Divergent perpetuities modulated by regime switches

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Abstract

In a divergent case, we obtain limit theorems for a suitably normalized sequence generated by a random linear recursion with dependent coefficients. The dependence structure of the coefficients is defined through an auxiliary random sequence that represents the current "state of Nature" (alternatively, current regime of the "underlying economy") and thus serves as an exogenous dynamic environment for the model. We assume that the exogenous environment is a stochastic process with long-range dependence, so that the current state of Nature reflects the whole past of the process.

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1 Introduction

The aim of this introduction is to present our model in general and discuss a specific set-up that is considered in this paper. The main results are stated in the subsequent Section 2 while their proofs are deferred to Section 3.

The model: stochastic difference equation. Consider the following linear recursion

$$X_n = M_n X_{n-1} + Q_n, \qquad n \in \mathbb{Z},\ X_n \in \mathbb{R}, \tag{1}$$

with random real coefficients Q_n and M_n. For simplicity, we will assume throughout that $X_0 = 0$. It follows then from (1) that $X_1 = Q_1$ and for $n \geq 2$,

$$X_n = \sum_{k=1}^{n-1} Q_k \prod_{j=k+1}^{n} M_j + Q_n. \tag{2}$$

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The stochastic difference equation (1) has a remarkable variety of both theoretical as well as real-world applications. We refer the reader to, for instance, [32, 73, 84] for a comprehensive survey of the literature.

For a general stationary and ergodic sequence of pairs $(Q_n, M_n)_{n\in\mathbb{Z}}$, conditions that ensure convergence of X_n in law to a unique stationary distribution can be found, for instance, in [13]. Under the assumption that $(Q_n, M_n)_{n\in\mathbb{Z}}$ form an i.i.d. sequence, the critical case when a unique invariant measure for X_n exists but is unbounded, is considered, for instance in [3, 16, 17]. The asymptotic properties of X_n in the case when it grows stochastically as $n \to \infty$ have been considered in [47, 69, 73], where it is assumed that the pairs of coefficients $(Q_n, M_n)_{n\in\mathbb{Z}}$ form an i.i.d. sequence. We recall that under mild integrability assumption on M_n, the distinction between the cases is according to the sign of $E(\log^+ |M_n|)$, where $\log^+ x := \max\{0, x\}$ for $x > 0$.

The aim of this paper. In the supercritical case considered in [47, 69, 73], X_n grows loosely speaking as a stretched exponential function of n, and the main results of the above papers are concerned with the asymptotic behavior of a properly normalized random variable $\log |X_n|$. The goal of this paper is to study (1) in a divergent case and obtain an extension of the results of [47, 69, 73] to a non-i.i.d. setting where the coefficients of the recursion are modulated by an exogenous dynamic environment as follows.

General framework: regime switching.

Definition 1.1. *The coefficients $(Q_n, M_n)_{n\in\mathbb{Z}}$ are said to be modulated by a sequence of random variables $(e_n)_{n\in\mathbb{Z}}$, each valued in a finite set $\mathcal{D}$, if there exist independent random variables $(Q_{n,i}, M_{n,i})_{n\in\mathbb{Z}, i\in\mathcal{D}} \in \mathbb{R}^2$ such that for a fixed $i \in \mathcal{D}$, $(Q_{n,i}, M_{n,i})_{n\in\mathbb{Z}}$ are i.i.d,*

$$Q_n = Q_{n,e_n} \qquad \text{and} \qquad M_n = M_{n,e_n}, \tag{3}$$

and $(Q_{n,i}, M_{n,i})_{n\in\mathbb{Z}, i\in\mathcal{D}}$ is independent of $(e_n)_{n\in\mathbb{Z}}$.

This definition introduces two levels of randomness into the model, the first one due to the *dynamic environment* $(e_n)_{n\in\mathbb{Z}}$ and the second one due to the randomness of the "endogenous variables" $M_{n,i}$ and $Q_{n,i}$. The process X_n defined by (1) often serves to model discrete-time dynamics of both the value as well as the volatility of financial assets and interest rates, see for instance [19, 39, 43, 82]. In this context, environment $e = (e_n)_{n\in\mathbb{Z}}$ can be interpreted as the sequence of switching states (or *regimes*) of the underlying economy. The idea of *regime shifts* or *regime switches* can be traced back at least to [33, 63], and was formally proposed in [42] to explain the cyclical feature of certain macroeconomic variables.

Dependence structure of the environment: general discussion and a background. It is remarked in [2] that"while the assumption of i.i.d. errors is convenient from the mathematical point of view, it is typically violated in regressions involving econometric variables". Typically, it is assumed in the regime-switching type of models that the environment (state of Nature) is represented by a stationary finite state Markov process (see, for instance, [19, 31, 56, 57, 79] and references therein). Testing a null hypothesis of a usual $AR(p)$ model versus a Markov switching framework is discussed, for instance, in [22, 44], using in particular classical examples of [42] and [40, 72] modeling, respectively, the postwar U. S. GNP growth rate and cartel market strategies.

Note that when $(e_n)_{n\in\mathbb{Z}}$ is a finite Markov chain, the sequence of triples $\big(M_n, Q_n, e_n)_{n\in\mathbb{Z}}$ constitutes a *Hidden Markov Model.* We remark that heavy tailed HMM as random coefficients of linear time-series models have been considered, for instance, in [18, 46, 80]. Recently, linear recursions (1) with Markov-dependent coefficients have been considered, for instance, in [4, 5, 18, 24, 27, 46, 78, 80].

Though a Markov setup in regime-switching models is tractable analytically and thus is a natural starting point, it appears that "nothing in the approach ... precludes looking at more general probabilistic specifications" [43]. For instance, an application of a general "unit root versus strongly mixing innovations" statistical test to the model (1) is discussed in Section 3 of the classical reference [71]. In fact, the Markov dynamics seems in general inadequate for modeling socioeconomic factors involved in financial applications of regime-switching autoregressive models (see, for instance, [23, 65] and references therein). While early regime-switching models assumed, in order to maintain the tractability of the theoretical framework, that the underlying Markov chain is stationary and the number of states is small (see, for instance, [31, 56, 57]), it has been proposed in more recent work to consider Markov models with a large number of highly connected states and to use a-prior Bayesian information (see, for instance, [19, 79]). Alternatively, one can replace the Markovian dynamics with that of *full shifts of finite type/chains of infinite order/chains with complete connections*, which are processes with long-range dependence (infinite, though a fading memory) preserving many key features of irreducible finite-state Markov chains [25, 34, 35, 37, 49].

Our set-up: long range dependence, chains of infinite order. In this paper we will consider a setting where a long-range dependence in random coefficients is possibly present. More precisely, we will assume the following:

Assumption 1.2. *The coefficients $(Q_n, M_n)_{n\in\mathbb{Z}}$ in (1) are modulated by a process $(e_n)_{n\in\mathbb{Z}}$ which is either*

(i) An irreducible Markov chain on a finite state $\mathcal{D} = \{1, \ldots, d\}$ for some $d \in \mathbb{N}$.

or

(ii) A C-chain as specified in Definition 1.3 below.

Definition 1.3. *A C-chain is a stationary process $(e_n)_{n\in\mathbb{Z}}$ taking values in a finite set (alphabet) $\mathcal{D}$ such that*

(i) For any $i_1, i_2, \ldots, i_n \in \mathcal{D}$, $P(e_1 = i_1, e_2 = i_2, \ldots, e_n = i_n) > 0$.

(ii) For any $i_0 \in \mathcal{D}$ and any sequence $(i_n)_{n\geq 1} \in \mathcal{D}^{\mathbb{N}}$, the following limit exists:

$$\lim_{n\to\infty} P(e_0 = i_0 | e_{-k} = i_k,\ 1 \leq k \leq n) = P(e_0 = i_0 | e_{-k} = i_k,\ k \geq 1),$$

where the right-hand side is a regular version of the conditional probabilities.

(iii) (fading memory) For $n \geq 0$ let

$$\gamma_n = \sup \left\{ \left| \frac{P(e_0 = i_0 | e_{-k} = i_k,\ k \geq 1)}{P(e_0 = j_0 | e_{-k} = j_k,\ k \geq 1)} - 1 \right| : i_k = j_k,\ k = 1, \ldots, n \right\}.$$

Then, the numbers γ_n are all finite and

$$\sum_{n=1}^{\infty} \gamma_n \log n < \infty \tag{4}$$

C-chains are a particular case of *chains of infinite order* (*chains with complete connections*) [25, 34, 35, 37, 49]. C-chains can be described as rapidly mixing full shifts, and alternatively defined as an essentially unique random process with a given transition function (g-measure) $P\big(X_0 = i_0 | X_k = i_k, k < 0\big)$ [6, 34, 35, 49]. Stationary distributions of these processes are Gibbs states in the sense of Bowen [11, 59]. Chains with complete connections were originally introduced by Onicescu and Mihoc in [67, 68]. The name "chains of infinite order" was proposed by Harris in [45]. For a review of applications of chains with complete connections see, for instance, classical references [45, 49, 52] and more recent [25, 37, 75, 85].

The particular form in which the processes are defined in Definition 1.3 is adapted from [59] (though the name "C-chains" is not used there). Alternatively, the process can be defined in terms of the g-functions which were introduced by Keane in [54]. A non-negative function g on $\mathcal{D} \times \prod_{n \leq -1} \mathcal{D}$ is called a g-function if

$$\sum_{i_0 \in \mathcal{D}} g(i_0 | i) = 1 \text{ for all } i = (i_{-1}, i_{-2}, \ldots) \in \prod_{n \leq -1} \mathcal{D}.$$

The quantity

$$\log \inf_{i_0, \hat{i}_0, i} \frac{g(i_0|i)}{g(\hat{i}_0|i)} = \log(1 + \gamma_n)$$

can be thought as a measure (modulus) continuity of g (see, for instance [6, 81]). It turns out that if a g-function has an appropriate modulus of continuity, then there exists a unique C-chain such that $P(e_0 = i_0 | e_{-k} = i_{-k},\ k \geq 1) = g(i_0|i)$, where $i = (i_{-1}, i_{-2}, \ldots)$. In fact, C-chains are chains with complete connections with the asymptotic decay of the modulus of continuity determined by (4).

Mixing properties and Markov representation of C-chains. For any C-chain $(e_n)_{n \in \mathbb{Z}}$ there exists a *Markov representation* (see [6, 36, 59]), that is a stationary irreducible Markov chain $(y_n)_{n \in \mathbb{Z}}$ in a countable state space $\mathcal{S}$ and a function $\zeta : \mathcal{S} \to \mathcal{D}$ such that

$$(e_n)_{n \in \mathbb{Z}} \overset{D}{=} \big(\zeta(y_n)\big)_{n \in \mathbb{Z}}, \tag{5}$$

where $\overset{D}{=}$ means equivalence of distributions. The Markov representation implies that the distributions of C-chains have a regeneration structure induced by the regeneration structure of the corresponding Markov chain y_n (see Section 3.1 below for details). The first regenerative structures for chains with complete connections were proposed by Lalley [59, 60] and Berbee [6]. See, for instance, [25, 36, 35] for extensions and generalizations of these results. In these constructions the Markov chain y_n is typically an array $(e_n, e_{n-1}, \ldots, e_{n-\tau_n})$ formed by a random number $\tau_n + 1$ of variables e_k. Lalley [59] refers to such Markov chains as list processes. Kalikow [53] refers to the essentially same class of processes as random Markov processes. Unfortunately, to the best of our knowledge, an exact relation between mixing properties of such Markov chains and that of the underlying C-chain is unknown. Also, notice that the identity (5) describes the equivalence of two random sequences in distribution only. In fact, C-chains constitute a more general class of processes than functions of a "nice" fast mixing Markov chain. We refer the reader to [6, 7, 59] for an interesting discussion of a relation between mixing conditions and the Markov representation.

It is not hard to see that (4) implies that

$$\sum_{n=1}^{\infty} \psi(2^n) < \infty, \tag{6}$$

where, for $n \in \mathbb{N}$,

$$\psi(n) := \sup \left\{ \left| \frac{P(e_k = i_i,\ k \geq 0 | e_k = i_k,\ k < 0)}{P(e_k = j_k,\ k \geq 0 | e_k = j_k,\ k < 0)} - 1 \right| : i_k = j_k,\ k \geq -n \right\}.$$

Thus C-chains are ψ-mixing sequence [6, 29]. Both, the Markov representation (5) as well as the mixing condition (6) are essential for our proofs. In particular, (6) ensures a mixing rate required to apply standard results from the literature and conclude that the key auxiliary result (a general functional central limit theorem), namely Proposition 2.4 below, holds true. The regeneration structure of the sequence e_n which is implied by (5), is not needed to derive Proposition 2.4 but is used in the proof of another crucial auxiliary result, namely Lemma 2.1 below. In particular, only for the purpose to ensure the existence of the Markov representation (and hence of the regeneration structure), we could replace (4) with a slightly weaker assumption $\sum_{n=1}^{\infty} \gamma_n < \infty$. We remark that the regeneration structure itself does not imply limit laws for partial sums of even a countable Markov chain (see, for instance, [26, 50, 51, 66] and references therein). For instance, Nagaev in the classical reference [66] (see also [64, 21]) used the strong Doeblin condition to ensure that an appropriate second moment condition holds for the regeneration times. The latter are essentially equivalent to assumptions on the mixing rate of the underlying Markov chain. For an example of a stationary strongly mixing countable Markov chain that does not satisfy the central limit theorem see [26].

Alternative models of the difference equation with dependent coefficients For different from the one considered in this paper examples of linear recursions with strongly dependent (non-Markov) coefficients see, for instance, [15, 20, 48, 62, 86] and references therein. We remark that though the models with innovations given by a martingale difference sequence are arguably the most popular in applications, "in practice, econometric and financial time series often exhibit long-range dependent structure (see, e.g., [76, 77] and [29]) which cannot be encompassed by the martingale difference setting..." [15].

Organization of the paper. The rest of the paper is organized as follows. Main results of the paper are stated and discussed in Section 2. Section 3 contains proofs of the main results.

2 Main results

It was first observed in [55] that in a "generic" convergent case the asymptotic behavior of X_n is similar to that of $\max_{1\le k\le n}\big\{Q_k\cdot|\Pi_k|\big\}$, where

$$\Pi_n := \prod_{j=1}^{n} |M_j|. \tag{7}$$

It has been shown in [47, 73] that this is also the case for a "generic" divergent setup with i.i.d. coefficients. In this paper we explore this phenomenon in the divergent case under Assumption 1.2.

Notice, that whenever $\mu := E(\log|M_n|) \in (0,\infty)$, the ergodic theorem suggests the asymptotic relations $|\Pi_n| \simeq e^{n\mu}$ and, consequently, $\Pi_n \simeq \Pi_n^*$ where

$$\Pi_n^* := \max_{1\le k\le n} |\Pi_k|. \tag{8}$$

Here and henceforth $f(n) \sim g(n)$ and $f(n) \simeq g(n)$ (as a rule, we omit "as $n\to\infty$") means, respectively, $\lim_{n\to\infty} f(n)/g(n) = 1$ and $\log f(n) \sim \log g(n)$.

In fact, Π_n typically dominates over Q_n, and we have

Lemma 2.1. *Let Assumption 1.2 hold. Suppose in addition that:*

(i) $\mu = E(\log|M_n|) > 0$.

(ii) $E\big(\log^+|Q_0|\big) < +\infty$, *where* $x^+ := \max\{x, 0\}$ *for* $x \in \mathbb{R}$.

Then,

$$\lim_{n\to\infty} \frac{\log|X_n|}{n} = \mu, \qquad \text{a. s.},$$

where $(X_n)_{n\in\mathbb{N}}$ *is defined by* (1) *with* $X_0 = 0$.

If $\mu = \infty$, Theorem 2.5 stated below shows that $\frac{\log|X_n|}{b_n}$ with a suitable normalization b_n converges in distributions to the same limit that $\frac{\log|\Pi_n|}{b_n}$ does. Roughly speaking, X_n grows as a stretched exponential sequence in that case. To ensure that $\frac{\log|\Pi_n|-a_n}{b_n}$ with suitable normalizing sequences a_n and b_n converges in distribution, we will impose the following assumption on the coefficients of (1). Recall that $f:\mathbb{R}\to\mathbb{R}$ is called regularly varying if $f(t) = t^\alpha L(t)$ for some $\alpha\in\mathbb{R}$ where $L(t)$ is a slowly varying function, that is $L(\lambda t) \sim L(t)$ for all $\lambda > 0$. The parameter α is called the *index* of the regular variation. In what follows we denote by $\mathcal{R}_\alpha$ the class of regularly varying real-valued functions with index α.

Assumption 2.2. *Let Assumption 1.2 hold. Suppose in addition that* $E\big(\log|M_0|\big) \in [0,\infty]$, $E\big(\log^+|Q_0|\big) < +\infty$, *and*

Either

(A1) There exist a constant $\alpha \in (0,2]$, *a function* $h \in \mathcal{R}_\alpha$, *and constants* $m_i^{(\eta)}$ *(with* $i \in \mathcal{D}$ *and* $\eta \in \{-1,1\}$*), such that* $\lim_{t\to\infty} h(t) \cdot P\big(\log|M_{n,i}| \cdot \eta > t\big) = m_i^{(\eta)}$. *Furthermore,* $\sum_{j\in\mathcal{D}} m_j^{(1)} > 0$.

or

(A2) $E\big[(\log|M_0|)^2\big] < \infty$.

Remark 2.3. *Notice that:*

(a) Assumption (A1) is the assumption that the random variables $\log|M_{n,i}|$ *have regularly varying tails. We refer the reader to the classical monograph [9] extensively discussing regular variation and its applications. For an extensive account of the theory of random walks with regularly varying increments we refer the reader to [10] and references therein.*

(b) The existence of $E\big(\log|M_0|\big)$ *implies that*

(i) $P(|M_0| = 0) = 0$.

(ii) If (A1) holds with $\alpha \in (0,1)$, *then necessarily* $m_j^{(-1)} = 0$ *for all* $j \in \mathcal{D}$.

Let $D[0,1]$ denote the set of real-valued càdlàg functions on $[0,1]$ equipped with the Skorokhod J_1-topology. It is well known that additive functional of Markov chains and rapidly mixing sequences obey the same functional stable limit theorems as partial sums of i.i.d. random variables under relatively mild additional "local dependence" assumptions (see, for instance, [58, 83] and references therein). We will next observe that if Assumption 2.2 holds, then with suitable normalizing constants a_n and b_n (*the same as in the case when* M_n *are independent*) the sequence of processes $(S^{(n)})_{n\in\mathbb{N}}$ defined by setting

$$S_t^{(n)} = \frac{\log \Pi_{[nt]} - a_{[nt]}}{b_n}, \qquad t \in [0,1], \tag{9}$$

converges weakly in $D[0,1]$ as $n \to \infty$ to a (Lévy) process $\xi = (\xi_t)_{t\in[0,1]}$ with stationary independent increments, and such that ξ_1 is distributed according to a stable law of index α whose domain of attraction includes $\log|M_0|$. Here and henceforth $[x]$ denotes the integer part of $x \in \mathbb{R}$, that is $[x] = \max\{z \in \mathbb{Z} : z \le x\}$. For an explicit form of a_n when $\alpha = 1$ and b_n, see for instance [14, Chapter 9]. We remark that

$$a_n = \begin{cases} n\mu & \text{if either (A1) with } \alpha > 1 \text{ or (A2) of Assumption 2.2 hold,} \\ 0 & \text{if (A1) with } \alpha \in (0,1) \text{ of Assumption 2.2 holds,} \end{cases} \tag{10}$$

and that $b_{[t]} \in \mathcal{R}_{1/\alpha}$ if (A1) holds, whereas $b_n = \sqrt{n}$ if (A2) is satisfied. Further, ξ is a Brownian motion if either (A1) with $\alpha = 2$ or (A2) are satisfied. For C-chains and aperiodic Markov chains, the weak convergence of $S^{(n)}$ to ξ in $D[0,1]$ follows, for instance, from the mixing property (6) and

- Corollary 5.9 in [58] if (A1) holds with $\alpha \in (0, 2)$;

- Theorem 1 in [12] if (A1) holds with $\alpha = 2$ and $E\big[(\log|M_0|)^2\big] = +\infty$;

- Theorem 7.7.11 in [30, p. 427] if (A2) holds;

Note that the results of [58] and the central limit theorem in [30, p. 427] can be applied to C-chain because by virtue of (iii) in Definition 1.3 the latter satisfy the ψ-mixing condition (which is stronger than the uniform or φ-mixing actually required to apply these general results) with exponential rate. If $(e_n)_{n\in\mathbb{Z}}$ is a periodic Markov chain, then the claim can be derived from the result for the aperiodic chains applied to the (one-dependent and uniformly mixing) random variables $\widetilde{M}_n$, $n \in \mathbb{Z}$, defined as follows: $\log M_n := \sum_{k=nd}^{(n+1)d-1} \log M_k$, where d is the period of the chain. The fact that the normalizing sequences a_n and b_n can be chosen as in the i.i.d. case for non-Gaussian limit laws, follows from (5.12) in [58] and (3.13) in [12].

Summarizing, we have:

Proposition 2.4. *Let Assumption 2.2 hold. Then, as $n \to \infty$,*

(a) $\frac{\log|\Pi_n| - a_n}{b_n} \Rightarrow \xi_1$,

(b) $\frac{\log|\Pi_n|^*}{b_n} \Rightarrow \sup_{0\le t\le 1} \xi_t$ *whenever* $a_n = 0$,

where $\Rightarrow$ stands for convergence of random variables in distribution.

In Section 3.2 we derive from this proposition the following:

Theorem 2.5. *Let Assumption 2.2 hold. If $E(\log|M_0|) > 0$, then, as $n \to \infty$,*

$$\frac{\log|X_n| - a_n}{b_n} \Rightarrow \xi_1, \tag{11}$$

where ξ_1, a_n, and b_n are the same as in the statement of Proposition 2.4.

Remark 2.6. *In the case of i.i.d. coefficients (Q_n, M_n), Theorem 2.5 is [47, Theorem 2] when (A2) is assumed. If (A1) is assumed with $\alpha > 1$, the theorem is an analogue of part (b) of [73, Theorem 2.1] where, as well, the assumption that (Q_n, M_n) form an i.i.d. sequence is made. In the case of (A1) with $\alpha \le 1$, the result appears to be new even in the context of i.i.d. coefficients.*

Under the conditions of Assumption 2.2 the sequence $(\log|\Pi_n|)_{n\ge 0}$ satisfies a law of iterated logarithm, which propagates into the following result.

Theorem 2.7. *Let Assumption 2.2 hold. Then:*

(a) If (A1) holds either with $\alpha \in (0, 1)$ or with $\alpha \in (1, 2)$, we have

$$\limsup_{n\to\infty} \frac{\log|X_n| - a_n}{b_n(\log n)^{1/\alpha+\varepsilon}} = \begin{cases} 0 & \text{if } \varepsilon > 0 \\ \infty & \text{if } \varepsilon < 0 \end{cases} \qquad \text{a. s.}$$

(b) If (A2) holds, we have

$$\limsup_{n\to\infty} \frac{\log|X_n| - \mu n}{\sqrt{2n\log\log n}} < \infty, \qquad \text{a. s.} \tag{12}$$

The proof of Theorem 2.7 given in Section 3.3 below follows the same lines as that of [74, Theorem 4].

The case $E(\log|M_0|) = 0$ is more challenging, and we are only able to treat it under the additional assumption that the coefficients Q_n and M_n are strictly positive. The following theorem carries over corresponding results of [73] and [47] for i.i.d. coefficients to the setup of this paper. When $\mu = 0$, the asymptotic behaviors of Π_n and Π_n^* are not anymore the same. Therefore, following [47], an extreme value theory for Π_n needs to be invoked in order to establish the asymptotic behavior of X_n.

We have:

Theorem 2.8. *Let Assumption 2.2 hold. Suppose in addition that:*

(i) $P(M_0 > 0, Q_0 > 0) = 1$.

(ii) $\lim_{n\to\infty} nP(\log Q_0 > b_n^{-1}) = 0$.

(iii) $E(\log M_0) = 0$.

Then, as $n \to \infty$,

$$\frac{\log X_n}{b_n} \Rightarrow \sup_{0\le t\le 1} \xi_t, \tag{13}$$

where ξ_t *and* b_n *are the same as in the statement of Proposition 2.4.*

Remark 2.9. *In the case of i.i.d. coefficients* (Q_n, M_n)*, Theorem 2.8 is a combination of part (i) of [47, Theorem 3] and part (c) of [73, Theorem 2.1]. We remark that part (ii) of [47, Theorem 3] gives an example of the situation where* Q_n *dominates over* Π_n*, and the asymptotic behavior of* X_n *is determined by the asymptotic behavior of the former sequence rather than the latter. It is not hard to see that using an extreme value theory for Markov chains (see for instance [8, 61]), a similar example can be given also in our Markovian setup. We leave the details to an interested reader.*

3 Proofs of the main results

This section includes the proofs of our main results stated above in Section 2. We remark that representations (14) and (20) which play the key role in some of our proofs have been previously used for a similar purpose in [47].

3.1 Proof of Lemma 2.1

For $n \geq 1$, let $\Pi_n = \prod_{j=1}^{n} M_j$. It follows from (2) that

$$X_n = \Pi_n \cdot \sum_{k=1}^{n} Q_k \prod_{j=1}^{k} \frac{1}{M_j}. \tag{14}$$

Let

$$R_n = \sum_{k=1}^{n} Q_k \prod_{j=1}^{k} \frac{1}{M_j}.$$

Assuming that $X_0 = 0$, we obtain

$$\log |X_n| = \log |\Pi_n| + \log |R_n|. \tag{15}$$

By the ergodic theorem,

$$\lim_{n\to\infty} \frac{1}{n} \log |\Pi_n| = \mu, \qquad \text{a. s.}$$

On the other hand, since $E\big(\log |M_0|^{-1}\big) = -E\big(\log |M_0|\big) < 0$, [13, Theorem 1] implies that the following limit exists with probability one:

$$R := \lim_{n\to\infty} R_n = \sum_{k=1}^{\infty} \frac{Q_k}{M_k} \cdot \prod_{j=1}^{k-1} \frac{1}{M_j}. \tag{16}$$

It follows that $\frac{1}{n} \log |R_n|$ converges to zero with probability one, provided that the distribution of R does not have an atom at zero. Thus, to complete the proof of Lemma 2.1 it suffices to show that

$$\lim_{\varepsilon\to 0} P(|R| \leq \varepsilon) = 0. \tag{17}$$

. We will deduce (17) from the corresponding result for i.i.d. coefficients (Q_n, M_n) (cf. Theorem 1.3 in [1], see also Theorem 2 in [41]). Toward this end we introduce the following regenerations structure based on cycles of an underlying Markov chain (cf., for instance, [36, 38]). For C-chains we will use Markov representation (5) whereas if the environment e_n of Assumption 1.2 is a Markov chain, we will identify in what follows y_n with e_n and $\mathcal{S}$ with $\mathcal{D}$. Fix $y^* \in \mathcal{S}$ and let $N_0 = 0$ and $N_i = \inf\{n > N_{i-1} : y_n = y^*\}$, $i \in \mathbb{N}$. The blocks $(y_{N_i}, \ldots, y_{N_{i+1}-1})$ are independent for $i \geq 0$ and identically distributed for $i \geq 1$. For $i \geq 0$, let

$$\begin{aligned} A_i &= Q_{N_i+1} \cdot \big(M_{N_i+1}\big)^{-1} + Q_{N_i+2} \cdot \big(M_{N_i+2} \cdot M_{N_i+1}\big)^{-1} \\ &\qquad + \ldots + Q_{N_{i+1}} \cdot \big(M_{N_{i+1}} M_{N_{i+1}-1} \cdots M_{N_i+1}\big)^{-1}, \\ B_i &= \big(M_{N_i+1} M_{N_i+2} \cdots M_{N_{i+1}}\big)^{-1}. \end{aligned}$$

The pairs (A_i, B_i) are independent for $i \geq 0$, identically distributed for $i \geq 1$ and

$$R = A_0 + \sum_{n=1}^{\infty} A_n \prod_{i=0}^{n-1} B_i := A_0 + B_0 R_1,$$

where the last identity serves as the definition of R_1. It follows from Theorem 1.3 in [1] applied to $(A_n, B_n)_{n\in\mathbb{Z}}$ that R_1 has a continuous distribution (possibly singular continuous). Therefore, for any pair of reals (A, B) such that $b \neq 0$, we have

$$\lim_{\varepsilon\to 0} P\big(|A + BR_1| \leq \varepsilon\big) = \lim_{\varepsilon\to 0} P\big(-\varepsilon \leq A + BR_1 \leq \varepsilon\big) = 0.$$

By the bounded convergence theorem this implies that

$$\lim_{\varepsilon\to 0} P(|R| \leq \varepsilon) = \lim_{\varepsilon\to 0} P\big(|A_0 + B_0 R_1| \leq \varepsilon\big) = 0,$$

and hence verifies (17). The proof of the lemma is completed. □

3.2 Proof of Theorem 2.5

Recall (15). By part (a) of Proposition 2.4, as $n \to \infty$,

$$\frac{\log |\Pi_n| - a_n}{b_n} \Rightarrow \xi_1.$$

Furthermore, (16) and (17) imply

$$\frac{1}{b_n} \cdot \log |R_n| \Rightarrow 0.$$

This completes the proof of the theorem. □

3.3 Proof of Theorem 2.7

Recall (15). Observe that (16) and (17) imply that

$$\lim_{n\to\infty} \frac{1}{c_n} \cdot \log |R_n| = 0, \qquad \text{a. s.},$$

for any sequence of reals c_n converging to $+\infty$. Therefore, in order to prove Theorem 2.7 it suffices to establish the corresponding law of iterated logarithm for $\log |\Pi_n| - a_n$.

For $i \in \mathcal{D}$ let $k_n(i)$ be the number of occurrences of i in the set $\{e_1, e_2, \ldots, e_n\}$. That is,

$$k_n(i) = \sum_{k=1}^{n} \mathbf{1}_{\{e_k = i\}}, \qquad n \in \mathbb{N},\ i \in \mathcal{D}.$$

Define $T_i(0) = 0$, $T_i(j) = \inf\{k > T_i(j-1) : e_k = i\}$, and set $\rho_{j,i} := M_{T_i(j),i}$. Then,

$$\log |\Pi_n| = \sum_{k=1}^{n} \log |M_k| = \sum_{i\in\mathcal{D}} \sum_{j=1}^{k_n(i)} \log |\rho_{j,i}|. \tag{18}$$

It follows from Definition 1.1 that

(i) The double-indexed sequence $(\rho_{j,i})_{i\in\mathcal{D},j\in\mathbb{Z}}$ is formed by independent random variables.

(ii) For every fixed $i \in \mathcal{D}$, the sequence $(\rho_{j,i})_{j\in\mathbb{Z}}$ consists of i.i.d. variables, each one distributed the same as $M_{0,i}$.

(iii) The sequences $(\rho_{j,i})_{i\in\mathcal{D},j\in\mathbb{Z}}$ and $(e_k)_{k\in\mathbb{Z}}$ are independent of each other.

For $n \in \mathbb{N}$ and $i \in \mathcal{D}$ let

$$A(n,i) = \begin{cases} n \cdot E\big(\log|M_{0,i}|\big) & \text{if} \quad \alpha > 1 \\ 0 & \text{if} \quad \alpha < 1 \end{cases}$$

and

$$S_{n,i} = \sum_{j=1}^{k_n(i)} \log|\rho_{j,i}| - A\big(k_n(i), i\big).$$

It follows from (18) that

$$\log|\Pi_n| - a_n = \sum_{i\in\mathcal{D}} S_{n,i} + \sum_{i\in\mathcal{D}} A\big(k_n(i), i\big) - a_n.$$

Let $\eta_n(i) = \mathbf{1}_{\{e_n=i\}} - E\big(\mathbf{1}_{\{e_n=i\}}\big)$. Then $k_n(i) - nP(e_0 = i) = \sum_{j=1}^n \eta_j(i)$. It follows for instance from Theorem 4 in [70] that if $\alpha > 1$, then

$$\limsup_{n\to\infty} \frac{\sum_{i\in\mathcal{D}} A\big(k_n(i), i\big) - a_n}{\sqrt{2n\ln\ln n}} \le \sum_{i\in\mathcal{D}} \limsup_{n\to\infty} \frac{E\big(\log|M_{0,i}|\big) \cdot \sum_{j=1}^n \eta_j(i)}{\sqrt{2n\ln\ln n}} < \infty, \qquad \text{a. s.}$$

Therefore, in order to complete the proof of the first part of Theorem 2.7 it suffices to show that

$$\limsup_{n\to\infty} \frac{\sum_{i\in\mathcal{D}} S_{n,i}}{b_n(\ln n)^{1/\alpha+\varepsilon}} = \begin{cases} 0 & \text{if} \quad \varepsilon > 0 \\ \infty & \text{if} \quad \varepsilon < 0 \end{cases} \qquad \text{a. s.}$$

Toward this end we first observe that by the law of iterated logarithm for heavy-tailed i.i.d. sequences (see Theorems 1.6.6 and 3.9.1 in [10]), for each $i \in \mathcal{D}$ we have

$$\limsup_{n\to\infty} \frac{S_{n,i}}{b_{k_n(i)}\big(\ln k_n(i)\big)^{1/\alpha+\varepsilon}} = \begin{cases} 0 & \text{if} \quad \varepsilon > 0 \\ \infty & \text{if} \quad \varepsilon < 0 \text{ and } m_i^{(1)} > 0, \end{cases} \qquad \text{a. s.}$$

where the constants $m_i^{(1)}$ are introduced in Assumption 2.2. Since by the ergodic theorem $\lim_{n\to\infty} k_n(i)/n = P(e_0 = i) > 0$, this yields

$$\limsup_{n\to\infty} \frac{S_{n,i}}{b_n(\ln n)^{1/\alpha+\varepsilon}} = \begin{cases} 0 & \text{if} \quad \varepsilon > 0 \\ \infty & \text{if} \quad \varepsilon < 0 \text{ and } m_i^{(1)} > 0 \end{cases} \qquad \text{a. s.}$$

To complete the proof of part (a) of the theorem it is therefore sufficient to show that for any $i, j \in \mathcal{D}$ and all $\delta \in (1/(2\alpha), 1/\alpha)$,

$$P\big(|S_{n,i}| > b_n(\ln n)^\delta, |S_{n,j}| > b_n(\ln n)^\delta \text{ i.o.}\big) = 0. \tag{19}$$

For $i \in \mathcal{D}$ let $\xi_i = 2 \cdot P(e_0 = i)$ and define events

$$E_{n,i} = \Big\{ \max_{1 \le m \le n\xi_i} \Big| \sum_{k=1}^{m} \log |M_{k,i}| - A(m,i) \Big| > b_n(\ln n)^\delta \Big\}, \qquad n \in \mathbb{N}.$$

Then

$$\begin{aligned} &P\big(|S_{n,i}| > b_n(\ln n)^\delta, |S_{n,j}| > b_n(\ln n)^\delta\big) \\ &\quad \le P\big(E_{n,i} \bigcap E_{n,j}\big) + P\big(k_n(i) > n\xi_i\big) + P\big(k_n(j) > n\xi_j\big) \\ &\quad = P(E_{n,i}) \cdot P(E_{n,j}) + P\big(k_n(i) > n\xi_i\big) + P\big(k_n(j) > n\xi_j\big). \end{aligned}$$

By the ergodic theorem, $\lim_{n\to\infty} P\big(k_n(i) > n\xi_i\big) = 0$. Furthermore, for any $A > 0$, $n_k = [A^k]$, and $\beta \in (0, \alpha)$ there exists a positive constant $C = C(A, \beta) > 0$ such that (see [10, p. 177]), $P(E_{n_k,i}) \le Ck^{-\delta\beta}$. Moreover, since $\delta > 1/(2\alpha)$, we can choose $\beta \in (0, \alpha)$ such that $2\delta\beta > 1$. Thus $P(E_{n_k,i}) \cdot P(E_{n_k,j}) \le C_1 k^{-\gamma}$, for some constant $\gamma > 1$ and a suitable constant $C_1 > 0$. A standard argument using the Borel-Cantelli lemma yields then (19) (see for instance p. 58 or p. 435 in [30]) . The proof of part (a) of Theorem 2.7 is therefore completed.

(b) The proof of the second part of Theorem 2.7 is very similar to the proof of the first part, and is therefore omitted. The only essential difference is that, as the basic law of iterated logarithm for i.i.d. sequences, we would use the standard Hartman and Wintner result with normalization $\sqrt{2n \ln \ln n}$ (Theorem 1.6.5 in [10]) rather than the heavy-tailed version with normalization $b_n(\ln n)^{1/\alpha}$ cited above (Theorem 1.6.6 in [10]). □

3.4 Proof of Theorem 2.8

Let $W_n = \max_{1 \le k \le n}\Big(Q_k \cdot \prod_{j=1}^{k-1} M_j\Big)$ and write

$$X_n^{\frac{1}{b_n}} = W_n^{\frac{1}{b_n}} \cdot \left(\frac{\sum_{k=1}^n Q_k \prod_{j=1}^{k-1} M_j}{W_n} \right)^{\frac{1}{b_n}} \tag{20}$$

First, observe that the second factor in (20) converges to one in distribution. Indeed, since $n^{\frac{1}{b_n}} = \exp\big(\frac{1}{b_n} \log n\big) \to_{n\to\infty} 1$, in virtue of the definition of W_n :

$$1 \le \left(\frac{\sum_{k=1}^n Q_k \prod_{j=1}^{k-1} M_j}{W_n} \right)^{\frac{1}{b_n}} \le n^{\frac{1}{b_n}} \to_{n\to\infty} 1.$$

Therefore, the asymptotic behavior of $b_n^{-1} \cdot \log X_n$ is determined by that of $b_n^{-1} \cdot \log W_n$. For $n \in \mathbb{N}$, denote $U_n = \log Q_k$ and $U_n^* = \max_{1 \le k \le n} U_k$. We then have

$$b_n^{-1} \cdot \log W_n = \frac{1}{b_n} \cdot \max_{1 \le k \le n} \Big(\log Q_k + \sum_{j=1}^{k-1} \log M_j \Big) = \frac{1}{b_n}(U_n^* + \log \Pi_{n-1}^*).$$

Therefore

$$U_n^* - |\log \Pi_{n-1}^*| \le b_n^{-1} \cdot \log W_n \le U_n^* + |\log \Pi_{n-1}^*|.$$

In virtue of part (b) of Proposition 2.4, in order to verify the claim of the theorem, it suffices to shows that

$$b_n^{-1} U_n^* \Rightarrow 0. \tag{21}$$

Toward this end, fix arbitrary constants $\varepsilon > 0$ and $\delta > 0$, and observe that condition $\lim_{n\to\infty} nP(\log Q_0 > b_n^{-1}) = 0$ in the statement of the theorem implies that

$$nP(\log Q_{0,i} > b_n \varepsilon) < \delta \tag{22}$$

for all $n \in \mathbb{N}$ large enough, say $n > n_{\varepsilon,\delta}$ for some $n_{\varepsilon,\delta} \in \mathbb{N}$, and any $i \in \mathcal{D}$. Indeed, for any $\varepsilon > 0$ and $i \in \mathcal{D}$ (recall that $\mathcal{D}$ is a finite set):

$$\limsup_{n\to\infty} nP(\log Q_{0,i} > b_n \varepsilon) \le \lim_{n\to\infty} [n/M] \cdot P(\log Q_{0,i} > b_{[n/M]}) = 0,$$

where $[x] := \max\{k \in \mathbb{Z} : k \le\}$ denotes the integer part of $x \in \mathbb{R}$, and M is a large enough integer chosen so that $M^{-\alpha} = \lim_{n\to\infty} b_{[n/M]}/b_n < \varepsilon$ if (A1) of Assumption 2.2 holds or, respectively, $M^{-1/2} = \lim_{n\to\infty} b_{[n/M]}/b_n < \varepsilon$ if (A2) of Assumption 2.2 holds.

For $n > n_{\varepsilon,\delta}$, (22) implies

$$P(U_n^* \le b_n \varepsilon) \ge P(U_{n_{\varepsilon,\delta}}^* \le b_n \varepsilon) \Big(1 - \frac{\delta}{n}\Big)^{n - n_{\varepsilon,\delta}}.$$

Therefore,

$$\liminf_{n\to\infty} P(U_n^* \le b_n \varepsilon) \ge e^{-\delta}.$$

Taking first δ and then ε to zero in the above inequality shows that (21) holds true. The proof of the theorem is therefore completed. □

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