$$

Therefore, replacing $\frac{\partial J_a}{\partial \varepsilon}$ with $\frac{J_a-1}{\varepsilon}$ in (19), we obtain

$$\frac{\partial V^*(a)}{\partial \varepsilon} > \frac{C(1-J_a^m)}{1-J_a} - \frac{C(1-mJ_a^{m-1}+(m-1)J_a^m)}{1-J_a} = CmJ_a^{m-1} \geq 0.$$

This completes the proof of the theorem. □

## 4. Population of Buyers. Randomized Entry Point for the Buyer.

In this section we aim to find the equilibrium policy $(c, n)$ for a single seller dealing with a population of walkers. Notice that according to Theorem 2.4, once the store is placed, the equilibrium price for a cookie is independent of the buyer's entry point and therefore is determined by the store placement only.

Assume that the buyers are independent of each other, and the starting position of each buyer is distributed uniformly on $\{1, \ldots, b-1\}$. Further, assume that the path of the random walk associated with the buyer with $X_0 = a$ is distributed according to $\mathbb{P}_{a,n}$ with $m = \infty$. It then follows from (2) that the problem is basically equivalent to its analogue with a single buyer whose initial position is uniformly distributed over the integers within $(0, b)$. In what follows we will therefore consider a slightly more general scenario, formally allowing $m < \infty$.

**Definition 4.1.** *The game* $\Gamma_{m,\text{unif}}$ *is the same as* $\Gamma_{m,a}$, *except that the buyer starts her random walk at a random integer point* $X_0$, *uniformly distributed over* $(0, b)$.

Let $V_c^{unif}(n)$ denote the expected revenue of a seller whose store is located at site $n$. For $n \in [1, b-1]$ we have

$$V_c^{unif}(n) = \frac{1}{b-1}\sum_{a=1}^{b-1} V_c(a,n) = \frac{V_c(n)}{b-1}\sum_{a=1}^{b-1} P_a(T_n < \mathcal{T})$$
$$= \frac{V_c(n)}{b-1}\Bigg[1 + \sum_{a=1}^{n-1} P_a(T_n < T_0) + \sum_{a=n+1}^{b-1} P_a(T_n < T_b)\Bigg].$$

For $p = q$ we obtain

$$1 + \sum_{a=1}^{n-1} P_a(T_n < T_0) + \sum_{a=n+1}^{b-1} P_a(T_n < T_b) = \sum_{a=1}^{n-1}\frac{a}{n} + \sum_{a=n}^{b-1}\frac{b-a}{b-n}$$
$$= \frac{n-1}{2} + b - \frac{(b-1)b-(n-1)n}{2(b-n)} = \frac{n-1}{2} + b - \frac{b+n-1}{2} = \frac{b}{2},$$

where we use the usual convention that $\sum_{a=n}^{m} x_a = 0$ for any sequence $x_a$ if $n > m$.

Using the same convention, for $p \neq q$ we obtain

$$1 + \sum_{a=1}^{n-1} P_a(T_n < T_0) + \sum_{a=n+1}^{b-1} P_a(T_n < T_b) = \sum_{a=1}^{n-1}\frac{\rho^a - 1}{\rho^n - 1} + \sum_{a=n}^{b-1}\frac{\rho^b - \rho^a}{\rho^b - \rho^n}$$
$$= \frac{\frac{\rho^n - 1}{\rho - 1} - 1}{\rho^n - 1} - \frac{n-1}{\rho^n - 1} + \frac{\rho^b(b-n)}{\rho^b - \rho^n} - \frac{\frac{\rho^b - \rho^n}{\rho - 1}}{\rho^b - \rho^n} = -\frac{n}{\rho^n - 1} + \frac{\rho^b(b-n)}{\rho^b - \rho^n}.$$

We summarize the above calculation in the following lemma. Recall $V_c(n)$ from Section 3.

**Lemma 4.2.** *Consider a game* $\Gamma_{m,\mathrm{unif}}$. *Then*

(a) *If* $p = q$, *we have*

$$V_c^{unif}(n) = \frac{V_c(n)b}{2(b-1)}.$$

(b) *If* $p \neq q$, *we have*

$$V_c^{unif}(n) = \frac{V_c(n)}{b-1}\cdot\left[-\frac{n}{\rho^n - 1} + \frac{\rho^b(b-n)}{\rho^b - \rho^n}\right].$$

Let $V^*_{unif}(n)$ denote the maximal expected revenue in $\Gamma_{m,\mathrm{unif}}$ of the store located at $n \in (0, b)$. That is $V^*_{unif}(n) = V^{unif}_{c^*(n)}(n)$, where $c^*(n)$ is defined in (13). Recall $J_n$ from (10). Combining Lemma 4.2 with (17) and part (a) of Theorem 3.3, we obtain

**Corollary 4.3.** *Consider a game* $\Gamma_{m,\mathrm{unif}}$. *Then*

(a) *If* $p = q$, *we have*

$$V^*_{unif}(n) = \frac{\varepsilon r(1 - J_n^m)}{(b-1)(1-J_n)}.$$

(b) *If* $p \neq q$, *we have*

$$V^*_{unif}(n) = \frac{r\varepsilon\rho^n(\rho^{-1} - \rho)}{1 - \rho^b}\cdot\frac{1 - J_n^m}{(b-1)(1-J_n)}\cdot\left[-\frac{n}{\rho^n - 1} + \frac{\rho^b(b-n)}{\rho^b - \rho^n}\right].$$

**Corollary 4.4.** *Let a real number $t \in [0,1]$ be fixed and let $(n_b)_{b\in\mathbb{N}}$ be any sequence of integers such that $lim_{b\to\infty} n_b/b = t$. Then*

$$\lim_{b\to\infty}\frac{V^*_{unif}(n_b)}{V^*(n_b)} = \begin{cases} 1-t & \text{if } \rho>1 \\ 1/2 & \text{if } \rho=1 \\ t & \text{if } \rho<1, \end{cases}$$

*where both $V^*_{unif}(n_b)$ and $V^*(n_b)$ are computed for arbitrary but always the same values of $m$ and $r$, which may or may not depend on $b$.*

We turn now to the study of the equilibrium location of the cookie store under Assumption 4.1. For $p=q$ we have

$$J_n = (p+\varepsilon)\Big(1-\frac{1}{b-n}\Big)+(q-\varepsilon)\Big(1-\frac{1}{n}\Big) = 1-\Big(\frac{p+\varepsilon}{b-n}+\frac{q-\varepsilon}{n}\Big).$$

For a real number $x\in(0,b)$ let $f(x)=\Big(\frac{p+\varepsilon}{b-x}+\frac{q-\varepsilon}{x}\Big)$. Then $\lim_{x\to 0} f(x)=\lim_{x\to b} f(x)=+\infty$ and $f(x)$ is minimal over $(0,b)$ when $f'(x)=0$, that is when $\frac{p+\varepsilon}{(b-x)^2}=\frac{q-\varepsilon}{x^2}$. This yields $(p-q+2\varepsilon)x^2+2bx(q-\varepsilon)-b^2(q-\varepsilon)=0$. The unique root of this equation which belongs to the interval $(0,b)$ is given by

$$\begin{aligned} x_0(\varepsilon) &= \frac{-2b(q-\varepsilon)+2b\sqrt{(q-\varepsilon)^2+(p-q+2\varepsilon)(q-\varepsilon)}}{2(p-q+2\varepsilon)} \\ &= b\frac{-(1-2\varepsilon)+\sqrt{1-4\varepsilon^2}}{4\varepsilon}. \end{aligned} \tag{20}$$

For the equilibrium location of the store $n^*_{unif}$ we have $|n^*_{unif}-x_0(\varepsilon)|<1$. Notice that $\lim_{\varepsilon\to 0} x_0(\varepsilon)=b/2$, $x_0(1/2)=0$, and

$$x_0'(\varepsilon)=\frac{1-\sqrt{1-4\varepsilon^2}}{4\varepsilon^2}-\frac{1}{\sqrt{1-4\varepsilon^2}}=\frac{1}{1+\sqrt{1-4\varepsilon^2}}-\frac{1}{\sqrt{1-4\varepsilon^2}}<0.$$

We next examine the optimal location of the store for the case $p\neq q$ and $m=\infty$. Let $A=(p+\varepsilon)(1-\rho)$ and $B=(q-\varepsilon)(1-\rho^{-1})$. Then it is routine to check, using the first derivative test, that $|n^*_{unif}-x_0(\varepsilon)|<1$, where $x_0(\varepsilon)\in(0,b)$ is the unique solution of the following equation:

$$x\ln\rho\cdot(A+B\rho^b)+[A+B\rho^b-Bb\rho^b\ln\rho]=\rho^x(A+B). \tag{21}$$

It is not hard to check that $\lim_{\varepsilon\to 0} x_0(\varepsilon)=\frac{b\rho^b}{\rho^b-1}-\frac{1}{\ln\rho}>0$, $\lim_{\varepsilon\to q} x_0(\varepsilon)=0$, and $x_0'(\varepsilon)<0$. The value of $x_0(\varepsilon)$ that solves (21) gives us insight as to which point will maximize the seller's expected revenue. We summarize the above calculations as follows.

**Lemma 4.5.** *Consider a game $\Gamma_{m,\mathrm{unif}}$. If $p\neq q$, assume in addition that $m=\infty$. Then $|n^*_{unif}-x_0(\varepsilon)|<1$ where for $p=q$, $x_0(\varepsilon)$ is given by (20), while for $p\neq q$, $x_0(\varepsilon)$ is determined as the unique positive solution to (21).*

**Corollary 4.6.** *Under the conditions of Lemma 4.5, $x_0(\varepsilon)$ is a decreasing function of the parameter $\varepsilon$. Furthermore, $\lim_{\varepsilon\to q} x_0(\varepsilon)=0$ and for a fixed $\rho>0$ we have*

(1) $\lim_{\varepsilon\to 0} x_0(\varepsilon)=b/2$ *for $p=q$.*
(2) $\lim_{\varepsilon\to 0} x_0(\varepsilon)=\frac{b\rho^b}{\rho^b-1}-\frac{1}{\ln\rho}>0$ *for $p\neq q$ and $m=\infty$.*

**Corollary 4.7.** *Under the conditions of Lemma 4.5, for fixed $\rho$, $r$, and $\varepsilon > 0$ we have*

(1) *The quotient $x_0(\varepsilon)/b$ is a decreasing, constant, or increasing function of $b$ according to whether $\rho$ is less, equal, or greater than one.*
(2) *If $\rho > 1$ (and $m = \infty$), then $\lim_{b\to\infty} n^*_{unif} = b - 1$ .*
(3) *If $\rho < 1$ (and $m = \infty$), then $\lim_{b\to\infty} x_0(\varepsilon) = \hat{x}_\varepsilon$ where $\hat{x}_\varepsilon$ is the unique positive solution to the equation $A(1 + x\ln\rho) = \rho^x(A + B)$.*

Corollary 4.6 implies that the range for the equilibrium store placement computed for all possible values of $\varepsilon$ and fixed $b, r$, and $\rho$, is the whole interval $(0, n_{\max})$ for some integer $n_{\max} \in (0, b)$. This is in stark contrast with the basic model, where the buyer's initial position is the major factor influencing the seller's decision regarding the optimal store placement. This can be heuristically explained recalling that the optimal store location is determined in the trade-off between the equilibrium price for a cookie and the expected number of visits to the store. The assumption that the buyer's entry point is spread uniformly over $(0, b)$ smooths out the influence of the "accessability" factor, and therefore implies that the price optimization gets more weight than it had for a "deterministically starting" buyer.

## 5. Risk Aversion

In this section we aim to compare the two-person game considered in Section 2 with a version where the buyer is risk-averse when making decisions under uncertainty. The main result of the section is stated in Theorem 5.2.

In this section we consider the following variation of the basic game.

**Definition 5.1.** *The game $\Gamma_{m,\mathrm{ra}}$ is the same as same as $\Gamma_{m,a}$ except that the buyer's goal is to maximize her utility function given, for some fixed constants $A \ge 0$ and $\alpha \in (0, 1)$, by*

$$U_c^{ra}(a,n) = \mathbb{E}_{a,n}\big(x - A\alpha^x\big), \tag{22}$$

*where $x = r \cdot \mathbf{1}_{\{\mathcal{T}=T_b\}} - c \cdot \eta_{n,m}$ is the total earnings of the buyer during the game (possibly negative). Here, as before, $c$ is the price taken by a seller for a cookie, $m$ is the number of cookies available at the store, and $\eta_{n,m}$ is introduced in (2).*

The individual utility function in the form (22) is a particularly popular choice in economics literature, used for modeling risk-averse behavior. See for instance [3, 4] for its axiomatic characterization. The utility function of the seller in this section is the same as the one in Section 3, namely the expected payment of the buyer to the seller, $\mathbb{E}_{a,n}(c \cdot \eta_{n,m})$. That is, in contrast to the buyer, the seller is risk-neutral.

The equilibrium price for a cookie $c^*_{ra}(a, n)$ can be determined as a solution for unknown variable $c$ to the equation

$$U_c^{ra}(a,n) = (r - A\alpha^r) \cdot P_a(T_b < T_0) - A \cdot P_a(T_0 < T_b), \tag{23}$$

which is the counterpart of (6) for a risk-averse buyer. Notice, that according to (22), $U_c^{ra}(a, n)$ is a decreasing function of the parameter $c$ with $\lim_{c\to\infty} U_c^{ra}(a, n) = -\infty$. Furthermore, $U_0^{ra}(a, n) = (r - A\alpha^r) \cdot \mathbb{P}_{a,n}(T_b < T_0) - A \cdot \mathbb{P}_{a,n}(T_0 < T_b)$, and hence

$$\begin{aligned} &U_0^{ra}(a,n) - (r - A\alpha^r) \cdot P_a(T_b < T_0) - A \cdot P_a(T_0 < T_b) \\ &\quad = \big[\mathbb{P}_{a,n}(T_b < T_0) - P_a(T_b < T_0)\big] \cdot \big[r + A(1 - \alpha^r)\big] > 0. \end{aligned}$$

Therefore, (23) has a unique positive solution. The main result of this section is stated in the following theorem. Recall $c^*(n)$ from (13).

**Theorem 5.2.** *Consider a game $\Gamma_{m,\mathrm{ra}}$. Then $c^*_{ra}(a,n) \le c^*(n)$.*

*Proof.* Let $R_{\mathcal{T}} = r\mathbf{1}_{\{T_b=\mathcal{T}\}}$. According to (23), $c^*_{ra}(a,n)$ is the unique solution for $c$ to the equation

$$E_a(x - A\alpha^x) = \mathbb{E}_{a,n}\big[(R_{\mathcal{T}} - c\eta_{n,m}) - A\alpha^{R_{\mathcal{T}} - c\eta_{n,m}}\big].$$

To avoid using two different expectation functionals, namely $E_a$ and $\mathbb{E}_{a,n}$, in the same equation, we can enlarge the probability space, where the random walk $(X_n)_{n\ge 0}$ is defined, to include a random walk $Y = (Y_n)_{n\ge 0}$ which is independent of $(X_n)_{n\ge 0}$, starts at $Y_0 = a$ with probability one, ignores cookies, and is distributed according to $P_a$. We will assume that the second walker is also rewarded \$$r$ if she reaches $b$. Let $y$ denote buyer's earnings, that is $y = r\cdot \mathbf{1}_{\{Y \text{ hits } b \text{ before } 0\}}$. Using this notation we obtain the equation for $c^*_{ra}(a,n)$ in the following form:

$$\mathbb{E}_{a,n}(y - A\alpha^y) = \mathbb{E}_{a,n}\big[(R_{\mathcal{T}} - c^*_{ra}(a,n)\cdot \eta_{n,m}) - A\alpha^{R_{\mathcal{T}} - c^*_{ra}(a,n)\cdot\eta_{n,m}}\big].$$

The latter is equivalent to

$$\begin{aligned} c^*_{ra}(a,n) &= \frac{\mathbb{E}_{a,n}(R_{\mathcal{T}} - y)}{\mathbb{E}_{a,n}(\eta_{n,m})} - A\cdot\frac{\mathbb{E}_{a,n}\big[\alpha^{R_{\mathcal{T}} - c^*_{ra}(a,n)\cdot\eta_{n,m}} - \alpha^y\big]}{\mathbb{E}_{a,n}(\eta_{n,m})} \\ &= c^*(n) - A\cdot\frac{\mathbb{E}_{a,n}\big[\alpha^{R_{\mathcal{T}} - c^*_{ra}(a,n)\cdot\eta_{n,m}} - \alpha^y\big]}{\mathbb{E}_{a,n}(\eta_{n,m})}. \end{aligned} \tag{24}$$

Therefore, the statement of the theorem is equivalent to the claim that (recall that two random walks under consideration are independent of each other),

$$\mathbb{E}_{a,n}\big[\alpha^{R_{\mathcal{T}} - y - c^*_{ra}(a,n)\cdot\eta_{n,m}}\big] > 1.$$

Hence it suffices to show that the above inequality holds. Toward this end, observe that $f(c) := \mathbb{E}_{a,n}\big(\alpha^{R_{\mathcal{T}} - y - c\eta_{n,m}}\big)$ is an increasing function of the parameter $c$. Therefore, if it would be the case that $c^*(n) \le c^*_{ra}(a,n)$ and $\mathbb{E}_{a,n}\big[\alpha^{R_{\mathcal{T}} - y - c^*_{ra}(a,n)\cdot\eta_{n,m}}\big] \le 1$, we would also have

$$\mathbb{E}_{a,n}\big[\alpha^{R_{\mathcal{T}} - y - c^*(n)\cdot\eta_{n,m}}\big] \le 1. \tag{25}$$

It follows from (8) that $\mathbb{E}_{a,n}\big[R_{\mathcal{T}} - y - c^*(n)\cdot\eta_{n,m}\big] = 0$, and hence (25) violates Jensen's inequality for the convex function $\alpha^x$. The proof of the theorem is therefore completed. $\square$

The intuitive explanation for the above result is as follows. While the walker described by the $(Y_n)_{n\ge 0}$ is risk-neutral and uses the expected earnings as her utility function, the first walker is "more skeptical" (risk-averse) and therefore she effectively values the expected earning less than its nominal value.

It is not hard to check that the proof of Theorem 2.4 goes through and hence its conclusion is in force for $\Gamma_{m,\mathrm{ra}}$. That is, for a fixed store location $n$, the maximal price $c^*_{ra}(a,n)$ that the buyer would be willing to pay for a cookie is independent of the value of $a$. This can also be derived directly from (24). Indeed, using the fact that

$$\mathbb{E}_{a,n}\big[(\alpha^{R_{\mathcal{T}} - c^*_{ra}(a,n)\cdot\eta_{n,m}} - \alpha^y)\mathbf{1}_{\{\eta_{n,m}=0\}}\big] = E_a(\alpha^{R_{\mathcal{T}}}\cdot\mathbf{1}_{\{\eta_{n,m}=0\}}) - E_a(\alpha^{R_{\mathcal{T}}}\cdot\mathbf{1}_{\{\eta_{n,m}=0\}}) = 0$$

and the Markov property, we obtain from (24) the following equation independent of $a$ :

$$c^*_{ra}(a,n) = c^*(n) - A\cdot\frac{\mathbb{E}_{a,n}\big[\alpha^{R_{\mathcal{T}} - c^*_{ra}(a,n)\cdot\eta_{n,m}} - \alpha^y\big]}{\mathbb{E}_{a,n}(\eta_{n,m})}$$

$$= \quad c^*(n) - A \cdot \frac{\mathbb{E}_{n,n}\big[\alpha^{R_{\mathcal{T}} - c^*_{ra}(a,n)\cdot\eta_{n,m}} - \alpha^y\big]}{\mathbb{E}_{n,n}(\eta_{n,m})}.$$

We can therefore simplify the notation $c^*_{ra}(a,n)$ to $c^*_{ra}(n)$. Since $\eta_{n,m}$ and $R_{\mathcal{T}}$ are independent random variables under $\mathbb{P}_{n,n}$, we obtain that $c^*_{ra}(n)$ is the unique solution of the equation

$$c^*_{ra}(n) = c^*(n) - A \cdot \frac{\mathbb{E}_{n,n}(\alpha^{-c^*_{ra}(n)\cdot\eta_{n,m}}) \cdot \mathbb{E}_{n,n}(\alpha^{R_{\mathcal{T}}}) - E_n(\alpha^{R_{\mathcal{T}}})}{\mathbb{E}_{n,n}(\eta_{n,m})}. \tag{26}$$

Though it seems impossible to determine the optimal location of the store from this equation analytically, it can be useful for numerical analysis since all the expectations appearing in the equation can be computed explicitly. We remark that, in virtue of Theorem 5.2, (26) yields the following lower bound for $c^*_{ra}(n)$ :

$$c^*_{ra}(n) \geq c^*(n) - A \cdot \frac{\mathbb{E}_{n,n}(\alpha^{-c^*(n)\cdot\eta_{n,m}}) \cdot \mathbb{E}_{n,n}(\alpha^{R_{\mathcal{T}}}) - E_n(\alpha^{R_{\mathcal{T}}})}{\mathbb{E}_{n,n}(\eta_{n,m})}. \tag{27}$$

The right-hand side is negative and thus the bound is trivial for $A$ large enough. When $A$ approaches infinity, $c^*_{ra}(n)$ converges to $c_\infty(n) > 0$ which is uniquely determined from the equation $\mathbb{E}_{n,n}(\alpha^{-c_\infty(n)\cdot\eta_{n,m}}) \cdot \mathbb{E}_{n,n}(\alpha^{R_{\mathcal{T}}}) = E_n(\alpha^{R_{\mathcal{T}}})$.

## 6. Time is Money

We next consider a model where the buyer values not only the size of the reward but also the time needed to achieve this reward. Time thus represents an opportunity cost of participating in the cookie game. For simplicity, we do not assume that the payoff is directly discounted or is subject to a "bias for the present" factorization, as say in [14]. More precisely, we impose in this section the following assumption regarding the buyer's utility function.

**Definition 6.1.** *The game $\Gamma_{m,\mathrm{time}}$ is the same as $\Gamma_{m,a}$ except that the buyer's goal is to maximize her utility function given, for a fixed constant $\Lambda > 0$, by*

$$U^{time}_c(a,n) = \mathbb{E}_{a,n}\big(x - \Lambda\mathcal{T}\big),$$

*where $x = r \cdot \mathbf{1}_{\{\mathcal{T}=T_b\}} - c \cdot \eta_{n,m}$ is the total earning of the buyer during the game (possibly negative).*

Our main result in this section is stated in Theorem 6.2, where the equilibrium price for a cookie is determined. The equilibrium cost structure can be then in principle used for finding the optimal store location. In general, the optimal placement does not necessarily coincide with the starting point of the buyer. For instance if $\Lambda$ is large enough, the buyer might be better off by avoiding the use of the cookies (at any positive price) in hopes to finishing the game quickly by exiting $[0,b]$ from the left. It can be shown that the optimal placement depends not only on the entry point $a$ and the relationship between the reward $r$ and the "implicit cost" $\Lambda$, but also on the number of cookies initially available at the store, $m$. Since there are many possible scenarios depending on the values of all the parameters involved, we will not pursue details here.

Let $c^*_{time}(n)$ be the equilibrium price for a cookie $\Gamma_{m,\mathrm{time}}$ when the store is placed at $n \in (0,b)$. Similarly to (2), we define

$$\eta_n(k) = \sum_{i=0}^{\min\{m,k\}} \mathbf{1}_{\{X_i=n\}}. \tag{28}$$

Notice that $\eta_n(\mathcal{T}) = \eta_n(\mathcal{T}-1) = \eta_{n,m}$. We have

**Theorem 6.2.** *Consider a game* $\Gamma_{m,\text{time}}$. *Then*

$$c^*_{time}(n) = \begin{cases} \dfrac{r\varepsilon\rho^n(\rho^{-1}-\rho)}{1-\rho^b} - \dfrac{\Lambda\varepsilon}{p-q}\cdot\left(\dfrac{b\rho^n(\rho^{-1}-\rho)}{1-\rho^b} - 2\right) & \text{if} \quad p \neq q \\ \\ \dfrac{2r\varepsilon}{b} - 2\varepsilon\Lambda(b-2n) & \text{if} \quad p = q. \end{cases}$$

*A negative value of* $c^*_{time}(n)$ *indicates that the walker will refrain from using cookies regardless of the price, and hence the seller is better off by not opening the store at location* $n$.

*Proof.*
(a) If $p \neq q$, let

$$M_k = X_k - k\cdot(p-q) - 2\varepsilon\cdot\eta_n(k-1), \qquad k \geq 0,$$

with the agreement that $\eta_n(-1) = 0$. Then $(M_k)_{k\geq 0}$ is martingale with respect to the natural filtration of the Markov chain formed by the pairs $(X_k, m_k)_{k\geq 0}$, where $m_k$ is the number of cookies left at the store by time $k$, as defined in Section 1. By the optional stopping theorem (see for instance Theorem 7.5 in [8, Section 4.7]),

$$\mathbb{E}_{a,n}(M_0) = a = \mathbb{E}_{a,n}(X_\mathcal{T}) - (p-q)\cdot\mathbb{E}_{a,n}(\mathcal{T}) - 2\varepsilon\cdot\mathbb{E}_{a,n}(\eta_{n,m}).$$

Therefore

$$\mathbb{E}_{a,n}(\mathcal{T}) - E_a(\mathcal{T}) = \frac{1}{p-q}\cdot\Big[\mathbb{E}_{a,n}(X_\mathcal{T}) - E_a(X_\mathcal{T}) - 2\varepsilon\cdot\mathbb{E}_{a,n}(\eta_{n,m})\Big].$$

The equilibrium price is defined from

$$\frac{r}{b}\cdot\mathbb{E}_{a,n}(X_\mathcal{T}) - c^*_{time}(a,n)\cdot\mathbb{E}_{a,n}(\eta_{n,m}) - \Lambda\cdot\mathbb{E}_{a,n}(\mathcal{T}) = \frac{r}{b}\cdot E_a(X_\mathcal{T}) - \Lambda\cdot E_a(\mathcal{T}).$$

That is

$$c^*_{time}(a,n) = \frac{1}{\mathbb{E}_{a,n}(\eta_{n,m})}\left[\frac{r}{b}\cdot\big(\mathbb{E}_{a,n}(X_\mathcal{T}) - E_a(X_\mathcal{T})\big) - \Lambda\cdot\big(\mathbb{E}_{a,n}(\mathcal{T}) - E_a(\mathcal{T})\big)\right],$$

and hence

$$\begin{aligned} c^*_{time}(a,n) &= \frac{1}{\mathbb{E}_{a,n}(\eta_{n,m})}\left[\left(\frac{r}{b} - \frac{\Lambda}{p-q}\right)\cdot\big(\mathbb{E}_{a,n}(X_\mathcal{T}) - E_a(X_\mathcal{T})\big) + \frac{2\varepsilon\Lambda}{p-q}\cdot\mathbb{E}_{a,n}(\eta_{n,m})\right] \\ &= \left(\frac{r}{b} - \frac{\Lambda}{p-q}\right)\cdot\frac{\mathbb{E}_{a,n}(X_\mathcal{T}) - E_a(X_\mathcal{T})}{\mathbb{E}_{a,n}(\eta_{n,m})} + \frac{2\varepsilon\Lambda}{p-q} = \left(1 - \frac{\Lambda b}{r(p-q)}\right)\cdot c^*(n) + \frac{2\varepsilon\Lambda}{p-q}. \end{aligned}$$

(b) If $p = q$, let

$$M_k = X_k^2 - k - 4\varepsilon\cdot n\cdot\eta_{n,m}(k-1), \qquad k \geq 0.$$

As before we convene that $\eta_{n,m}(-1) = 0$. Then $(M_k)_{k\geq 0}$ is martingale with respect to the natural filtration of the Markov chain $(X_k, m_k)_{k\geq 0}$. Hence

$$a^2 = \mathbb{E}_{a,n}(X^2_\mathcal{T}) - \mathbb{E}_{a,n}(\mathcal{T}) - 4\varepsilon\cdot n\cdot\mathbb{E}_{a,n}(\eta_{n,m}),$$

and thus

$$\mathbb{E}_{a,n}(\mathcal{T}) - E_a(\mathcal{T}) = \big[\mathbb{E}_{a,n}(X^2_\mathcal{T}) - E_a(X^2_\mathcal{T}) - 4\varepsilon\cdot n\cdot\mathbb{E}_{a,n}(\eta_{n,m})\big].$$

The equilibrium price is defined from the identity

$$\frac{r}{b} \cdot \mathbb{E}_{a,n}(X_{\mathcal{T}}) - c^*_{time}(n) \cdot \mathbb{E}_{a,n}(\eta_{n,m}) - \Lambda \mathbb{E}_{a,n}(\mathcal{T}) = \frac{r}{b} \cdot E_a(X_{\mathcal{T}}) - \Lambda E_a(\mathcal{T}).$$

That is

$$c^*_{time}(n) = \frac{1}{\mathbb{E}_{a,n}(\eta_{n,m})} \left[ \frac{r}{b} \cdot \big( \mathbb{E}_{a,n}(X_{\mathcal{T}}) - E_a(X_{\mathcal{T}}) \big) - \Lambda \cdot \big( \mathbb{E}_{a,n}(\mathcal{T}) - E_a(\mathcal{T}) \big) \right],$$

and hence

$$\begin{aligned} c^*_{time}(n) &= \frac{1}{\mathbb{E}_{a,n}(\eta_{n,m})} \big[ (r - \Lambda b^2) \cdot \big( \mathbb{P}_{a,n}(X_{\mathcal{T}}) - P_a(X_{\mathcal{T}}) \big) + 4\varepsilon n \Lambda \cdot \mathbb{E}_{a,n}(\eta_{n,m}) \big] \\ &= \left( 1 - \frac{\Lambda b^2}{r} \right) \cdot c^*(n) + 4\varepsilon n \Lambda, \end{aligned}$$

as required. □

## 7. Chain of Stores Associated with the 1-Excited Random Walk

One is prompted to study the buyer-seller game described in Section 2 for more complex initial configurations of cookies (store placements). In particular, it is interesting to compare the effect of the "cookie store perturbation" on the underlying random walk in different models. In what follows we focus on finding the equilibrium price for a cookie when $X_0 = 1$ and exactly one cookie is placed at each integer site within the interval $(0, b)$. The corresponding random walk $(X_k)_{k \geq 0}$ is usually referred to as the *1-excited* random walk on $\mathbb{Z}$ (see, for instance, [2, 5]). Our main results in this section are stated in Theorems 7.1 and 7.2, see also two remarks concluding the section.

Let $\mathcal{P}_k$ be the probability that the 1-excited random walk starting at $X_0 = 1$ will reach site $k > 0$ before hitting 0. Our results in this section rely on an explicit formula for $\mathcal{P}_k$ and its asymptotic analysis. These quantities are fundamental for the random walk theory. They have been discussed in [2], based on different type of argument than ours.

Let $U^{we}_c(b)$ (here *we* abbreviates "weakly excited") denote the expected earnings of the buyer when the price for a cookie is $c > 0$ and she is using the cookies. We will denote by $c^*_{we}(b)$ the subgame perfect equilibrium price for a cookie for a buyer performing the 1-excited random walk on $[0, b]$ with absorbing boundaries, starting at $X_0 = 1$. Since $\mathcal{P}_k - \mathcal{P}_{k+1}$ is the probability that the random walk started at $X_0 = 1$ will reach $k$ but never $k+1$ before the ruin at 0, we have

$$U^{we}_c(b) = \mathcal{P}_b \cdot [r - c(b-1)] - \sum_{k=1}^{b-1} (\mathcal{P}_k - \mathcal{P}_{k+1}) \cdot ck.$$

Similarly to (8), we have

$$c^*_{we}(b) = \frac{r \big[ \mathcal{P}_b - P_1(T_b < T_0) \big]}{\mathcal{P}_b \cdot (b-1) + \sum_{k=1}^{b-1} (\mathcal{P}_k - \mathcal{P}_{k+1}) \cdot k}. \tag{29}$$

**Theorem 7.1.** *If* $p = q$*, we have* $\lim_{b \to \infty} b c^*_{we}(b) = 2r\varepsilon$.

*Proof.* We have

$$\mathcal{P}_{k+1} = \mathcal{P}_k \cdot \big[ p + \varepsilon + (q - \varepsilon) P_{k-1}(T_{k+1} < T_0) \big], \tag{30}$$

which implies for $p=q$,

$$\mathcal{P}_{k+1}=\mathcal{P}_k\cdot\Big[p+\varepsilon+(q-\varepsilon)\frac{k-1}{k+1}\Big]=\frac{\mathcal{P}_k\cdot(k+2\varepsilon)}{k+1}.$$

Thus

$$\mathcal{P}_k=\frac{1}{k!}\prod_{j=1}^{k-1}(j+2\varepsilon)=\frac{1}{k}\prod_{j=1}^{k-1}\Big(1+\frac{2\varepsilon}{j}\Big),\qquad k=1,\dots,b,$$

with the usual convention that $\prod_{k=1}^{0}a_k=1$ for any reals $a_k$. It follows from (29) that

$$c^*_{we}(b)=\frac{r\big[\mathcal{P}_b-b^{-1}\big]}{\mathcal{P}_b\cdot(b-1)+\sum_{k=1}^{b-1}(\mathcal{P}_k-\mathcal{P}_{k+1})\cdot k}.$$

Observe that

$$(\mathcal{P}_k-\mathcal{P}_{k+1})\cdot k=\mathcal{P}_k\frac{k(1-2\varepsilon)}{k+1}. \tag{31}$$

We will next show that

$$\lim_{n\to\infty}n^{1-2\varepsilon}\mathcal{P}_n=c_\varepsilon>0 \text{ for some constant } c_\varepsilon>0. \tag{32}$$

Let $f_n=n^{1-2\varepsilon}\mathcal{P}_n$. Then

$$\begin{aligned}\frac{f_{n+1}}{f_n}&=\frac{(n+1)^{1-2\varepsilon}(n+2\varepsilon)}{(n+1)n^{1-2\varepsilon}}=\frac{(n+1)^{-2\varepsilon}}{n^{-2\varepsilon}}\frac{n+2\varepsilon}{n}=\Big(1+\frac{1}{n}\Big)^{-2\varepsilon}\cdot\Big(1+\frac{2\varepsilon}{n}\Big)\\&<\Big(1+\frac{2\varepsilon}{n}\Big)^{-1}\cdot\Big(1+\frac{2\varepsilon}{n}\Big)=1.\end{aligned}$$

Therefore, $f_n$ is an increasing sequence. On the other hand, using convexity of the function $g(x)=1/x$ and the inequality $1+x\le e^x$, $x\in\mathbb{R}$, we obtain

$$\begin{aligned}f_n&=n^{1-2\varepsilon}\mathcal{P}_n=n^{-2\varepsilon}\prod_{j=1}^{n-1}\Big(1+\frac{2\varepsilon}{j}\Big)\le n^{-2\varepsilon}\exp\Big(\sum_{j=1}^{n-1}2\varepsilon j^{-1}\Big)\\&<n^{-2\varepsilon}\exp\Big(2\varepsilon+2\varepsilon\int_1^{n-1}x^{-1}dx\Big)<e^{2\varepsilon}<\infty.\end{aligned}$$

Therefore, $f_n$ converges to a finite non-zero limit when $n$ approaches infinity. Furthermore, according to (32), $f_n$ is a regularly varying at infinity sequence of index $-(1-2\varepsilon)$ [6]. This implies $\lim_{b\to\infty}(b^2f_b)^{-1}\sum_{k=1}^{b}k^2f_k(k+1)^{-1}=(2\varepsilon)^{-1}$ [6, Theorem 6]. This observation along with (31) imply

$$\begin{aligned}\lim_{b\to\infty}b\cdot c^*_{we}(b)&=\lim_{b\to\infty}\frac{br\big(\mathcal{P}_b-b^{-1}\big)}{\mathcal{P}_b\cdot(b-1)+\sum_{k=1}^{b-1}(\mathcal{P}_k-\mathcal{P}_{k+1})\cdot k}\\&=\lim_{b\to\infty}\frac{br\mathcal{P}_b}{\mathcal{P}_b\cdot(b-1)+(2\varepsilon)^{-1}(\mathcal{P}_{b-1}-\mathcal{P}_b)\cdot(b-1)b}\\&=\lim_{b\to\infty}\frac{br\mathcal{P}_b}{\mathcal{P}_b\cdot(b-1)+(2\varepsilon)^{-1}\mathcal{P}_{b-1}(b-1)(1-2\varepsilon)}=\frac{r}{1+(2\varepsilon)^{-1}(1-2\varepsilon)}=2\varepsilon r.\end{aligned}$$

The proof of the theorem is completed. $\square$

For $p \neq q$, recurrence relation (30) implies

$$\begin{aligned}\mathcal{P}_{k+1} &= \mathcal{P}_k \cdot \left[p + \varepsilon + (q-\varepsilon)\frac{\rho^{k-1}-1}{\rho^{k+1}-1}\right] = \mathcal{P}_k \frac{\rho^k - 1 + \varepsilon(\rho^{k+1}-\rho^{k-1})}{\rho^{k+1}-1} \\ &= \mathcal{P}_k \left(1 + \varepsilon \frac{\rho^{k+1}-\rho^{k-1}}{\rho^k - 1}\right)\frac{\rho^k-1}{\rho^{k+1}-1}.\end{aligned}$$

Thus $\mathcal{P}_1 = 1$ and

$$\mathcal{P}_k = \frac{\rho-1}{\rho^k-1}\prod_{j=1}^{k-1}\left(1+\varepsilon\frac{\rho^{j+1}-\rho^{j-1}}{\rho^j-1}\right), \qquad k = 2, \ldots, b.$$

In this case

$$c^*_{we}(b) = \frac{r\left(\mathcal{P}_b - \frac{\rho-1}{\rho^b-1}\right)}{\mathcal{P}_b \cdot (b-1) + \sum_{k=1}^{b-1}(\mathcal{P}_k - \mathcal{P}_{k+1})\cdot k}. \tag{33}$$

Observe that

$$\begin{aligned}(\mathcal{P}_k - \mathcal{P}_{k+1}) &= \mathcal{P}_k\left(1 - \frac{\rho^k - 1 + \varepsilon(\rho^{k+1}-\rho^{k-1})}{\rho^{k+1}-1}\right) = \mathcal{P}_k\left(\frac{\rho^{k+1}-\rho^k - \varepsilon(\rho^{k+1}-\rho^{k-1})}{\rho^{k+1}-1}\right) \\ &= \mathcal{P}_k \cdot \rho^{k-1}\left(\frac{\rho(\rho-1)-\varepsilon(\rho^2-1)}{\rho^{k+1}-1}\right). \qquad (34)\end{aligned}$$

It follows from (33) that $c^*_{we}(b) \leq \frac{r\mathcal{P}_b}{\mathcal{P}_b\cdot(b-1)} = \frac{r}{b-1}$. The following theorem shows that this bound is asymptotically tight for $\rho < 1$, regardless the value of $\varepsilon$.

**Theorem 7.2.**
(a) *If* $\rho > 1$, *we have* $\lim_{b\to\infty}\left(\frac{\rho}{1+\varepsilon(\rho-\rho^{-1})}\right)^b c^*_{we}(b) = c_\varepsilon$ *for some constant* $c_\varepsilon \in (0,\infty)$.
(b) *If* $\rho < 1$, *we have* $\lim_{b\to\infty} bc^*_{we}(b) = r$.

*Proof.*
(a) Assume that $\rho > 1$. We will first show that

$$\lim_{n\to\infty}\left(\frac{\rho}{1+\varepsilon(\rho-\rho^{-1})}\right)^n \mathcal{P}_n = \tilde{c}_\varepsilon \text{ for some constant } \tilde{c}_\varepsilon \in (0,\infty). \tag{35}$$

Notice that $\frac{\rho}{1+\varepsilon(\rho-\rho^{-1})} > 1$ because $\varepsilon < q$. Let $f_n = \left(\frac{\rho}{1+\varepsilon(\rho-\rho^{-1})}\right)^n\mathcal{P}_n$. Then

$$\begin{aligned}\frac{f_{n+1}}{f_n} &= \frac{\rho}{1+\varepsilon(\rho-\rho^{-1})}\cdot\frac{\rho^n-1}{\rho^{n+1}-1}\cdot\left(1+\varepsilon\frac{\rho^{n+1}-\rho^{n-1}}{\rho^n-1}\right) \\ &= \frac{\rho}{1+\varepsilon(\rho-\rho^{-1})}\cdot\frac{\rho^n-1+\varepsilon(\rho^{n+1}-\rho^{n-1})}{\rho^{n+1}-1} = \frac{\rho^{n+1}-\rho+\varepsilon(\rho^{n+2}-\rho^n)}{\rho^{n+1}-1+\varepsilon(\rho^{n+2}-\rho^n-\rho+\rho^{-1})} < 1.\end{aligned}$$

To verify the inequality in the last line above write

$$\begin{aligned}\rho^{n+1}-\rho+\varepsilon(\rho^{n+2}-\rho^n) < \rho^{n+1}-1+\varepsilon(\rho^{n+2}-\rho^n-\rho+\rho^{-1}) \qquad &\text{if and only if} \\ \varepsilon(\rho-\rho^{-1}) < \rho - 1 \qquad \text{if and only if} \qquad \varepsilon(\rho+1) < \rho.\end{aligned}$$

The last inequality is true because $\varepsilon < q$. Thus, we have proved that $f_n$ is a decreasing sequence. On the other hand, since $\rho^{n-1}\frac{\rho-1}{\rho^n-1} > \frac{\rho-1}{\rho}$, we obtain

$$\begin{aligned} f_n &= \left(\frac{\rho}{1+\varepsilon(\rho-\rho^{-1})}\right)^{n-1}\mathcal{P}_n \geq \frac{\rho-1}{\rho}\cdot\left(\frac{1}{1+\varepsilon(\rho-\rho^{-1})}\right)^{n-1}\prod_{j=1}^{n-1}\left(1+\varepsilon\frac{\rho^{j+1}-\rho^{j-1}}{\rho^j-1}\right) \\ &\geq \frac{\rho-1}{\rho}\cdot\left(\frac{1}{1+\varepsilon(\rho-\rho^{-1})}\right)^{n-1}\prod_{j=1}^{n-1}\left(1+\varepsilon(\rho-\rho^{-1})\right) > \frac{\rho-1}{\rho} > 0. \end{aligned}$$

Therefore, $f_n$ is a bounded away from zero decreasing sequence, and hence $\lim_{n\to\infty} f_n$ exists and is strictly positive and finite. Notice that $\frac{\rho}{1+\varepsilon(\rho-\rho^{-1})} < \rho$, and hence

$$\lim_{n\to\infty}\mathcal{P}_b\left(\frac{\rho-1}{\rho^b-1}\right)^{-1} = \infty.$$

Therefore, due to (34) and (35), the following limit exists and is strictly positive and finite

$$\begin{aligned} \lim_{b\to\infty}\left(\frac{\rho}{1+\varepsilon(\rho-\rho^{-1})}\right)^b c^*_{we}(b) &= \lim_{b\to\infty}\frac{\left(\frac{\rho}{1+\varepsilon(\rho-\rho^{-1})}\right)^b r\mathcal{P}_b}{\mathcal{P}_b\cdot(b-1)+\sum_{k=1}^{b-1}(\mathcal{P}_k-\mathcal{P}_{k+1})\cdot k} \\ &= \frac{r\tilde{c}_\varepsilon}{\sum_{k=1}^{\infty}(\mathcal{P}_k-\mathcal{P}_{k+1})\cdot k} := c_\varepsilon \in (0,\infty). \end{aligned}$$

(b) We now turn to the case $\rho < 1$. In virtue of (33) and (34) it suffices to show that

$$\lim_{n\to\infty}\mathcal{P}_n = \lim_{n\to\infty}(1-\rho)\prod_{j=1}^{n-1}\left(1+\varepsilon\frac{\rho^{j+1}-\rho^{j-1}}{\rho^j-1}\right) = \hat{c}_\varepsilon \text{ for some constant } \hat{c}_\varepsilon \in (0,\infty).$$

Let $f_n = \prod_{j=1}^{n-1}\left(1+\varepsilon\frac{\rho^{j+1}-\rho^{j-1}}{\rho^j-1}\right)$. Then $f_n$ is an increasing sequence. On the other hand,

$$\begin{aligned} f_n &= \prod_{j=1}^{n-1}\left(1+\varepsilon\frac{\rho^{j+1}-\rho^{j-1}}{\rho^j-1}\right) \leq \prod_{j=1}^{n-1}\left(1+\varepsilon\frac{\rho^{j+1}-\rho^{j-1}}{\rho-1}\right) \leq \exp\left(\sum_{j=1}^{\infty}\varepsilon\frac{\rho^{j+1}-\rho^{j-1}}{\rho-1}\right) \\ &= \exp\left(\varepsilon\frac{1+\rho}{1-\rho}\right) < \infty. \end{aligned}$$

Therefore, $f_n$ is a bounded and creasing sequence, and hence $\lim_{n\to\infty}\mathcal{P}_n = (1-\rho)\lim_{n\to\infty} f_n$ exists and is strictly positive and finite. This completes the proof of the theorem. ☐

**Remark 7.3.** *We notice that Theorem 7.1 and Theorem 7.2 can be alternatively stated as follows. We will write $a_n \sim b_n$ when $\lim_{n\to\infty} a_n/b_n = 1$ for two sequences of real numbers $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$.*

(a) *If $p = q$ and $r$ depends on $b$ such that $r(b) \sim cb$ for some constant $c \in (0,\infty)$, then $\lim_{b\to\infty} c^*_{we}(b) = 2c\varepsilon$.*

(b) *If $\rho > 1$ and $r$ depends on $b$ such that $r(b) \sim c\left(\frac{\rho}{1+\varepsilon(\rho-\rho^{-1})}\right)^b$ for some constant $c \in (0,\infty)$, then $\lim_{b\to\infty} c^*_{we}(b) = c_\varepsilon$ for some constant $c_\varepsilon \in (0,\infty)$.*

(c) *If $\rho < 1$ and $r$ depends on $b$ such that $r(b) \sim cb$ for some constant $c \in (0,\infty)$, then $\lim_{b\to\infty} c^*_{we}(b) = c$.*

**Remark 7.4.** *Let $V^*_{we}(b)$ denote the expected revenue of the seller at the equilibrium. Then $V^*_{we}(b) = r \cdot [\mathcal{P}_b - b^{-1}] \sim r\mathcal{P}_b$ as $b$ goes to infinity. Thus the asymptotic for $\mathcal{P}_b$ found in the course of the proof of Theorems 7.1 and 7.2 (see also* [2] *for a heuristic derivation) yields the asymptotic for $V^*_{we}(b)$. More precisely, for some strictly positive constants $c_\varepsilon$, $\tilde{c}_\varepsilon$, and $\hat{c}_\varepsilon$ we have, as $b$ goes to infinity,*

(a) *If $p = q$, then $V^*_{we}(b) \sim c_\varepsilon b^{-(1-2\varepsilon)}$.*
(b) *If $\rho > 1$, then $V^*_{we}(b) \sim \tilde{c}_\varepsilon \big(\frac{\rho}{1+\varepsilon(\rho-\rho^{-1})}\big)^{-b}$.*
(c) *If $\rho < 1$, then $V^*_{we}(b) \sim \hat{c}_\varepsilon$.*

## 8. Conclusion

We explored a simple game-theoretic modification of the gambler's ruin problem. The underlying random walk is defined through a single-point perturbation of the transition probabilities of the regular nearest-neighbor random walk on $\mathbb{Z}$, either recurrent or transient. The perturbation is the same as the one in the excited (cookie) random walks model, except being localized to a single point. Informally, the deformation of the transition kernel can be described as a store that provides an instant increase in probability in the positive direction when the buyer visits the store. The price of a cookie is determined in the game (negotiation) between the buyer (walker) and the seller (store's owner). The equilibrium price can vary, depending on the store's location. The seller chooses the location to maximize her expected revenue. The goal of the buyer in the game is to maximize her expected earning which is expressed in terms of a utility function. An analytical equation for the equilibrium price, given the starting point of the walker and the store's location, is derived for several interesting choices of the utility function, including risk-neutral behavior model, risk averse behavior model, and a model including an opportunity cost represented by time spent in the game. The difference between the equilibrium price policies associated with different utility functions is quite intuitive. The equilibrium price of the cookie has a nice scaling property when the range of the interval approaches infinity. Thus the price is a natural characteristics capturing the global effect of the "cookie store perturbation" on the regular random walk. In fact, the structure of the equilibrium price is closely related to the structure of exit probabilities (and local times) of the underlying (both, perturbed and not perturbed) random walks. For comparison, we include similar asymptotic results for 1-excited random walk in our analysis. In principle, the spatial distribution of the equilibrium price allows us to recover the optimal store location. The optimal store placement coincides with the buyer's starting point for the basic model of a risk neutral buyer, whereas in other cases it can be determined with the help of numerical analysis. In a future work we consider continuous-time version of the problems studied in this paper by replacing the nearest-neighbor random walk with a drifted Brownian motion. In a paper in preparation we enrich the game-theoretic component of the basic game by including a third player, modeling both duopoly competition and a state regulation of the market.

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